

# SEQUENCES OF KNOTS AND THEIR LIMITS

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## Abstract

This work concerns the possibility of taking limits of sequences of knots of increasing crossing number. We show that whenever there is a knot invariant which takes values on a closed metric space these limits may exist in the appropriate closure of a quotient space of knots constructed using the given invariant. We construct examples of such situations.

## 1 Theoretical foundations

Consider a knot invariant, say  $f$ , which takes values on a closed metric space,  $M$ , i.e.

$$f : \mathcal{K} \longrightarrow M$$

where  $\mathcal{K}$  denotes the collection of knots. Consider the (equivalence) relation on  $\mathcal{K}$ :

$$K \sim K' \quad \stackrel{\text{def.}}{\Leftrightarrow} \quad f(K) = f(K')$$

i.e., two knots are in the same class if they have the same ( $f$ ) invariant. The collection of all these equivalent classes is denoted by  $\mathcal{K}^\sim$ . Figure 1 shows how  $\mathcal{K}$ ,  $M$ , and  $\mathcal{K}^\sim$  relate to each other. In Figure 1  $\sim$  is the map that takes a knot

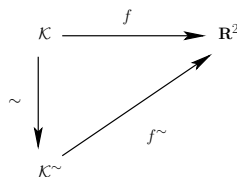


Figure 1: The  $f$  map and a relevant factorization of it

to its equivalence class and  $f^\sim$  is the map that takes an equivalence class to the value of the  $f$  invariant of any of its representatives. Since  $f^\sim$  is injective it embeds  $\mathcal{K}^\sim$  into the closed metric space  $M$ . In particular it can be regarded as a metric subspace of  $M$ . Taking the closure of  $\mathcal{K}^\sim$  with respect to this topology we obtain  $\overline{\mathcal{K}^\sim}$  which is what we referred to above as the appropriate closure of a quotient space of knots. In this way, suppose there is a sequence of knots, say  $(K_n)$ , such that the sequence  $(f(K_n))$  converges. Then, the sequence  $([K_n])$  converges to an element of  $\overline{\mathcal{K}^\sim}$ , say  $K_\infty$ . We call such  $K_\infty$  *hyperfinite knots*.

## 2 Concrete example

We now present an example of a knot invariant which takes values on a closed metric space: the CJKLS invariant. For further information on these topics we refer the reader to [1] and references therein.

There is a CJKLS invariant for each choice of

- (finite) Quandle,  $X$
- (finite) Abelian group,  $A$
- Quandle 2-cocycle,  $\phi$

A quandle is a set,  $X$ , equipped with a binary operation,  $*$ , satisfying the following axioms. For any  $a, b, c \in X$  (i)  $a * a = a$  (idempotency); (ii) there is a unique  $x$  such that  $x * b = a$  (right-invertibility); and (iii)  $(a * b) * c = (a * c) * (b * c)$  (self-distributivity). An example of a quandle is any group equipped with conjugation for the  $*$  operation. Here is another example of a quandle which will pave the way for the sequel. Any (oriented) diagram of a given knot  $K$  gives rise to a presentation of a quandle by regarding the arcs as generators and reading relations at the crossings of the sort  $a * b = c$  where  $b$  is the over-arc and  $c$  is the under-arc the co-orientation at the over-arc points to. The quandle so presented is a knot invariant; it is called the knot quandle (of the given knot). Now fix a finite quandle  $X$ . The number of homomorphisms (a.k.a. colorings) from the knot quandle to  $X$  is again an invariant of  $K$ . Finally a quandle 2-cocycle  $\phi$  is map from  $X \times X$  to an abelian group  $A$  such that  $\phi(a, a) = 1$  and  $\phi(a, b)\phi(a * b, c) = \phi(a, c)\phi(a * c, b * c)$ . The CJKLS invariant of  $K$ , denoted  $Z(K)$ , is then a sum over each coloring,  $C$ , of the products of the factors  $\phi(a_C, b_C)^{\epsilon_\tau}$  ( $\epsilon_\tau = \pm 1$ ) over the crossings,  $\tau$ , of a diagram of  $K$ . Figure 2 illustrates the assignments of  $\phi$  to the crossings  $\tau$ .



Figure 2: The two possible evaluations of  $\phi$  at a crossing ( $a_C$  and  $b_C$  are part of an overall coloring  $C$ ).

$$Z(K) := \sum_{\text{colorings by } X, C} \prod_{\text{crossings}, \tau} \phi_\tau^{\epsilon_\tau}(a_C, b_C)$$

In this poster, the choice of  $X$ ,  $A$ , and  $\phi$  for the CJKLS invariant is:

- $X = S_4 \cong \mathbb{Z}_2[T, T^{-1}]/(T^2 + T + 1)$        $a * b = Ta + (1 - T)b$       (in the quotient)
- $A = \mathbb{Z}_2 \cong (t \mid t^2 = 1)$
- 

$$\phi(a, b) = \begin{cases} 1, & \text{if } a = b \text{ or } a = T \text{ or } b = T \\ t, & \text{otherwise} \end{cases}$$

# THE CJKLS INVARIANT OF THE TREFOIL

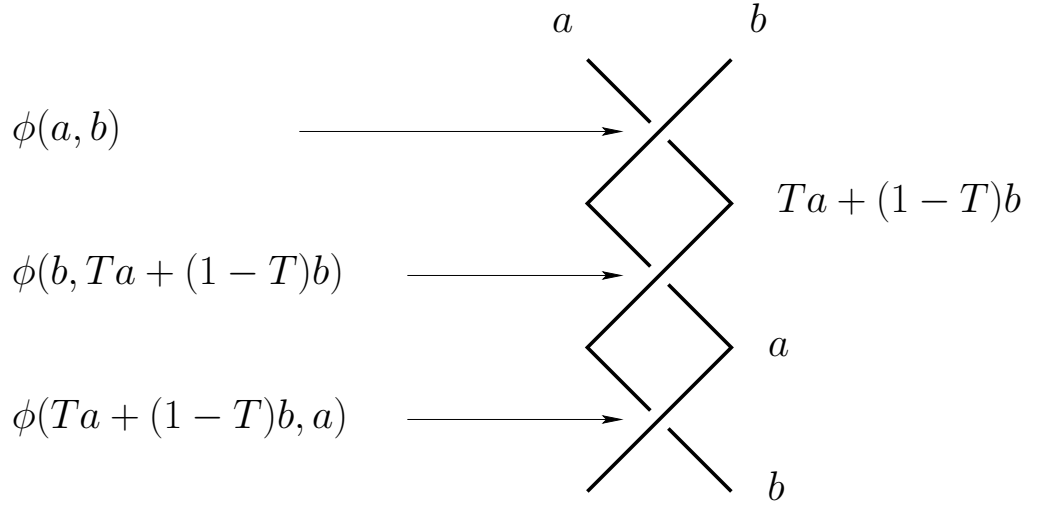


Figure 3: The colorings and evaluation of the 2-cocycle at crossings for the trefoil

**SET**

$$\Phi(a, b) := \phi(a, b) \cdot \phi(b, Ta + (1 - T)b) \cdot \phi(Ta + (1 - T)b, a) = \begin{cases} t, & \text{if } a \neq b \\ 1, & \text{if } a = b \end{cases} =: t^{\bar{\delta}_{a,b}}$$

**SO**

$$Z(\mathbf{Trefoil}) = \sum_{a,b \in \{0,1,T,1+T\}} t^{\bar{\delta}_{a,b}} = 4(1 + 3t)$$

We remark that (any) CJKLS invariant takes on values in the ring algebra  $\mathbb{Z}[A]$  which embeds in  $\mathbb{R}^{|A|}$  - the coefficient affecting the  $i$ -th group element becomes the  $i$ -th coordinate. In this way, the CJKLS invariant takes on values on a closed metric space. In the case under study this is  $\mathbb{R}^2$  and

$$Z(\mathbf{Trefoil}) = (4, 12)$$

**THE  $K_n$  SEQUENCE OF KNOTS**  
**(INCREASING CROSSING NUMBER)**

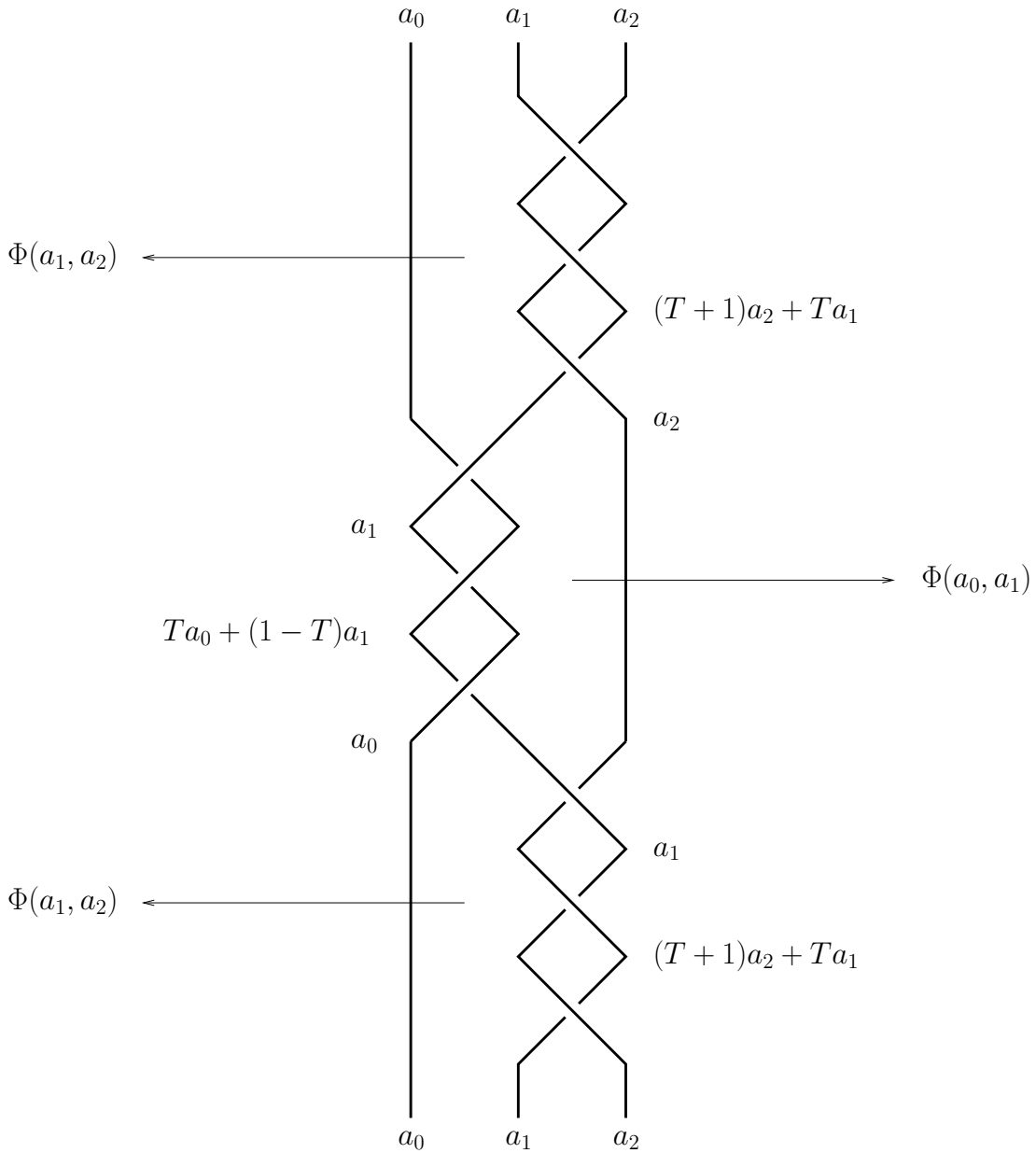


Figure 4:  $K_2$ , upon closure of the braid, endowed with a coloring by  $S_4$

$$\begin{aligned}
 Z(K_2) &= \sum_{a_0, a_1, a_2 \in \{0, 1, T, 1+T\}} \Phi(a_1, a_2) \Phi(a_0, a_1) \Phi(a_1, a_2) = \sum_{a_0, a_1, a_2 \in \{0, 1, T, 1+T\}} t^{\bar{\delta}_{a_0, a_1}} = \\
 &= 4^2(1 + 3t) \quad \longleftrightarrow \quad (4^2, 4^2 \cdot 3)
 \end{aligned}$$

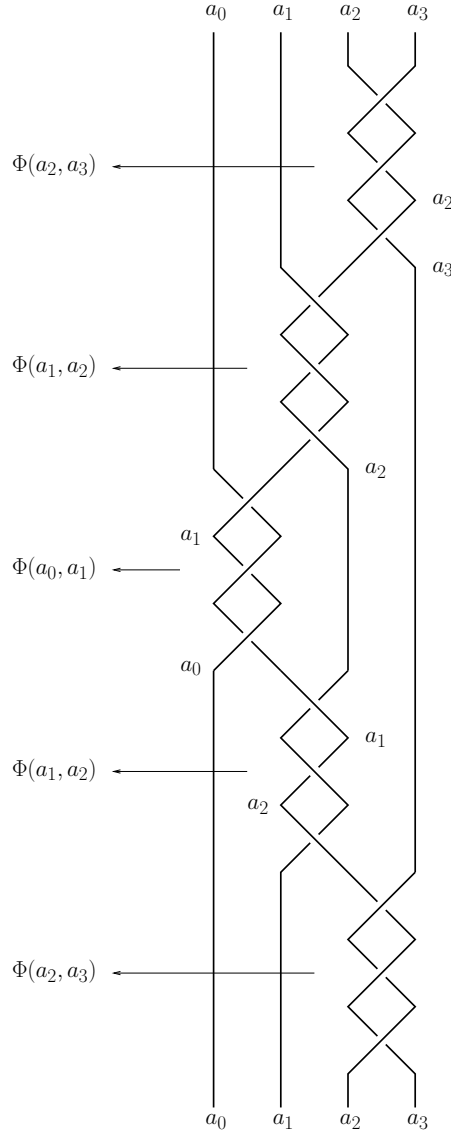


Figure 5:  $K_3$ , upon closure of the braid, endowed with a coloring by  $S_4$

$$\begin{aligned}
 Z(K_3) &= \sum_{a_0, \dots, a_3 \in \{0, 1, T, 1+T\}} \Phi(a_2, a_3) \Phi(a_1, a_2) \Phi(a_0, a_1) \Phi(a_1, a_2) \Phi(a_2, a_3) = \\
 &= \sum_{a_0, \dots, a_3 \in \{0, 1, T, 1+T\}} t^{\bar{\delta}_{a_0, a_1}} = 4^3(1 + 3t) \iff (4^3, 4^3 \cdot 3)
 \end{aligned}$$

So  $K_n$  is obtained by “adjoining  $\sigma_n^{(-1)^{n+1}3}$  above and below  $K_{n-1}$ ”.  
**In this way,**

$$Z(K_n) = 4^n(1 + 3t) \iff (4^n, 4^n \cdot 3)$$

Clearly this sequence does not converge...

The CJKLS invariant has the structure of a partition function or state-sum.

In Statistical Mechanics the logarithm of the partition function in the Helmholtz representation gives the Free Energy (modulo multiplication by a factor). In the Statistical Mechanics of Exactly Solved Models, the Thermodynamical Limit states that the Free Energy divided by the number of particles of the system and taking the limit as this number goes to infinity yields a number.

We mimic these ideas in order to obtain limits in our example. We take logarithms in each coordinate of the CJKLS invariant, then divide by the crossing number of the knot to obtain what we call the *Free Energy per crossing*, denoted  $f$  (we remark that this is still a knot invariant):

$$f(K_n) = \left( \frac{\ln(4^n)}{6n-3}, \frac{\ln(4^n \cdot 3)}{6n-3} \right) = \left( \frac{2n \ln(2)}{6n-3}, \frac{2n \ln(2) + \ln(3)}{6n-3} \right) \xrightarrow{n \rightarrow \infty} \left( \frac{\ln(2)}{3}, \frac{\ln(2)}{3} \right)$$

In this way we obtained a hyperfinite knot whose  $f$  invariant is  $\left( \frac{\ln(2)}{3}, \frac{\ln(2)}{3} \right)$

In [1] we present infinitely many distinct hyperfinite knots using the same approach.

One of our current research topics is to understand what happens to the limits of the sequences of knots as we vary  $X$ ,  $A$ , or  $\phi$

## References

- [1] P. Lopes, *Hyperfinite knots via the CJKLS invariant in the thermodynamic limit*, Chaos, Solitons and Fractals, **34** (2007) 1450-1472