

# SYMMETRIES IN $k$ -SYMPLECTIC FIELD THEORIES

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## SUMMARY

$k$ -symplectic geometry provides the simplest geometric framework for describing certain class of first-order classical field theories. Using this description we analyze different kinds of symmetries for the Hamiltonian and Lagrangian formalisms of these field theories, including the study of conservation laws associated to them, and stating Noether's theorem in different situations.

## HAMILTONIAN $k$ -SYMPLECTIC CASE

### GEOMETRIC ELEMENTS. $k$ -SYMPLECTIC HAMILTONIAN SYSTEMS

Let  $Q$  be a  $n$ -dimensional differentiable manifold,  $(T_k^1)^*Q = T^*Q \oplus \cdot^k \cdot \oplus T^*Q$ , its  $k$ -cotangent bundle with projection  $\tau^*: (T_k^1)^*Q \rightarrow Q$ . Natural coordinates on  $(T_k^1)^*Q$  are  $(q^i, p_i^A)$ ;  $1 \leq i \leq n$ ,  $1 \leq A \leq k$ .

The *canonical  $k$ -symplectic structure* in  $(T_k^1)^*Q$  is  $(\omega^A, V)$ , where  $V = \ker(\tau^*)$ , and  $\omega^A = (\tau_A^*)^*\omega = -d(\tau_A^*)^*\theta = -d\theta^A$ ; being  $\omega = -d\theta$  the canonical symplectic structure in  $T^*Q$  ( $\theta \in \Omega^1(T^*Q)$  is the Liouville 1-form), and  $\tau_A^*: (T_k^1)^*Q \rightarrow T^*Q$  the projection on the  $A^{\text{th}}$ -copy  $T^*Q$  of  $(T_k^1)^*Q$ . Locally

$$\omega^A = -d\theta^A = -d(p_i^A dq^i) = dq^i \wedge dp_i^A.$$

Being  $\varphi: Q \rightarrow Q$  a diffeomorphism, its *canonical prolongation* to  $(T_k^1)^*Q$  is  $(T_k^1)^*\varphi: (T_k^1)^*Q \rightarrow (T_k^1)^*Q$

$$(T_k^1)^*\varphi(\alpha_{1q}, \dots, \alpha_{kq}) = (T^*\varphi(\alpha_{1q}), \dots, T^*\varphi(\alpha_{kq})) \quad , \quad (\alpha_{1q}, \dots, \alpha_{kq}) \in (T_k^1)^*Q, \quad q \in Q.$$

Let  $Z \in \mathfrak{X}(Q)$ , with local 1-parametric group of transformations  $h_s: Q \rightarrow Q$ , the *canonical lift* of  $Z$  to  $(T_k^1)^*Q$  is  $Z^C \in \mathfrak{X}((T_k^1)^*Q)$  generated by  $(T_k^1)^*(h_s): (T_k^1)^*Q \rightarrow (T_k^1)^*Q$ . Locally, if  $Z = Z^i \frac{\partial}{\partial q^i}$  then

$$Z^C = Z^i \frac{\partial}{\partial q^i} - p_j^A \frac{\partial Z^j}{\partial q^k} \frac{\partial}{\partial p_k^A}.$$

**Definition 1.** Let  $M$  be a differentiable manifold and its  $k$ -tangent bundle  $T_k^1 M = TM \oplus \cdot^k \cdot \oplus TM$ .

• A  $k$ -vector field on  $M$  is a section  $\mathbf{X}: M \rightarrow T_k^1 M$  of  $\tau$ .

A  $k$ -vector field  $\mathbf{X}$  defines a family of vector fields  $X_1, \dots, X_k \in \mathfrak{X}(M)$  by  $X_A = \tau_A \circ \mathbf{X}$ , where  $\tau_A: T_k^1 M \rightarrow TM$  is the projection on the  $A^{\text{th}}$ -copy  $TM$  of  $T_k^1 M$ .

• An integral section of  $\mathbf{X}$  at a point  $q \in M$ , is a map  $\psi: U_0 \subset \mathbb{R}^k \rightarrow M$ , with  $0 \in U_0$ , such that  $\psi(0) = q$ ,  $\psi_*(t) \left( \frac{\partial}{\partial t^A} \Big|_t \right) = X_A(\psi(t))$ , for every  $t \in U_0$ .

A  $k$ -vector field  $\mathbf{X}$  on  $M$  is *integrable* if there is an integral section passing through every point of  $M$ .

$$\text{Locally,} \quad \psi^{(1)}(t^1, \dots, t^k) = \left( \psi^i(t^1, \dots, t^k), \frac{\partial \psi^j}{\partial t^A}(t^1, \dots, t^k) \right).$$

Let  $H: (T_k^1)^*Q \rightarrow \mathbb{R}$  be a *Hamiltonian function*. The family  $((T_k^1)^*Q, \omega^A, H)$  is a  $k$ -symplectic Hamiltonian system. The *Hamilton-de Donder-Weyl (HDW) equations* are

$$\frac{\partial H}{\partial q^i} \Big|_{\psi(t)} = - \sum_{A=1}^k \frac{\partial \psi_i^A}{\partial t^A} \Big|_t, \quad \frac{\partial H}{\partial p_i^A} \Big|_{\psi(t)} = \frac{\partial \psi^i}{\partial t^A} \Big|_t, \quad (1)$$

where  $\psi: \mathbb{R}^k \rightarrow (T_k^1)^*Q$ ,  $\psi(t) = (\psi^i(t), \psi_i^A(t))$ , is a solution.

Let  $\mathfrak{X}_H^k((T_k^1)^*Q)$  be the set of  $k$ -vector fields on  $(T_k^1)^*Q$  which are solutions to the equations

$$\sum_{A=1}^k i(X_A)\omega^A = dH.$$

If  $\mathbf{X} \in \mathfrak{X}_H^k((T_k^1)^*Q)$  is integrable, and  $\psi: \mathbb{R}^k \rightarrow (T_k^1)^*Q$  is an integral section of  $\mathbf{X}$ , then  $\psi(t) = (\psi^i(t), \psi_i^A(t))$  is a solution to the HDW equations (1).

## SYMMETRIES AND CONSERVATION LAWS

**Definition 2.** A *conservation law (or a conserved quantity) for the HDW equations (1)* is a map  $\mathcal{F} = (\mathcal{F}^1, \dots, \mathcal{F}^k): (T_k^1)^*Q \rightarrow \mathbb{R}^k$  such that the divergence of  $\mathcal{F} \circ \psi = (\mathcal{F}^1 \circ \psi, \dots, \mathcal{F}^k \circ \psi): U_0 \subset \mathbb{R}^k \rightarrow \mathbb{R}^k$  is zero for every solution  $\psi$  to the Hamilton-de Donder-Weyl equations (1); that is  $\sum_{A=1}^k \frac{\partial (\mathcal{F}^A \circ \psi)}{\partial t^A} = 0$ .

**Proposition 1.** If  $\mathcal{F} = (\mathcal{F}^1, \dots, \mathcal{F}^k): (T_k^1)^*Q \rightarrow \mathbb{R}^k$  is a conservation law, then for every integrable

$k$ -vector field  $\mathbf{X} = (X_1, \dots, X_k) \in \mathfrak{X}_H^k((T_k^1)^*Q)$  we have  $\sum_{A=1}^k L(X_A)\mathcal{F}^A = 0$ .

**Definition 3.** Let  $((T_k^1)^*Q, \omega^A, H)$  be a  $k$ -symplectic Hamiltonian system.

1 (a) A *symmetry* is a diffeomorphism  $\Phi: (T_k^1)^*Q \rightarrow (T_k^1)^*Q$  such that, for every solution  $\psi$  to the HDW equations (1), we have  $\Phi \circ \psi$  is also a solution to these equations.

(b) An *infinitesimal symmetry* is a vector field  $Y \in \mathfrak{X}((T_k^1)^*Q)$  whose local flows are local symmetries.

2 (a) A *Cartan or Noether symmetry* is a diffeomorphism  $\Phi: (T_k^1)^*Q \rightarrow (T_k^1)^*Q$  such that:

(i)  $\Phi^*\omega^A = \omega^A$ , (ii)  $\Phi^*H = H$  (up to a constant).

(b) An *infinitesimal Cartan symmetry* is a vector field  $Y \in \mathfrak{X}((T_k^1)^*Q)$  such that:

(i)  $L(Y)\omega^A = 0$ , (ii)  $L(Y)H = 0$ .

If  $\Phi = (T_k^1)^*\varphi$  for some  $\varphi: Q \rightarrow Q$ , the (Cartan) symmetry  $\Phi$  is said to be *natural*.

If  $Y = Z^C$  for some  $Z \in \mathfrak{X}(Q)$ , the infinitesimal (Cartan) symmetry  $Y$  is said to be *natural*.

**Remarks:** ★ If  $\Phi: (T_k^1)^*Q \rightarrow (T_k^1)^*Q$  is a Cartan symmetry, then it is a symmetry.

★ If  $\mathbf{X} = (X_1, \dots, X_k) \in \mathfrak{X}_H^k((T_k^1)^*Q)$ , then  $\Phi_*\mathbf{X} = (\Phi_*X_1, \dots, \Phi_*X_k) \in \mathfrak{X}_H^k((T_k^1)^*Q)$ .

**Proposition 2.** Let  $Y \in \mathfrak{X}((T_k^1)^*Q)$  be an infinitesimal Cartan symmetry. Then, for every  $p \in (T_k^1)^*Q$ , there is an open neighbourhood  $U_p \ni p$ , such that:

1. There exist  $f^A \in C^\infty(U_p)$ , unique up to constant functions, such that  $i(Y)\omega^A = df^A$  (on  $U_p$ ).

2. There exist  $\zeta^A \in C^\infty(U_p)$ , verifying that  $L(Y)\theta^A = d\zeta^A$ , on  $U_p$ ; and then  $f^A = i(Y)\theta^A - \zeta^A$  (up to constant functions on  $U_p$ ).

**Theorem 1. (Noether's theorem):** Let  $Y \in \mathfrak{X}((T_k^1)^*Q)$  be an infinitesimal Cartan symmetry.

1. For every  $p \in (T_k^1)^*Q$ , there is an open neighborhood  $U_p$  such that the functions  $f^A = i(Y)\theta^A - \zeta^A$  define a conservation law  $f = (f^1, \dots, f^k)$  on  $U_p$ .

2. For every  $\mathbf{X} = (X_1, \dots, X_k) \in \mathfrak{X}_H^k((T_k^1)^*Q)$ , we have  $\sum_{A=1}^k L(X_A)f^A = 0$  (on  $U_p$ ).

## LAGRANGIAN $k$ -SYMPLECTIC CASE

### GEOMETRIC ELEMENTS

Let  $Q$  be a  $n$ -dimensional differentiable manifold, and  $T_k^1 Q = TQ \oplus \cdot^k \cdot \oplus TQ$  its  $k$ -tangent bundle with natural projection  $\tau: T_k^1 Q \rightarrow Q$ . Natural coordinates on  $T_k^1 Q$  are  $(q^i, v_A^i)$ .

For  $Z_q \in T_q Q$ , the *vertical  $A$ -lift* at  $(v_{1q}, \dots, v_{kq}) \in T_k^1 Q$  is the vector  $(Z_q)^{VA}$  tangent to  $\tau^{-1}(q) \subset T_k^1 Q$ ,

$$(Z_q)^{VA}(v_{1q}, \dots, v_{kq}) = \frac{d}{ds}(v_{1q}, \dots, v_{A-1q}, v_{Aq} + sZ_q, v_{A+1q}, \dots, v_{kq})|_{s=0}.$$

Locally, if  $X_q = a^i \frac{\partial}{\partial q^i} \Big|_q$ , then  $(Z_q)^{VA}(v_{1q}, \dots, v_{kq}) = a^i \frac{\partial}{\partial v_A^i} \Big|_{(v_{1q}, \dots, v_{kq})}$ .

The *canonical  $k$ -tangent structure* on  $T_k^1 Q$  is the set  $(S^1, \dots, S^k)$  of  $(1, 1)$ -tensor fields defined by

$$S^A(w_q)(Z_{w_q}) = (\tau_*(w_q)(Z_{w_q}))^{VA}(w_q) \quad , \quad \text{for } w_q \in T_k^1 Q, \quad Z_{w_q} \in T_{w_q}(T_k^1 Q).$$

The *Liouville vector field*  $\Delta \in \mathfrak{X}(T_k^1 Q)$ , is the infinitesimal generator of the flow

$$\psi: \mathbb{R} \times T_k^1 Q \rightarrow T_k^1 Q \quad , \quad \psi(s, v_{1q}, \dots, v_{kq}) = (e^s v_{1q}, \dots, e^s v_{kq}),$$

Locally,  $S^A = \frac{\partial}{\partial v_A^i} \otimes dq^i$ , and  $\Delta = \sum_{A=1}^k \Delta_A = \sum_{A=1}^k v_A^i \frac{\partial}{\partial v_A^i}$ .

Being  $\varphi: Q \rightarrow Q$  a diffeomorphism, its *canonical prolongation* to  $T_k^1 Q$  is  $T_k^1 \varphi: T_k^1 Q \rightarrow T_k^1 Q$  given by

$$T_k^1 \varphi(v_{1q}, \dots, v_{kq}) = (\varphi_*(q)v_{1q}, \dots, \varphi_*(q)v_{kq}) \quad , \quad (v_{1q}, \dots, v_{kq}) \in (T_k^1)_q Q, \quad q \in Q.$$

Let  $Z \in \mathfrak{X}(Q)$ , with local 1-parametric group  $h_s: Q \rightarrow Q$ , the *canonical lift* of  $Z$  to  $(T_k^1)_q Q$  is  $Z^C \in \mathfrak{X}(T_k^1 Q)$

generated by  $T_k^1 h_s: T_k^1 Q \rightarrow T_k^1 Q$ . Locally, if  $Z = Z^i \frac{\partial}{\partial q^i}$ , then  $Z^C = Z^i \frac{\partial}{\partial q^i} + v_A^j \frac{\partial Z^k}{\partial q^j} \frac{\partial}{\partial v_A^k}$ .

**Definition 4.** A *second order partial differential equation (SOPDE)* is a  $k$ -vector field  $\Gamma$  in  $T_k^1 Q$  which is a section of the projection  $T_k^1 \tau: T_k^1(T_k^1 Q) \rightarrow T_k^1 Q$ ; that is,  $T_k^1 \tau \circ \Gamma = \text{Id}_{T_k^1 Q}$ .

Locally, a SOPDE  $\Gamma = (\Gamma_1, \dots, \Gamma_k)$  is given by  $\Gamma_A(q^i, v_A^i) = v_A^i \frac{\partial}{\partial q^i} + (\Gamma_A)^j_B \frac{\partial}{\partial v_B^j}$ ,  $(\Gamma_A)^j_B \in C^\infty(T_k^1 Q)$ .

**Proposition 3.** If  $\psi$  is an integral section of an integrable SOPDE  $\Gamma$ , then  $\psi = \phi^{(1)}$ , being  $\phi^{(1)}$  the first prolongation of  $\phi = \tau \circ \psi$ , and  $\phi$  is a solution to the system  $\frac{\partial^2 \phi^i}{\partial t^A \partial t^B}(t) = (\Gamma_A)^j_B \left( \phi^j(t), \frac{\partial \phi^i}{\partial t^C}(t) \right)$ .

Conversely, if  $\phi: \mathbb{R}^k \rightarrow Q$  is a solution to this system, then  $\phi^{(1)}$  is an integral section of  $\Gamma$ .

## $k$ -SYMPLECTIC LAGRANGIAN SYSTEMS

Let  $L: T_k^1 Q \rightarrow \mathbb{R}$  be a Lagrangian. The *generalized Euler-Lagrange equations* for  $L$  are:

$$\sum_{A=1}^k \frac{\partial}{\partial t^A} \Big|_t \left( \frac{\partial L}{\partial v_A^i} \Big|_{\psi(t)} \right) = \frac{\partial L}{\partial q^i} \Big|_{\psi(t)} \quad , \quad v_A^i(\psi(t)) = \frac{\partial \psi^i}{\partial t^A} \quad (2)$$

whose solutions are maps  $\psi: \mathbb{R}^k \rightarrow T_k^1 Q$ . Observe that  $\psi(t) = \phi^{(1)}(t)$ , for some  $\phi = \tau \circ \psi$ .

We introduce the forms  $\theta_L^A = dL \circ S^A \in \Omega^1(T_k^1 Q)$ ,  $\omega_L^A = -d\theta_L^A \in \Omega^2(T_k^1 Q)$ , and the *Energy Lagrangian function*  $E_L = \Delta(L) - L \in C^\infty(T_k^1 Q)$ . Locally

$$\theta_L^A = \frac{\partial L}{\partial v_A^i} dq^i \quad , \quad \omega_L^A = \frac{\partial^2 L}{\partial q^j \partial v_A^i} dq^i \wedge dq^j + \frac{\partial^2 L}{\partial v_B^j \partial v_A^i} dq^i \wedge dv_B^j \quad , \quad E_L = v_A^i \frac{\partial L}{\partial v_A^i} - L.$$

The Lagrangian  $L: T_k^1 Q \rightarrow \mathbb{R}$  is *regular* if the matrix  $\left( \frac{\partial^2 L}{\partial v_A^i \partial v_B^j} \right)$  is not singular at every point of  $T_k^1 Q$ .

This condition is equivalent to say that  $(\omega_L^1, \dots, \omega_L^k; V)$  is a  $k$ -symplectic structure, where  $V = \ker \tau_*$ .

The family  $(T_k^1 Q, \omega_L^A, E_L)$  is called a  *$k$ -symplectic Lagrangian system*.

Let  $\mathfrak{X}_L^k(T_k^1 Q)$  the set of  $k$ -vector fields  $\Gamma = (\Gamma_1, \dots, \Gamma_k)$  in  $T_k^1 Q$ , which are solutions to the equation

$$\sum_{A=1}^k i(\Gamma_A)\omega_L^A = dE_L. \quad (3)$$

Locally, if  $\Gamma_A = (\Gamma_A)^i \frac{\partial}{\partial q^i} + (\Gamma_A)^j_B \frac{\partial}{\partial v_B^j}$  and  $L$  is regular, then  $\Gamma = (\Gamma_1, \dots, \Gamma_k)$  is a solution to (3) iff

$$\frac{\partial^2 L}{\partial q^j \partial v_A^i} v_A^j + \frac{\partial^2 L}{\partial v_A^i \partial v_B^j} (\Gamma_A)^j_B = \frac{\partial L}{\partial q^i} \quad , \quad (\Gamma_A)^i = v_A^i.$$

Thus, if  $\Gamma \in \mathfrak{X}_L^k(T_k^1 Q)$  then it is a SOPDE and, if it is integrable, its integral sections are first prolongations of maps  $\phi: \mathbb{R}^k \rightarrow Q$  which are solutions to the Euler-Lagrange equations (2).

## SYMMETRIES AND CONSERVATION LAWS

Definitions 2, 3, and Propositions 1, 2 are also applied to the Lagrangian case, just considering  $(T_k^1 Q, \omega_L^A, E_L)$  as a Hamiltonian system with Hamiltonian function  $E_L$ . Furthermore:

**Theorem 2. (Lagrangian Noether's theorem):** Let  $Y \in \mathfrak{X}(T_k^1 Q)$  be an infinitesimal Cartan symmetry.

1. For every  $p \in T_k^1 Q$ , there is an open neighborhood  $U_p$  such that the functions  $f^A = i(Y)\theta_L^A - \zeta^A$  define a conservation law  $f = (f^1, \dots, f^k)$  on  $U_p$ .

In particular, if  $Y = Z^C \in \mathfrak{X}(T_k^1 Q)$  is an infinitesimal natural Cartan symmetry then the functions  $f^A = Z^{VA}(L) - \zeta^A$  define a conservation law on  $U_p$ .

2. For every  $\Gamma = (\Gamma_1, \dots, \Gamma_k) \in \mathfrak{X}_L^k(T_k^1 Q)$ , we have  $\sum_{A=1}^k L(\Gamma_A)f^A = 0$  (on  $U_p$ ).

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