Modular class and integrability in Poisson and related geometries

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Outline

1. Modular class on Poisson manifolds
   - Curl operator
   - Properties of Modular class

2. Modular classes on Lie algebroids
   - Definition
   - Examples
   - Cohomology pairing
   - Generalizing the curl operator

3. Modular classes of Nijenhuis operators
   - Relative modular class
   - Poisson-Nijenhuis algebroids
   - Jacobi-Nijenhuis algebroids

4. References
It was used by...

- J. L. Koszul (1985)
- J. P. Dufour and A. Haraki (1991)
- J. Grabowski, G. Marmo and A. M. Perelomov (1993)

Classify quadratic Poisson structures
What is the Curl Operator?

On a Poisson manifold \((M, \pi)\) choose a volume form \(\mu\):

\[
\phi : \mathfrak{X}^k(M) \xrightarrow{\sim} \Omega^{n-k}(M), \quad P \xrightarrow{} i_P \mu
\]
What is the Curl Operator?

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The **curl operator** is defined as

\[
D_\mu : \mathfrak{X}^k(M) \rightarrow \Omega^{n-k}(M) \rightarrow \Omega^{n-k+1}(M) \rightarrow \mathfrak{X}^{k-1}(M)
\]

\[
P \xrightarrow{\phi} i_P \mu \xrightarrow{d} d i_P \mu \xrightarrow{\phi^{-1}} D_\mu P
\]

curl of \(P\)
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  \]
  
  
  \[
  P \xrightarrow{\phi} i_P \mu \xrightarrow{d} d i_P \mu \xrightarrow{\phi^{-1}} D_\mu P
  \]
  
  \text{curl of } P

- The curl of a vector field is its divergence with respect to \(\mu\):

\[
D_\mu X = \text{div}_\mu(X) = \frac{\mathcal{L}_X \mu}{\mu}, \quad X \in \mathfrak{x}^1(M).
\]
Definition

The modular vector field of $(M, \pi)$ is the curl of a Poisson bivector $\pi$:

$$X_\mu = D_\mu \pi = \phi^{-1}(\text{di}_\pi \mu).$$
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Remark

This modular vector field depends on the choice of \(\mu\).
Weinstein geometric approach

Question

Given \((M, \pi)\) is there a volume form \(\mu\) invariant under the flow of all hamiltonian vector fields, i.e., such that:

\[
\mathcal{L}_{X_f} \mu = 0, \forall f \in C^\infty(M)\
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The modular vector field relative to \(\mu\) gives:

\[X_{\mu}f = \mathcal{L}_{X_f}\mu/\mu = \text{div}_{\mu} \pi^\#(df).\]
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- \(X_\mu\) is a 1-cocycle in Poisson cohomology

\[ d_\pi X_\mu = \mathcal{L}_{X_\mu} \pi = 0. \]
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\]

- If \(\nu = f\mu\) then: \(X_\nu = X_\mu + \underbrace{\pi^\#(-d \ln |f|)}_{\text{Hamiltonian vector field}}\)
Modular Class

Definition

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Remarks

- The Poisson cohomology groups are usually very hard to compute.
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Remarks

- The Poisson cohomology groups are usually very hard to compute.
- The modular class is a basic invariant, with geometric meaning, which is easy to determine in many examples.
Examples

- Any **symplectic manifold** is unimodular: The Liouville form is invariant under all Hamiltonian flows.
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- \( M = \mathfrak{g}^* \) with Lie-Poisson structure \( \pi \):
  For a translation invariant measure \( \mu \), the modular vector field is the modular character of \( \mathfrak{g} \):

\[
\xi(g) = \text{Tr} \text{ad}_g, \quad g \in \mathfrak{g}.
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Examples

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  For a translation invariant measure $\mu$, the modular vector field is the modular character of $\mathfrak{g}$:

  $$\xi(g) = \text{Tr} \text{ad}_g, \quad g \in \mathfrak{g}.$$  

- For a regular Poisson manifold, with transversally oriented symplectic foliation $\mathcal{F}$, one can show:

  $$i^*(\text{mod } (\mathcal{F})) = \text{mod } (\pi), \quad i^* : H^1(\mathcal{F}) \to H^1_\pi(M).$$
Cohomology and Homology

- Poisson cohomology $H^\bullet(M, \pi)$: $d_\pi = [\pi, -]$;
Cohomology and Homology

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- Poisson homology $H_\bullet(M, \pi)$: $\delta_\pi = [i_\pi, d]$;
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**Theorem ((Evens, Lu and Weinstein, 1998), (Xu, 1999))**

*If $(M, \pi)$ is unimodular then*

$$H_\bullet(M, \pi) \cong H^{n-\bullet}(M, \pi).$$
Holonomy and the modular class

Theorem (Ginzburg and Golubev, 2001)

*For any cotangent loop* $a : I \rightarrow T^* M$:

$$\det h(a) = \exp \left( \int_a \mod (\pi) \right).$$

Remarks
- $h$ denotes the linear Poisson holonomy (defined using parallel transport relative to contravariant version of the Bott connection).
- Integration of Poisson vector field over cotangent paths is invariant under cotangent homotopies.
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Following Evans, Lu and Weinstein (1998)

Substitute for volume form: $\eta \otimes \mu$ section of the line bundle $Q_A = \bigwedge^{\text{top}} A \otimes \bigwedge^{\text{top}} (T^* M)$. 
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**Definition**

The **modular form** of the Lie algebroid $A$ with respect to $\eta \otimes \mu$ is the $A$-form $\xi_A \in \Omega^1(A)$ defined by

$$\langle \xi_A, X \rangle \eta \otimes \mu = [X, \eta] \otimes \mu + \mu \otimes \mathcal{L}_{\rho(X)} \mu, \quad X \in \Gamma(A).$$
Substitute for volume form: \( \eta \otimes \mu \) section of the line bundle \( Q_A = \Lambda^{\text{top}} A \otimes \Lambda^{\text{top}} (T^* M) \).

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\[ \nabla : \Gamma(A) \times \Gamma(Q_A) \rightarrow \Gamma(Q_A) \]

\( (X, \eta \otimes \mu) \rightarrow [X, \eta] \otimes \mu + \eta \otimes \mathcal{L}_{\rho(X)} \mu \)

is a representation of the Lie algebroid \( A \) on \( Q_A \).
Following Evans, Lu and Weinstein (1998)

- $\xi_A$ is a 1-cocycle in the Lie algebroid cohomology

\[ d_A \xi_A = 0 \]
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- $\xi_A$ is a 1-cocycle in the Lie algebroid cohomology
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- If $\eta' \otimes \mu' = f \eta \otimes \mu$ is another section of $Q_A$ then
  \[ \xi'_A = \xi_A - d_A \log f \quad \text{coboundary} \]
Following Evans, Lu and Weinstein (1998)

- \( \xi_A \) is a 1-cocycle in the Lie algebroid cohomology
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**Definition**

\([\xi_A] \in H^1(A)\) is the **modular class** of the Lie algebroid \( A \).
Tangent Bundle: $A = TM$

Assume $M$ is orientable.

- $\Gamma(Q_{TM}) = \mathfrak{X}^{\text{top}}(M) \otimes \Omega^{\text{top}}(M)$
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Assume $M$ is orientable.

- $\Gamma(Q_{TM}) = \mathcal{X}^{\text{top}}(M) \otimes \Omega^{\text{top}}(M)$

- $P \otimes \mu$ such that $\langle \mu, P \rangle = 1$
Tangent Bundle: $A = TM$

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- $\xi_{TM} = 0$
Tangent Bundle: \( A = TM \)

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If \( M \) is orientable then \( \text{mod } TM = 0 \).
A Lie algebra $\mathfrak{g}$ is a Lie algebroid over one point.
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- $Q_g = \wedge^{\text{top}} g$
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- The modular form is the adjoint character

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\langle \xi_g, g \rangle = \text{Tr} \, \text{ad}_g, \quad g \in g.
\]

\[
\text{mod } g = \text{mod} \left( g^*, \pi_{\text{Lie}} \right)
\]
Foliation: $A = TF$

- $TF$: integrable subbundle of $TM$ with orientable conormal bundle $N$. 
Foliation: $A = T\mathcal{F}$

- $T\mathcal{F}$: integrable subbundle of $TM$ with orientable conormal bundle $N$.
- $Q_A \cong \wedge^{\text{top}} N$
Foliation: $\mathcal{A} = T\mathcal{F}$

- $T\mathcal{F}$: integrable subbundle of $TM$ with orientable conormal bundle $N$.
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Poisson manifold: $A = T^*M$

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- $(M, \pi)$ is a Poisson manifold
- $T^*M$ is a Lie algebroid over $M$ with anchor $\rho = \pi^\#$ and Lie bracket

$$[\alpha, \beta] = \mathcal{L}_{\pi^\# \alpha} \beta - \mathcal{L}_{\pi^\# \beta} \alpha - d\pi(\alpha, \beta), \quad \alpha, \beta \in \Omega^1(M).$$
Poisson manifold: $A = T^*M$

- $(M, \pi)$ is a Poisson manifold
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$$\text{mod } T^*M = 2\text{mod } (\pi)$$
Cohomology and Homology

- Pairing between $H^\bullet(A, d_A)$ and $H^{r-\bullet}(A, Q_A)$, with
  - trivial coefficients
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- Pairing between $H^\bullet(A, d_A)$ and $H_{r-\bullet}(A, Q_A)$, with:
  - trivial coefficients
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- $(M, \pi)$ Poisson manifold.

  Pairing $H_\bullet(M)$ and $H_{n-\bullet}(M, \pi)$
P. Xu (1999) relates the study of modular vector fields on Poisson manifold with the notion of Gerstenhaber algebra.
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Extended to Lie-Rienart algebras by Huebschmann (1999).
Generators of Gerstenhaber algebras

Definition

An operator \( \partial \) of degree \(-1\) is a **generator of the Gerstenhaber algebra** \( \mathcal{X}(A) \) if, for \( P \in \mathcal{X}^{p}(A) \), \( Q \in \mathcal{X}(A) \),

\[
[P, Q] = (-1)^{p} (\partial(P \wedge Q) - \partial P \wedge Q - (-1)^{p} P \wedge \partial Q).
\]

If \( \partial^{2} = 0 \) then \( \mathcal{X}(A) \) is **exact** or a Batalin-Vilkosky algebra.
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If \( \partial^2 = 0 \) then \( \mathcal{X}(A) \) is **exact** or a Batalin-Vilkosky algebra.

Let \( \partial \) and \( \partial' \) be two generators of the Gerstenhaber algebra \( \mathcal{X}(A) \), such that \( \partial^2 = \partial'^2 = 0 \). Then

\[
\partial - \partial' = i_\alpha,
\]

for a closed 1-form \( \alpha \in \Gamma(A^*) \).

**P. Xu (1999)**
What happens with triangular Lie bialgebroids?

- $P \in \mathfrak{x}^2(A)$ a Poisson bivector: $[P, P] = 0$;
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- $P \in \mathfrak{x}^2(A)$ a Poisson bivector: $[P, P] = 0$;
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- $P \in \mathfrak{x}^2(A)$ a Poisson bivector: $[P, P] = 0$;
- $A^*$ is also a Lie algebroid;
- $\partial_P = [i_P, d_A] = i_A d_A - d_A i_A$ is a generator of $\Omega(A) = \mathfrak{x}(A^*)$. 

Let $\mu$ be a top-section of $A^*$. It defines another generator of $\Omega(A)$: $\partial_P, \mu = -\phi d_P \phi - 1$ where $\phi Q = i_Q \mu$, $Q \in \mathfrak{x}(A)$. 

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Modular class and integrability in Poisson and related geometries
What happens with triangular Lie bialgebroids?

- $P \in \mathfrak{x}^2(A)$ a Poisson bivector: $[P, P] = 0$;
- $A^\ast$ is also a Lie algebroid;
- $\partial_P = [i_P, d_A] = i_A d_A - d_A i_A$ is a generator of $\Omega(A) = \mathfrak{x}(A^\ast)$.

Let $\mu$ be a top-section of $A^\ast$. It defines another generator of $\Omega(A)$:

$$\partial_{P, \mu} = -\phi d_P \phi^{-1}$$

where $\phi Q = i_Q \mu$, $Q \in \mathfrak{x}(A)$. 

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The **modular vector field** of a triangular Lie bialgebroid associated with $\mu$ is the vector field $X_\mu$ given by

$$\partial_{P,\mu} - \partial_P = i_{X_\mu}.$$
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The **modular vector field** of a triangular Lie bialgebroid associated with $\mu$ is the vector field $X_\mu$ given by

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$\text{mod} (A, P) = [X_\mu] \in H^1(A^*)$ is the **modular class** of $(A, P)$.

If $(A, P)$ unimodular then

$$H_\bullet (A, \delta_P) \cong H_{\text{top} - \bullet} (A, d_A).$$
Relation between the two generalizations

If \((M, \pi)\) is Poisson manifold:

\[
\text{mod} \left( TM, \pi \right) = \text{mod} \left( \pi \right) = \frac{1}{2} \text{mod} \ T^* M.
\]
Relation between the two generalizations

- If $(M, \pi)$ is Poisson manifold:
  \[ \text{mod} (TM, \pi) = \text{mod} (\pi) = \frac{1}{2} \text{mod} T^*M. \]

- For general triangular bialgebroids
  \[ \text{mod} (A, P) \neq \frac{1}{2} \text{mod} A^* \]
The **relative modular class** of the Lie algebroid morphism \( \varphi : A \to B \) is

\[
\text{mod } \varphi(A, B) = \text{mod } A - \varphi^*(\text{mod } B)
\]
The relative modular class of the Lie algebroid morphism \( \varphi : A \rightarrow B \) is

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\text{mod} \varphi(A, B) = \text{mod} A - \varphi^*(\text{mod} B)
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\text{mod}^{\rho^A}(A, TM) = \text{mod} A.
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The relative modular class of the Lie algebroid morphism \( \varphi : A \to B \) is

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\text{mod} \varphi(A, B) = \text{mod} A - \varphi^* (\text{mod} B)
\]

- \( \text{mod}^{\rho_A}(A, TM) = \text{mod} A \).
- \( \text{mod}^{P\#}(A^*, A) = 2 \text{mod} (A, P) \).
Nijenhuis operators

- $N \equiv \text{Nijenhuis operator of } A;$
Nijenhuis operators

- $N \equiv \text{Nijenhuis operator of } A$;
- $A_N = (A, \rho \circ N, [ , ]_N)$ is a Lie algebroid,

$$[X, Y]_N = [NX, Y] + [X, NY] - N[X, Y], \quad X, Y \in \mathfrak{x}^1(A).$$
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Theorem

Fix $\eta \otimes \mu$ section of $Q_A \wedge_{\text{top}} (A) \otimes \wedge_{\text{top}} (T^* M)$. Then

$$\xi_{A_N} = d_A (\text{Tr } N) + N^* \xi_A.$$

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The 1-form

$$\xi_{AN} - N^* \xi_A = d_A \text{Tr} N$$

is independent of the choice of section of $Q_A$. 

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The 1-form

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is independent of the choice of section of \( Q_A \).

\([d_A \text{Tr} N] \in H^1(AN)\) is the relative modular class of the Lie algebroid morphism \( N : AN \rightarrow A \).
\((A, P, N)\): Poisson-Nijenhuis Lie algebroid
- $N^*$ is a Nijenhuis operator of the dual Lie algebroid $(A^*, [\ , \ ]_P, \rho \circ P^\#)$.
- $N^*$ is a Nijenhuis operator of the dual Lie algebroid

$$\left( A^*, [\cdot, \cdot]^P, \rho \circ P^\# \right).$$

- The relative modular class of $N^*$ has the canonical representative $d_P(\text{Tr} N^*)$, so that:

$$\text{mod}^{N^*}(A_{N^*}^*, A^*) = [d_P(\text{Tr} N^*)] = [d_P(\text{Tr} N)].$$
• $N^*\ is\ a\ Nijenhuis\ operator\ of\ the\ dual\ Lie\ algebroid\ 
\quad (A^*\ ,\ [\ ,\ ]_P\ ,\ \rho\ \circ\ P^\#).$

• The relative modular class of $N^*$ has the canonical representative $d_P(\text{Tr}\ N^*)$, so that:
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\mod^{N^*}(A^*_{N^*}, A^*) = [d_P(\text{Tr} N^*)] = [d_P(\text{Tr} N)].
\]

**Definition**

The **modular vector field** of the Poisson-Nijenhuis Lie algebroid $(A, P, N)$ is
\[
X_{(N,P)} = \xi_{A^*_{N^*}} - N\xi_{A^*} = d_P(\text{Tr} N) = -P^\#(d_A\text{Tr} N) \in \Gamma(A).
\]
Theorem

Suppose \( N \) is non-degenerated Nijenhuis operator.

- \( X_{(N,P)} \) is a \( d_{NP} \)-coboundary;
Suppose $N$ is non-degenerated Nijenhuis operator.

- $X_{(N,P)}$ is a $d_{NP}$-coboundary;
- A hierarchy of vector fields

\[ X^{i+j}_{(N,P)} = N^{i+j-1} X_{(N,P)} = d_{NiP} h_j = d_{NjP} h_i, \quad (i, j \in \mathbb{Z}) \]

where $h_0 = \ln(\det N)$ and $h_i = \frac{1}{i} \text{Tr} \ N^i$, \quad (i \neq 0);
Theorem

Suppose $N$ is non-degenerated Nijenhuis operator.

- $X_{(N,P)}$ is a $d_{NP}$-coboundary;
- A hierarchy of vector fields

$$X^{i+j}_{(N,P)} = N^{i+j-1} X_{(N,P)} = d_{N^i P} h_j = d_{N^j P} h_i, \quad (i, j \in \mathbb{Z})$$

where $h_0 = \ln(\det N)$ and $h_i = \frac{1}{i} \mathrm{Tr} N^i, \quad (i \neq 0)$;
- A hierarchy of vector fields on $M$ given by:

$$X_{i+j} = -\pi^j_i dh_j = -\pi^i_j dh_i \quad (i, j \in \mathbb{Z})$$

where $\pi_j$ are Poisson structures on $M$. 
The hierarchy of modular vector fields $X_{(Ni,P)}$ is generated by the Nijenhuis operator $N$. 
The hierarchy of modular vector fields $X_{(N^i,P)}$ is generated by the Nijenhuis operator $N$.

BUT
The hierarchy of modular vector fields $X_{(N^i,P)}$ is generated by the Nijenhuis operator $N$.

BUT

The covered hierarchy of bi-Hamiltonian vector fields on $M$ may not be generated by any Nijenhuis operator.
The two canonical examples

- **Tangent bundle:**

- **Lie algebra \((g, r, N)\):**
  \(N\) defines a sequence of modular forms on \(g^*\), \(\xi_{g^*_{Nk}}\), which are associated with the higher brackets on \(g^*\).

**Relation between modular forms of \(g^*\)**

\[
\xi_{g^*_{Nk}} = N^k \xi_{g^*_{N^*}}.
\]
A Jacobi algebroid is a Lie algebroid $A$ equipped with a closed 1-form $\phi$.
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**Schouten-Jacobi bracket** on $\mathfrak{x}(A)$:

$$[P, Q]^{\phi} = [P, Q] + (p - 1)P \wedge i_{\phi} Q - (-1)^{p-1}(q - 1)i_{\phi} P \wedge Q,$$

for $P \in \mathfrak{x}^p(A)$, $Q \in \mathfrak{x}^q(A)$. 

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**Jacobi algebroid**
Consider the Lie algebroid $\hat{A} = A \times \mathbb{R}$ over $M \times \mathbb{R}$, with Lie bracket

$$[X, Y]_{\hat{A}} = [X, Y], \quad X, Y \in \mathfrak{x}(A),$$
Consider the Lie algebroid $\hat{A} = A \times \mathbb{R}$ over $M \times \mathbb{R}$, with Lie bracket
\[ [X, Y]_{\hat{A}} = [X, Y], \quad X, Y \in \mathfrak{x}(A), \]
and anchor
\[ \hat{\rho}(X) = \rho(X) + \langle \phi, X \rangle \frac{\partial}{\partial t}, \quad X \in \mathfrak{x}^1(A). \]
Consider the Lie algebroid \( \hat{A} = A \times \mathbb{R} \) over \( M \times \mathbb{R} \), with Lie bracket

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\hat{\rho}(X) = \rho(X) + \langle \phi, X \rangle \frac{\partial}{\partial t}, \quad X \in \mathfrak{x}^1(A).
\]  

_induced Lie algebroid structure from \( A \) by \( \phi \)._
Poissonization

\( \phi = \hat{dt}; \)
Poissonization

- $\phi = \hat{dt}$;

- $\tilde{P} = e^{-(p-1)t}P, \quad P \in \mathfrak{x}^p(A)$;
Poissonization

- $\phi = \hat{dt}$;

- $\tilde{P} = e^{-(p-1)t}P, \quad P \in \mathfrak{x}^p(A)$;

- $[\tilde{P}, \tilde{Q}]_{\hat{A}} = [P, Q]_{\phi}$;
Poissonization

\[ \phi = \hat{dt}; \]

\[ \tilde{P} = e^{-(p-1)t} P, \quad P \in \mathfrak{x}^p(A); \]

\[ \left[ \tilde{P}, \tilde{Q} \right]_{\hat{A}} = \left[ P, Q \right]_{\phi}; \]

\[ [P, P]_{\phi} = 0 \iff \tilde{P} = e^{-t}P \text{ a Poisson bivector of } \hat{A} \]
Poissonization

\[ \phi = \hat{dt}; \]

\[ \tilde{P} = e^{-(p-1)t}P, \quad P \in \mathfrak{x}^p(A); \]

\[ \left[ \tilde{P}, \tilde{Q} \right]_{\hat{A}} = \tilde{[P, Q]}_{\phi}; \]

\[ [P, P]_{\phi} = 0 \iff \tilde{P} = e^{-t}P \text{ a Poisson bivector of } \hat{A} \]

\[ (\hat{A}^*, [\ , \ ]_{\tilde{P}}, \hat{\rho} \circ \tilde{P}^\#) \] is a Lie algebroid
Poissonization

\[ \hat{\alpha} = e^t \alpha, \quad \alpha \in \Omega^1(A); \]
Poissonization

1. \( \hat{\alpha} = e^{t\alpha}, \quad \alpha \in \Omega^1(A); \)
2. \[ [\hat{\alpha}, \hat{\beta}]_P = e^t(\mathcal{L}_{P^\#\alpha} \beta - \mathcal{L}_{P^\#\beta} \alpha - d^\phi P(\alpha, \beta)) \]

\( [\hat{\alpha}, \hat{\beta}]_P \)
Poissonization

- $\hat{\alpha} = e^t \alpha$, $\alpha \in \Omega^1(A)$;
- $\left[ \hat{\alpha}, \hat{\beta} \right]_\tilde{P} = e^t \left( \mathcal{L}^\phi_{\mathcal{P}^\#} \alpha \beta - \mathcal{L}^\phi_{\mathcal{P}^\#} \beta \alpha - d^\phi \mathcal{P}(\alpha, \beta) \right)$
- $(A^*, [\ , \ ]_P, \rho \circ \mathcal{P}^\#)$ is a Lie algebroid;
Poissonization

\( \hat{\alpha} = e^t \alpha, \quad \alpha \in \Omega^1(A); \)

\[
\left[ \hat{\alpha}, \hat{\beta} \right]_{\tilde{P}} = e^t \left( \mathcal{L}_{P^\#}^{\phi} \alpha \beta - \mathcal{L}_{P^\#}^{\phi} \beta \alpha - d^{\phi} P(\alpha, \beta) \right)_{[\alpha, \beta]_P}
\]

\( (A^*, \{ , \}_P, \rho \circ P^\#) \) is a Lie algebroid;

\( N \) Nijenhuis operator of \( A \) \( \implies \) \( N \) Nijenhuis operator of \( \hat{A} \)
Poissonization

- $\widehat{\alpha} = e^t \alpha, \quad \alpha \in \Omega^1(A)$;
- $\left[\widehat{\alpha}, \widehat{\beta}\right]_\tilde{P} = e^t \left(\mathcal{L}_{\tilde{P}^\#}^\phi \alpha \beta - \mathcal{L}_{\tilde{P}^\#}^\phi \beta \alpha - d^\phi P(\alpha, \beta)\right)$
- $(A^*, [\ , \ ]_P, \rho \circ P^\#)$ is a Lie algebroid;
- $N$ Nijenhuis operator of $A \iff N$ Nijenhuis operator of $\hat{A}$

Definition

$(A, \phi, P, N)$ is a **Jacobi-Nijenhuis algebroid** if $(\hat{A}, \tilde{P}, N)$ is a Poisson-Nijenhuis algebroid.
$(\hat{A}, \tilde{P}, N)$ has a modular vector field

$$\hat{X}_{(N, \tilde{P})} = \xi_{\hat{A}^*} - N\xi_{\hat{A}^*} = e^{-t}d_P(\text{Tr} N).$$
(\(\hat{A}, \tilde{P}, N\)) has a modular vector field

\[
\hat{X}_{(N, \tilde{P})} = \xi_{\hat{A}^* - N\xi_{\hat{A}^*} = e^{-t}dP(\text{Tr } N)}.
\]

**Definition**

The **modular vector field** of the Jacobi-Nijenhuis algebroid \((A, \phi, P, N)\) is

\[
X_{(N, P)} = \xi_{A^* - N\xi_A}.
\]
\((\hat{A}, \tilde{P}, N)\) has a modular vector field

\[
\hat{X}_{(N,\tilde{P})} = \xi_{\hat{A}_N^*} - N\xi_{\hat{A}^*} = e^{-t}d_P(\text{Tr} N).
\]

**Definition**

The **modular vector field** of the Jacobi-Nijenhuis algebroid \((A, \phi, P, N)\) is

\[
X_{(N,P)} = \xi_{A_N^*} - N\xi_{A^*}.
\]

\[\text{mod}^{(N,P)} A = \left[X_{(N,P)}\right] \text{ the modular class of } (A, \phi, P, N).\]
If $N$ is non-degenerated, then

1. $\hat{X}_{(N,\bar{P})}^i = N^{i+j-1} \hat{X}_{(N,\bar{P})} = d_{N^i\bar{P}} h_j = d_{N^i\bar{P}} h_i \in \Gamma \hat{A}$

2. $X_{(N,P)}^{i+j} = N^{i+j-1} X_{(N,P)} = d_{N_i P} h_j = d_{N_i P} h_i \in \Gamma (A)$

$$h_0 = \ln(\det N) \quad \text{and} \quad h_i = \frac{1}{i} \text{Tr} N^i, \quad (i \neq 0, \quad i, j \in \mathbb{Z}).$$

These hierarchies cover two hierarchies, one on $M \times \mathbb{R}$ and another one on $M$. 


Homework

