The art of modeling in mechanics

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Contents

§1. Examples of modeling : mechanics → mathematics

The fundamental principles are

$$\text{conservation of}\begin{cases}\text{energy} & \dfrac{\Delta E}{\Delta t} = P \sim \text{external power}\\[2mm] \text{momentum} & \dfrac{\Delta p}{\Delta t} = F \sim \text{ext force}\\[2mm] \text{angular momentum} & \dfrac{\Delta L}{\Delta t} = \tau \sim \text{ext torque.}\end{cases}$$

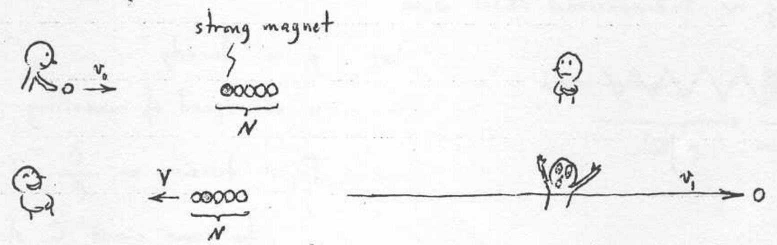
We think of $\frac{\Delta}{\Delta t}$ in general and not merely of $\frac{d}{dt}$, because our problems are not necessarily differentiable or even continuous.

Ex 1. Magnetic croquet

Phenomenon

before :

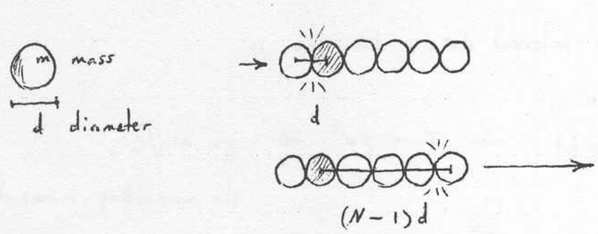
after :



tiny input

huge output

Model



attachment energy (magnetic potential) $\dfrac{\mu}{d^2}$

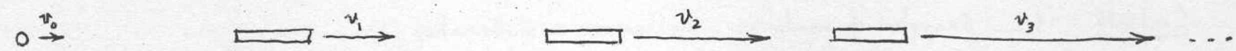
detachment energy $\dfrac{\mu}{(N-1)^2 d^2}$

Analysis Comparing before and after the interaction, conservation laws give

$$\left.\begin{aligned} \text{energy :}\quad & \tfrac{1}{2} m v_0^2 + \frac{\mu}{d^2} = \tfrac{1}{2} N m V^2 + \tfrac{1}{2} m v_1^2 + \frac{\mu}{(N-1)^2 d^2} \\ \text{momentum :}\quad & m v_0 = -NmV + m v_1 \end{aligned}\right\} \Rightarrow$$

$$\Rightarrow\quad v_0^2 + \frac{2\mu}{m}\frac{1}{d^2} = \frac{(v_1 - v_0)^2}{N} + v_1^2 + \frac{2\mu}{m}\frac{1}{(N-1)^2 d^2}.$$

The exact expression of v_1 in terms of v_0 is possible, but let us rather look for a simpler approximate expression by assuming $N \gg 1$. Then $v_0^2 + \frac{2\mu}{m}\frac{1}{d^2} \sim v_1^2$, and upon iterating i times in sequence,



we have $v_i^2 \sim v_0^2 + \frac{2\mu}{m}\frac{i}{d^2}$. If we write $m \sim \rho d^3$ (ρ density of the balls) and start from a negligible $v_0 \sim 0$, then

magnet iteration

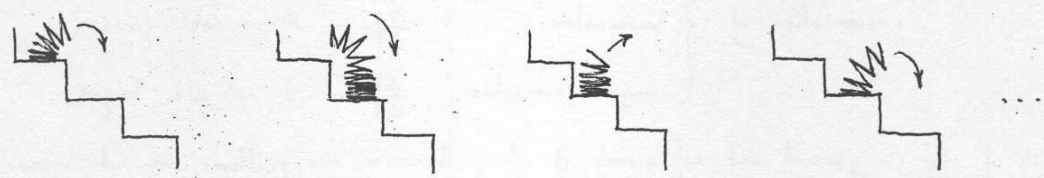
$$v_i \sim \left(\frac{\mu}{\rho}\,\frac{i}{d^5}\right)^{1/2}.$$

final speed

inertia size

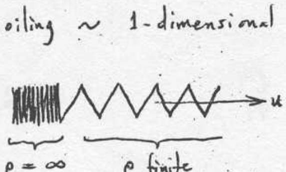
Ex. 2. Slinky

Phenomenon



A surprising observation is that descent time/step is independent of the height of each step.

Model Uncoiling $\sim$ 1-dimensional shock wave



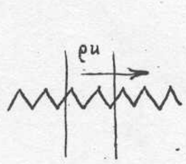
Let ρ = density

u = speed of uncoiling

T = tension $= \frac{C}{\rho}$ (Hooke's law)

for some const C of dimension $\frac{\text{length}^2}{\text{time}^2}$.

Analysis

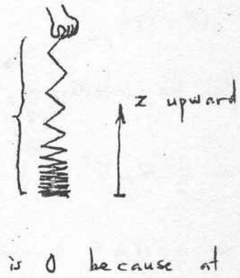


The momentum across interval Δx in time Δt is

$$T\Delta t = \rho u\,\Delta x$$
$$= \rho u \cdot u\Delta t \;\Rightarrow\; T = \rho u^2 \;\Rightarrow\; \rho u = \sqrt{C},$$

the uncoiling momentum is const.

Determination of C: When we hang the slinky,



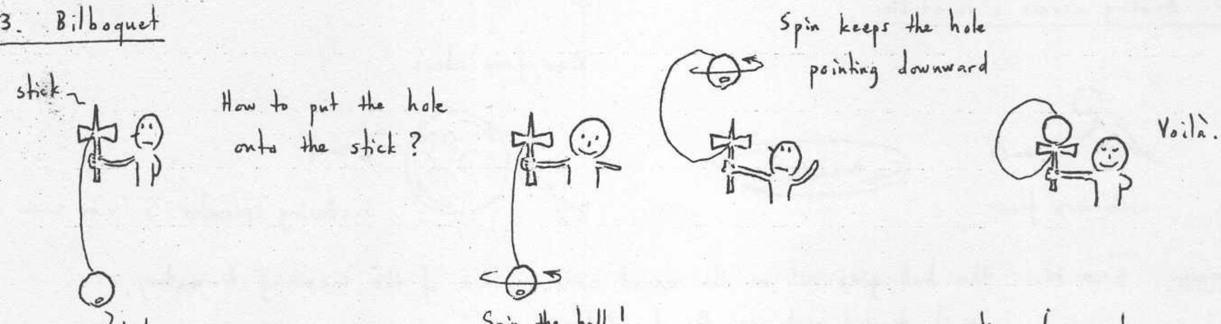
$$\frac{\partial}{\partial z} T = \rho g$$

$$\Rightarrow \frac{1}{\rho}\frac{\partial}{\partial z}\left(\frac{1}{\rho}\right) = \frac{g}{C}$$

$$\Rightarrow \frac{1}{2}\left(\frac{1}{\rho}\right)^2 = \frac{gl}{C} \quad (\text{const of integration is } 0 \text{ because at } z=0,\ \frac{1}{\rho} = \frac{T}{C} = 0)$$

$$\Rightarrow \text{total mass of slinky } m = \int_0^l \rho\, dz = \sqrt{\frac{2Cl}{g}}. \quad \text{So descent time/step} = \frac{m}{\sqrt{C}} = \sqrt{\frac{2l}{g}}.$$

Ex. 3. Bilboquet



... conservation of angular momentum.

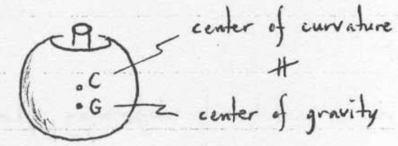
Ex. 4. Tippy top

Phenomenon

Spin a boiled egg ... It stands up!

A spinning body in frictional contact with a floor tends to raise its center of gravity.

Model



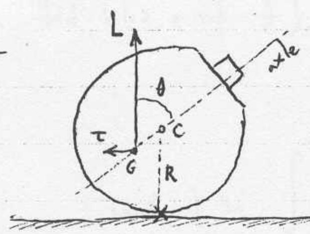
shell spherical of radius R

$$\frac{CG}{R} \ll 1.$$

$$\text{friction} = \mu \times \text{normal force}$$

μ — dimensionless coefficient, usually about $\frac{1}{3}$

Analysis



The angular momentum L remains approximately vertical and passes through G. As the top spins about L, the point of contact (marked ×) is directly below C (not below G) and slides <u>into</u>, therefore the friction points <u>out of</u>, the plane of the picture. This friction $\mu m g$ exerts a torque τ at G. If $\frac{CG}{R} \ll 1$, then τ is approximately horizontal.

Looking at components along the axle,

$$\frac{d}{dt}(L\cos\theta) = -\tau\sin\theta.$$

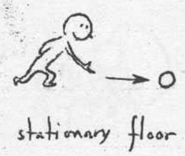
Over a short time-scale $L \sim$ const, so $\dot{\theta} \sim \frac{\tau}{L}$ and the tip-over time T is $T \sim \frac{\pi}{\dot{\theta}} \sim \frac{\pi L}{\tau}$.

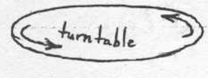
Now $L \sim \underbrace{\frac{2}{5} m R^2}_{\text{moment of inertia}} \underbrace{\omega}_{\text{angular velocity}}$ and $\tau \sim R \mu m g$, so we estimate for Tadashi's tippy top

$$T \sim \frac{\pi \cdot \frac{2}{5} m \cdot R^2 \omega}{R \mu m g} = \frac{\pi \cdot \frac{2}{5} R\omega}{\mu g} \sim \frac{\pi \cdot \frac{2}{5} \cdot 2\,\text{cm} \cdot 2\pi \times 30\,\frac{1}{\text{sec}}}{\frac{1}{3} \cdot 980\,\frac{\text{cm}}{\text{sec}^2}}$$

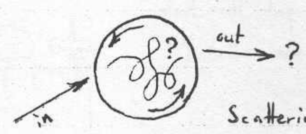
$\sim$ just under 2 sec, in agreement with observation.

Ex. 5. Bowling across a turntable



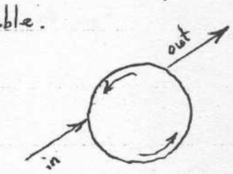


View from above

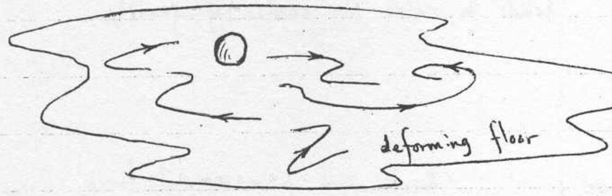


Scattering operator S : in $\mapsto$ out

Theorem $S = \mathrm{id}$. The ball goes out on the exact continuation of the incoming trajectory, as if it did not see the turntable.



More general problem



A ball of radius r, mass m, moment of inertia $\hat{I} m r^2$ rolls on an arbitrarily deforming floor.

dimensionless coefficient $\hat{I} = \begin{cases} \frac{2}{3} & \text{for a spherical shell} \\ \frac{2}{5} & \text{for a solid ball.} \end{cases}$

We work only with the horizontal components of various quantities.
Introduce the complex notation

$$\left.\begin{aligned} z &= x + iy = \text{position of ball's center} \\ \omega &= \omega_x + i\omega_y = \text{angular velocity around center} \\ R &= R_x + iR_y = \text{floor's reaction at point of contact} \\ D(t,z) &= \text{floor's deformation velocity at instant } t, \text{ position } z \end{aligned}\right\} \text{all functions with values in } \mathbb{C} \simeq \mathbb{R}^2.$$

We have the conservation laws of

$$\text{momentum}: \quad \frac{d}{dt}(m\dot{z}) = R \qquad (1)$$

$$\text{angular momentum}: \quad \frac{d}{dt}(\hat{I} m r^2 \omega) = -irR \qquad (2)$$

$$\text{rolling constraint}: \quad \dot{z} + ir\omega = D \qquad (3)$$

[Note that in (2) the torque due to R around the ball's center is $\begin{pmatrix} 0 \\ 0 \\ -r \end{pmatrix} \wedge \begin{pmatrix} R_x \\ R_y \\ 0 \end{pmatrix} = \begin{pmatrix} rR_y \\ -rR_x \\ 0 \end{pmatrix} \longleftrightarrow$
$\longleftrightarrow rR_y - irR_x = -ir(R_x + iR_y) = -irR$.]

Eliminate R then ω to get

$$\left.\begin{matrix}(1)\\(2)\end{matrix}\right\} \Rightarrow \left.\ddot{z} = \hat{I}\, ir\dot{\omega} \quad \begin{matrix}\\(3)\end{matrix}\right\} \Rightarrow (1+\hat{I})\ddot{z} = \hat{I}\dot{D} \Rightarrow$$

$$\Rightarrow \quad \dot{z}(t) - \frac{\hat{I}}{1+\hat{I}} D(t, z(t)) = \text{const} = \dot{z}(0) - \frac{\hat{I}}{1+\hat{I}} D(0, z(0)). \qquad (4)$$

The conservation law (4) we have just discovered is remarkable : it is not due to any obvious symmetry.

Back to the turntable problem

A turntable spinning at const angular velocity Ω is modeled by $D(t,z) = i\Omega z$. Then (4) has the solution

$$z(t) = \underbrace{z(0) - \frac{1+\hat{I}}{\hat{I}}\,\frac{\dot z(0)}{i\Omega}}_{\text{center}} + \underbrace{\frac{1+\hat{I}}{\hat{I}}\,\frac{\dot z(0)}{i\Omega}}_{\text{radius}}\, e^{\frac{\hat{I}}{1+\hat{I}}\, i\Omega t} \qquad (5)$$

angular frequency

a uniform circular motion while the ball is on the table. Outside the table $D=0$, and (4) gives

$$\dot z(\text{going out}) = \dot z(\text{coming in}),$$

which means that the outgoing trajectory is parallel to the incoming trajectory. But these trajectories are connected by a circular arc inside the table by (5), and moreover the motion has a time-reversible symmetry. Hence the conclusion of Theorem.

How realistic is the rolling constraint (3) ?

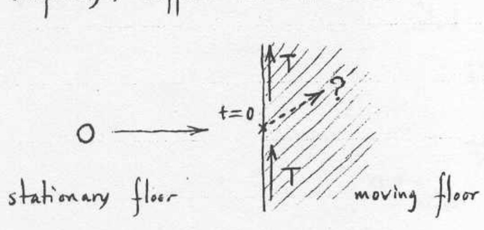
Sliding is prevented as long as the horizontal reaction is kept in check by the friction :

$$|R| \leqslant \mu m g . \qquad (6)$$

In the case of a turntable, substitute (4) with $D = i\Omega z$ in (1) and (6) becomes $|\Omega \dot z| \leqslant \frac{1+\hat{I}}{\hat{I}}\,\mu g$.
For $\hat{I} \sim \frac{2}{5}$, $\mu \sim \frac{1}{3}$, $\dot z \sim 50\ \frac{\text{cm}}{\text{sec}}$, $|\Omega|$ is allowed to be at most $2\pi \cdot 3\ \frac{1}{\text{sec}}$, or 3 turns/sec.
Thus we should not spin the turntable too fast.

When (3) breaks, how bad is the error of the model?

For simplicity, suppose the ball crosses a discontinuity of D at $t=0$:



$$D(t \leqslant 0, z) = 0$$
$$D(t > 0, z) = \text{const } T.$$

Such a discontinuity inflicts a shock ($R = \frac{\text{acceleration}}{\text{mass}} = \infty$) and the no-slide condition (6) is violated. So the ball must slide. The relative velocity between the points of contact is

$$v = \dot z + ir\omega - T, \qquad v(0) = -T.$$

The reaction R is against the direction of v :

$$R = -\mu m g \frac{v}{|v|} .$$

From these and (1), (2), $\dot v = \frac{R}{m} + \frac{R}{\hat{I} m} = -\frac{1+\hat{I}}{\hat{I}}\,\mu g \frac{v}{|v|}$. Since $\dot v$ is parallel to v, as a $\mathbb{C}$-valued function v has const argument : $\frac{v(t)}{|v(t)|} = \frac{v(0)}{|v(0)|} = -\frac{T}{|T|}$. The modulus of v varies as $|v(t)| = |T| - \frac{1+\hat{I}}{\hat{I}}\,\mu g t$; so the ball stops sliding $v(t_*) = 0$ and transits to rolling after time

$$t_* = \frac{\hat{I}}{1+\hat{I}}\,\frac{|T|}{\mu g} .$$

For a turntable of radius 30 cm spinning gently at $\Omega \sim \pi/\text{sec}$, the speed on the rim is $|T| \sim \pi \cdot 30 \frac{\text{cm}}{\text{sec}}$, so a ball crossing into the turntable slides during

$$t_* \sim \frac{\frac{2}{5}}{1+\frac{2}{5}} \frac{\pi \cdot 30}{\frac{1}{3} \cdot 980} \text{ sec} \sim 0.1 \text{ sec},$$

an acceptable error.

Integrating (1) with $R = \mu g \frac{T}{|T|}$, we find

$$z(t_*) = z(0) + \dot{z}(0)\, t_* + \mu g \frac{T}{|T|} \frac{t_*^2}{2}.$$

Let us compare this against the idealized model where the ball is assumed to adjust instantaneously to the shock, satisfying the rolling constraint (3) despite violating (6). By (4), $\dot{z}(t) - \frac{\hat{I}}{1+\hat{I}} T = \dot{z}(0)$, which integrates to

$$z(t_*) = z(0) + \left(\dot{z}(0) + \frac{\hat{I}}{1+\hat{I}} T \right) t_* .$$

The discrepancy between the realistic $z(t_*)$ and the idealized $z(t_*)$ is

$$\left| \mu g \frac{T}{|T|} \frac{t_*^2}{2} - \frac{\hat{I}}{1+\hat{I}} T t_* \right| = \frac{\hat{I}}{1+\hat{I}} \frac{|T|}{2} t_* \sim 1 \text{ cm},$$

again an acceptable error.

Ex. 6. Golfer's chagrin

Roll a ball on the inside of a vertical cylinder. We assume the ball rolls and never slides. Then the motion of the ball's center is

horizontally a uniform revolution, and
vertically a harmonic oscillation (!).

Moreover, the ratio of the frequencies of these two periodic motions is $\sqrt{\frac{7}{2}}$, independently of the initial conditions or the parameters such as mass, radii, gravity...

This may explain why a golf ball sometimes pops out of the hole.

The system, as an integrable problem, seems maximally degenerate (ratio of frequencies frozen from one invariant torus to the next). For modeling and analysis, please read [Gualtieri, Tokieda, et al. Amer. J. of Phys. 74 (2006) 497 – 501].

In summary, to decide how good any model is, we need to

– simplify the model courageously: unless the model is simpler than the phenomenon, there is no point in having a model;

– assess whether the assumptions in the model are physically realistic;

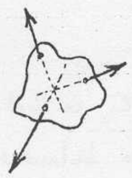
– estimate orders of magnitude of quantities of physical interest, and check the estimates against experiments.

§2. Examples of reverse modeling : mathematics → mechanics

There are many, not well-known, applications of mechanics to mathematics. As an illustration, in this section we use statics to prove various theorems on triangles.

—//—

Lemma



If 3 forces keep a rigid body in equilibrium, then

i) their sum is 0, and

ii) their 'lines of action' meet in a point.

sum = 0

but not in equilibrium

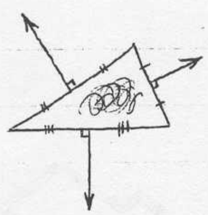
Proof i) means the body does not translate, ii) means it does not rotate. □

Theorem Given a triangle, the following lines meet in a point :

1) perpendicular bisectors of the 3 edges (circumcenter)
2) perpendicular projections of the 3 vertices on their opposite edges (orthocenter)
3) internal bisectors of the 3 angles (incenter)
4) internal bisector of any angle and external bisectors of the other angles (excenter)
5) 3 medians (barycenter).

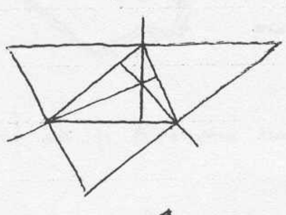
Proof

1)



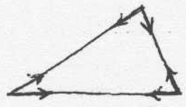
Fill with gas. The pressures on the edges keep the triangle in equilibrium. So Lemma applies to the 3 forces shown.

2)



orthocenter = circumcenter of the 'dual' triangle, so this reduces to 1).

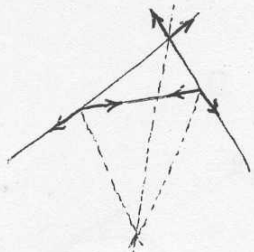
3)



Make 6 forces as shown act on the vertices ; they are in equilibrium. Each pair

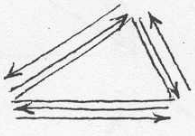
can be replaced by a single force directed along the angle bisector.

4)

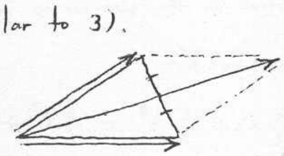


Similar to 3).

5)



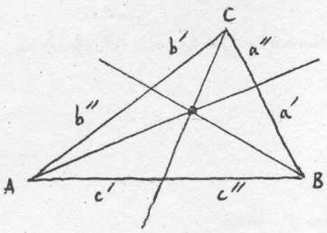
Again similar to 3). Each pair



can be replaced by a single force directed along the median.

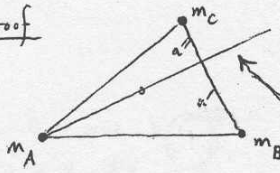
□

Čeva's theorem (generalization of 5)



$$\frac{a'}{a''}\,\frac{b'}{b''}\,\frac{c'}{c''} = 1.$$

Proof

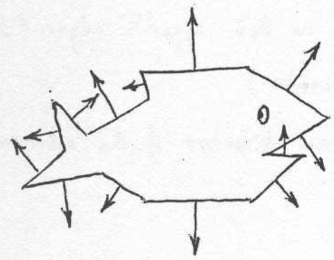


Choose weights m_A, m_B, m_C and place them at A, B, C so that their center of gravity is at the 'Čeva point'. Since the triangle balances on this line,

$$m_B\,a' = m_C\,a'' \Rightarrow \frac{a'}{a''} = \frac{m_C}{m_B}.$$

Cyclically permuting and multiplying, we find $\dfrac{a'}{a''}\,\dfrac{b'}{b''}\,\dfrac{c'}{c''} = \dfrac{m_C}{m_B}\,\dfrac{m_A}{m_C}\,\dfrac{m_B}{m_A} = 1.$ □

Theorem (generalization of 1)

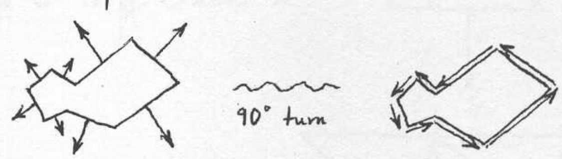


Given any closed polyhedron in $\mathbb{R}^n$, on each face draw an outward normal vector whose length is equal to the area of that face.

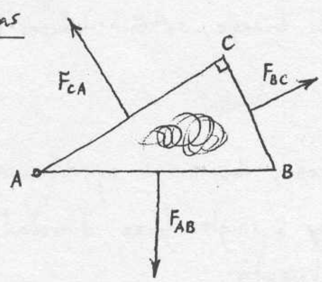
Then the sum of these vectors is 0.

Proof Fill with gas. The pressures on the faces are in equilibrium. □

Remark In $\mathbb{R}^2$, this is easy to prove directly by turning each normal vector by 90°.



Pythagoras



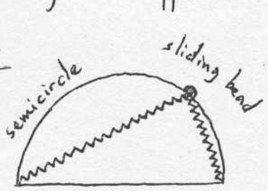
Fill with gas at pressure p. The torques around A of the forces on the edges balance, so

$$F_{AB}\,\frac{AB}{2} = F_{BC}\,\underbrace{\frac{BC}{2}}_{\text{here we use the hypothesis}} + F_{CA}\,\frac{CA}{2}.$$

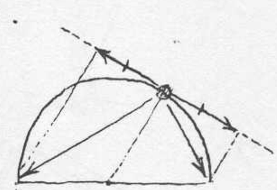
But $F_{AB} = p \cdot AB$, etc.

The same argument applied to an arbitrary triangle gives the cosine law. □

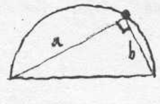
Another proof



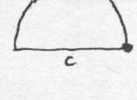
'zero-length' spring: force ∝ length.



Every position on the semicircle is an equilibrium, so the elastic energy is independent of the position:

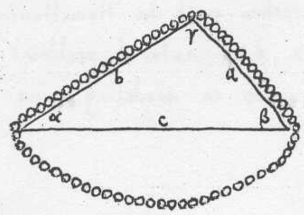


here $\frac{k}{2}a^2 + \frac{k}{2}b^2 = \frac{k}{2}0^2 + \frac{k}{2}c^2$ here

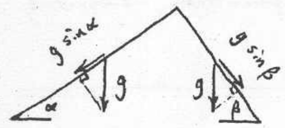


. □

Sine law



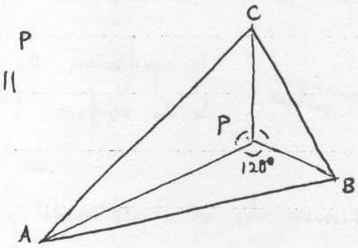
Hang a chain of density ρ. Since it is in equilibrium, parts of the chain on sides b, a balance:



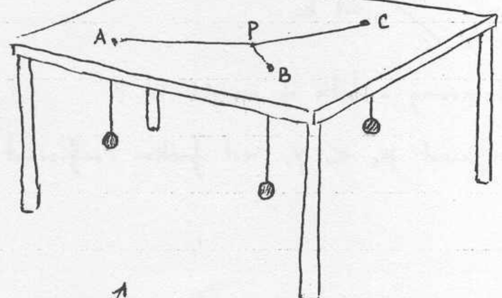
$$\rho b g \sin\alpha = \rho a g \sin\beta \Rightarrow \frac{\sin\alpha}{a} = \frac{\sin\beta}{b},$$

and similarly $\dfrac{\sin\alpha}{a} = \dfrac{\sin\beta}{b} = \dfrac{\sin\gamma}{c}$. □

Theorem Given a triangle ABC, the point (brachistocenter) P such that PA + PB + PC is minimal has the property that all 3 angles around P are 120°.



Proof

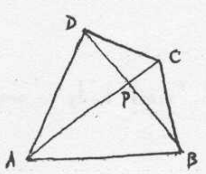


Hang 3 equal weights through holes at A, B, C. The weights pull as far down as they can, thereby minimizing the total length of the strings above the table.

The node thus settles at P. But

3 equal forces are in equilibrium ⟺ they are 120° apart. □

Theorem The brachistocenter of a quadrilateral is at the intersection of the diagonals.



PA + PB + PC + PD minimal.

In summary, it is useful to model not only mechanics by mathematics, but also mathematics by mechanics:

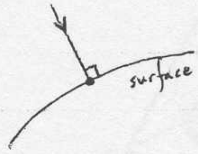
- there are theorems whose traditional proofs are difficult, which are explained easily by mechanics;
- especially theorems of type 'something additive = 0' or 'something multiplicative = 1' are promising candidates for modeling by statics;
- though not discussed in these lectures, much of complex analysis can be derived from ideal fluid dynamics, while some of number theory may be thermodynamics with imaginary temperature (theta function ⟷ partition function).

§3. Theoretical issues in modeling

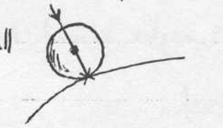
Some mathematicians vaguely assume that a non-dissipative mechanical system must be Hamiltonian. In real life, however, mechanics is far richer and more varied. As seen in §1, natural problems are often non-differentiable, and even when differentiable, surprising issues arise in modeling, as we see in this section.

'Particle' and 'rigid body' are suspicious concepts

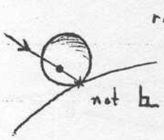
Ex.



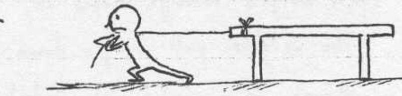
In equilibrium, the reaction that keeps a particle on a surface is supposed perpendicular to the surface, because we model the situation by $\lim\limits_{\text{radius} \to 0}$ ball



But there is no reason why we should model with a symmetric ball:



Ex.

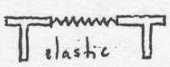


What force will start moving a table of weight W?

Sliding friction coefficient $\mu_s < \mu_r$ rest friction coefficient.

Answer 1: $\underset{\text{front legs}}{\mu_r \frac{W}{2}} + \underset{\text{back legs}}{\mu_r \frac{W}{2}}$.

Answer 2: If



, only $\mu_s \frac{W}{2} + \mu_r \frac{W}{2}$ independently of elasticity. Thus

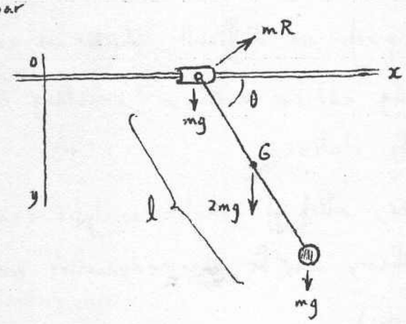
$$\lim_{\text{elastic} \to \text{rigid}} \left(\mu_s \frac{W}{2} + \mu_r \frac{W}{2} \right) = \mu_s \frac{W}{2} + \mu_r \frac{W}{2} < \text{Answer 1}.$$

Models of dynamics based unquestioningly on the concepts of 'particle' and 'rigid body' are not quite well-defined.

Non-existence, non-uniqueness, finite-time singularity in dynamics

Ex. Consider a cylinder

sliding on a bar with friction coefficient μ, joined to a pendulum of the same mass:



$$\text{momentum}: \quad \ddot{x} + \frac{d^2}{dt^2}(x + l\cos\theta) = R_x \qquad (1)$$

$$\frac{d^2}{dt^2}\, l \sin\theta = R_y + 2g \qquad (2)$$

$$\text{angular momentum}: \quad \frac{l^2}{2}\ddot{\theta} = -\frac{l}{2}(R_y \cos\theta - R_x \sin\theta) \qquad (3)$$

The components of the reaction are related by

$$R_x = -\varepsilon\mu R_y \qquad (4)$$

where ε is chosen $+1$ or -1 so as to make $\varepsilon \dot{x} R_y > 0$.

Using (4), eliminate R_x from (1), (2), (3) and solve the resulting 3 eqns for the 3 unknowns $R_y, \ddot{x}, \ddot{\theta}$. We find

$$(1 + \cos^2\theta + \varepsilon\mu \sin\theta\cos\theta)\, R_y = -\ell\dot{\theta}^2 \sin\theta - 2g \qquad (5)$$

(we will not need the expressions for $\ddot{x}, \ddot{\theta}$). Give the initial conditions

$x(0)$ arbitrary, $0 < \theta(0) < \frac{\pi}{2}$, $\dot{x}(0)$ and $\dot{\theta}(0)$ arbitrary.

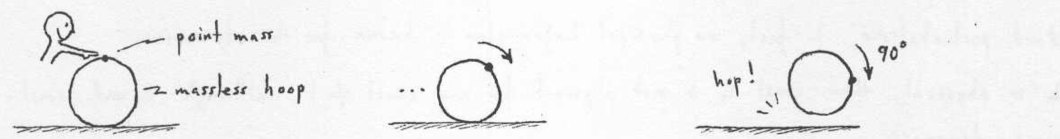
Also suppose μ large enough that $1 + \cos^2\theta(0) - \mu\sin\theta(0)\cos\theta(0) < 0$. Now using (5) we check

$\dot{x}(0) > 0 \Rightarrow$ impossible to choose ε to satisfy (4) (non-existence)

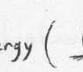
$\dot{x}(0) < 0 \Rightarrow$ choice of ε in (4) ambiguous (non-uniqueness).

<u>Remark</u> $\mu > 2\sqrt{2}$ is 'large enough', but such a friction coefficient is difficult to design in experiments.

<u>Ex.</u>



It is easy to predict that, if the rolling constraint lasts, some singularity must occur before 180°:

energy() > 0, energy() $=$ $\underbrace{\text{kinetic}}_{0}$ $+$ $\underbrace{\text{potential}}_{0}$ $= 0$

rolling means velocity at point of contact is zero.

Please read [Amer. Math. Monthly 104 (1997) 152–154].

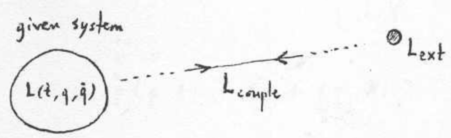
<u>Non-existence, non-uniqueness of Lagrangian</u>

<u>Questions</u> * How can we tell whether a given Lagrangian is 'physical'?

* It is well-known that Lagrangians $L, \tilde{L}$ differ by $L - \tilde{L} = \frac{d}{dt} f \rightarrow$ same dynamics.

What about $\leftarrow$?

The idea is to <u>probe</u> the given system by coupling it to an external perturbation.



Physically it is natural to demand that $L + L_{couple} + L_{ext}$ 'respond correctly'

to any <u>standard perturbation</u>: $L_{ext} = \frac{1}{2} M \dot{r}^2$,

$L_{couple} = -U(t, q, r)$

$\rightarrow 0$ as $|r| \rightarrow \infty$.

<u>Ex.</u> Two Lagrangians $L = \frac{1}{2} m(\dot{q}_1^2 + \dot{q}_2^2) - V(q_1 - q_2)$, $\tilde{L} = m\dot{q}_1 \cdot \dot{q}_2 + V(q_1 - q_2)$ both lead to the same dynamics

$$\begin{cases} m\ddot{q}_1 = -\nabla_{q_1} V(q_1 - q_2) \\ m\ddot{q}_2 = \nabla_{q_1} V(q_1 - q_2) \, . \end{cases}$$

L is the usual Lagrangian for a 2-body problem (equal masses). Is $\tilde{L}$ 'physical'? Perturb with $U(t, q_1, q_2, r) = U(q_1 - r)$. Then $\tilde{L} + L_{couple} + L_{ext}$ leads to

$$\begin{cases} m\ddot{q}_1 = -\nabla_{q_1} V(q_1 - q_2) \\ m\ddot{q}_2 = \nabla_{q_1} V(q_1 - q_2) - \nabla_{q_1} U(q_1 - r) \\ M\ddot{r} = -\nabla_r U(q_1 - r) . \end{cases}$$

The force $-\nabla_{q_1} U(q_1 - r)$ does not affect $\ddot{q}_1$ which it should affect, but instead affects $\ddot{q}_2$ which it should not affect. So $\tilde{L}$ does not respond correctly to standard perturbations.

<u>Ex.</u> $L = e^{\frac{\gamma}{m}t}\left(\frac{1}{2}m\dot{q}^2 - V(q)\right)$ leads to $m\ddot{q} = -\nabla V(q) \underbrace{- \gamma\dot{q}}$

Stokes drag: body moving through honey $Re \ll 1$.

When coupled to a perturbation U, the dynamics should be modified to $m\ddot{q} = -\nabla V(q) - \gamma\dot{q} - \nabla U$, but this $L + L_{couple} + L_{ext}$ leads to $m\ddot{q} = -\nabla V(q) - \gamma\dot{q} - e^{-\frac{\gamma}{m}t}\nabla U$. So L does not respond correctly to standard perturbations. In fact, no physical Lagrangian is known for this dynamics.

<u>Ex.</u> If L is physical, then const.L is not physical for any const $\neq 1$, although L and const.L lead to the same dynamics.

We now answer the uniqueness question.

<u>Theorem</u> Lagrangians L, $\tilde{L}$ that lead to the same dynamics and respond correctly to every standard perturbation U differ at most by some $\frac{d}{dt} f$.

<u>Proof</u> For an arbitrary $q(t)$, choose $U(t, q, r)$ standard so that the given $q(t)$ satisfies

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = -\frac{\partial U}{\partial q_i} \qquad \forall i = 1, \dots, n .$$

Perturb L, $\tilde{L}$ with this U. The hypothesis of correct response means that $q(t)$ satisfies also the eqn with $\tilde{L}$ replacing L. By subtracting we deduce that, letting $D = L - \tilde{L}$,

$$\frac{d}{dt}\frac{\partial D}{\partial \dot{q}_i} - \frac{\partial D}{\partial q_i} = \sum_j \frac{\partial^2 D}{\partial \dot{q}_i \partial \dot{q}_j}\ddot{q}_j + \sum_j \frac{\partial^2 D}{\partial \dot{q}_i \partial q_j}\dot{q}_j + \frac{\partial^2 D}{\partial \dot{q}_i \partial t} - \frac{\partial D}{\partial q_i} = 0 \quad \forall i \qquad (*)$$

is an identity satisfied by every $q(t)$. Since $\ddot{q}_j$ appears in only one place in $(*)$, its coefficient must be zero:

$$\frac{\partial^2 D}{\partial \dot{q}_i \partial \dot{q}_j} = 0 \qquad \forall i, j$$

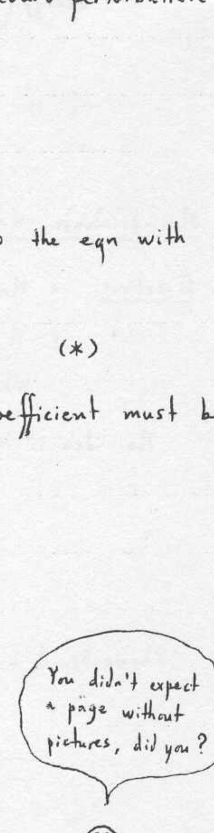
$$\Rightarrow \quad D \text{ has the form } F_0(t, q) + \sum_i F_i(t, q)\dot{q}_i .$$

Substituting in $(*)$ we find

$$\frac{\partial F_\alpha}{\partial q_\beta} = \frac{\partial F_\beta}{\partial q_\alpha} \qquad \forall \alpha, \beta = 0, 1, \dots, n \quad (q_0 = t)$$

$$\Rightarrow \exists f \text{ such that } F_\alpha = \frac{\partial f}{\partial q_\alpha}, \text{ that is}$$

$$D = \frac{\partial f}{\partial t} + \sum_i \frac{\partial f}{\partial q_i}\dot{q}_i = \frac{d}{dt} f . \qquad \square$$



Lack of correspondence between Lagrangian and Hamiltonian

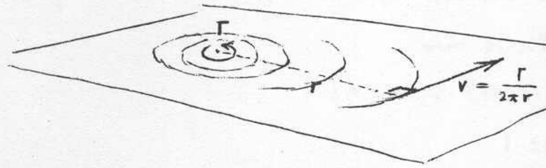
Lagrangian and Hamiltonian formalisms are supposed to be 'dual' via the Legendre transform, but this is not always true.

Lagrangian without Hamiltonian

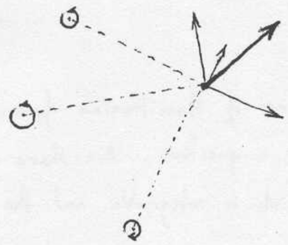
This is a textbook remark. Given a Lagrangian $L(q, \dot q)$, we try to invert $p_i = \dfrac{\partial L}{\partial \dot q_i}$ to express $\dot q_i = \dot q_i(q, p)$ and form the Hamiltonian $H(q,p) = p \cdot \dot q(q,p) - L(q, \dot q(q,p))$. For the inversion to be possible, we need $\det\left(\dfrac{\partial^2 L}{\partial \dot q_i \partial \dot q_j}\right) \neq 0$. When $\det\left(\dfrac{\partial^2 L}{\partial \dot q_i \partial \dot q_j}\right) = 0$, the Lagrangian formalism does not map to the Hamiltonian formalism.

Hamiltonian without Lagrangian

I think the most interesting example is the theory of point vortices in a 2-dimensional ideal fluid (Kirchhoff 1876).



A 'point vortex' induces a flow and carries the ambient points as shown. If there are many vortices, they are carried by mutual interactions and move around on the plane. This dynamics turns out to be Hamiltonian:



$z_1(t), \dots, z_N(t) \in \mathbb{C} \simeq \mathbb{R}^2$ positions of vortices at time t

$\Gamma_1, \dots, \Gamma_N \in \mathbb{R}$ their vorticities (of dimension $\frac{\text{length}^2}{\text{time}}$)

phase space $\underbrace{\mathbb{C}^N \setminus \text{diagonals}}_{\text{not a cotangent bundle}}$, $\underbrace{\omega = \frac{i}{2} \sum_\alpha \Gamma_\alpha \, dz_\alpha \wedge d\bar z_\alpha}_{\text{weighted symplectic form}}$

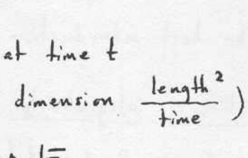
Hamiltonian $\quad H = -\dfrac{1}{2\pi} \sum_{\alpha < \beta} \Gamma_\alpha \Gamma_\beta \log |z_\alpha - z_\beta|$

eqns of motion $\quad \dot z_\alpha = + \dfrac{1}{2\pi i} \sum_{\beta : \beta \neq \alpha} \dfrac{\Gamma_\beta}{\bar z_\beta - \bar z_\alpha}, \quad \alpha = 1, \dots, N.$

$\log|z - c|$ is Green's function for the Laplacian $-\nabla^2$ on $\mathbb{C}$. The theory generalizes to other Kähler manifolds, with or without boundary, provided we can write explicitly Green's function for the Laplacian on that geometry.

Ex. Sector

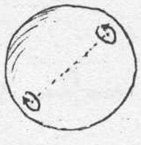
By the method of images, this is equivalent to n copies of the vortex and n copies of the anti-vortex on $\mathbb{C}$:



Green's function is $\displaystyle\sum_{\alpha=0}^{n-1} \log\left|z - e^{i\frac{2\pi}{n}\alpha} c\right| - \sum_{\alpha=0}^{n-1} \log\left|z - e^{-i\frac{2\pi}{n}\alpha} \bar c\right|$ (here $n = 6$)

$$= \log \left| \frac{z^n - c^n}{z^n - \bar c^n} \right|.$$

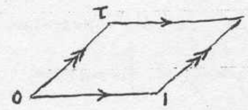
Ex. Sphere



Since the manifold is closed, by Stokes's formula the sum of vorticities must be zero. Therefore when we place a vortex on a sphere, we automatically create an anti-vortex at the antipodal point.

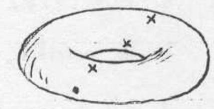
Green's function $\log \tan \frac{\theta}{2}$ is singular at $\theta = 0$ (vortex) and $\theta = \frac{\pi}{2}$ (anti-vortex).

Ex. Torus



$\mathbb{C}/\mathbb{Z} + \tau\mathbb{Z}$, $\quad \operatorname{Im}\tau > 0$.

On a torus, each point has 3 natural choices of 'antipode' (for example 0 is antipodal to $\frac{1}{2}$, $\frac{1}{2} + \frac{\tau}{2}$, $\frac{\tau}{2}$):



Green's function is obtained by periodizing Green's function for $\mathbb{C}$ then renormalizing:

$$\log \left| \frac{\vartheta_k(z|\tau)}{\vartheta_0(z|\tau)} \right| \quad \text{where } k = 1, 2, 3 \text{ correspond to the 3 'antipodal choices'.}$$

The model of point vortices has a beautiful dynamics and is a source of examples and counter-examples in the geometry of momentum maps. Please read

[C.R. Acad. Sci. Paris **333** série I (2001) 943–946]

[J. Fluid Mech. **460** (2002) 83–92]

[Topology **42** (2003) 833–844]

[SIAM J. Appl. Dyn. Sys. **2** (2003) 417–430].

The best introduction is still the chapter on vortex motion in H. Lamb, Hydrodynamics.

Question of integrability

The orthodox definitions of integrability are attached to a particular choice of Hamiltonian formalism, rather than to the underlying dynamics. Physically this is reckless, and raises a question. Are there examples of the same dynamics being modeled by 2 Hamiltonian formalisms, one of which is integrable and the other is not?

In summary, when modeling mechanics we are wise to avoid unconscious assumptions and

— remember that the fundamental principles are conservation laws, and that variational principles or even differential eqns of motion are lucky superstructures if they happen to exist;

— in the Lagrangian case, use a physical Lagrangian;

— think of particles and rigid bodies as limits of continuous media (in general, relax and perturb the model).

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