Symmetries in k-symplectic field theories

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Summary

k-symplectic geometry provides the simplest geometric framework for describing certain class of first-order classical field theories. Using this description we analyze different kinds of symmetries for the Hamiltonian and Lagrangian formalisms of these field theories, including the study of conservation laws associated to them, and stating Noether's theorem in different situations.

Hamiltonian k-symplectic case

Geometric elements. k-symplectic Hamiltonian systems

Let Q be a n-dimensional differentiable manifold, $(T^1_k)^*Q = T^*Q\oplus \overset{k}{\dots} \oplus T^*Q$, its *$k$-cotangent bundle* with projection $\tau^*\colon (T^1_k)^*Q \to Q$. Natural coordinates on $(T^1_k)^*Q$ are (q^i, p^A_i); $1 \leq i \leq n$, $1 \leq A \leq k$.

The *canonical k-symplectic structure* in $(T^1_k)^*Q$ is (ω^A, V), where $V = \ker(\tau^*)_*$, and $\omega^A = (\tau^*_A)^*\omega = -\mathrm{d}(\tau^*_A)^*\theta = -\mathrm{d}\theta^A$; being $\omega = -\mathrm{d}\theta$ the canonical symplectic structure in T^*Q ($\theta \in \Omega^1(T^*Q)$ is the Liouville 1-form), and $\tau^*_A\colon (T^1_k)^*Q \to T^*Q$ the projection on the A^{th}-copy T^*Q of $(T^1_k)^*Q$. Locally

$$\omega^A = -\mathrm{d}\theta^A = -\mathrm{d}(p^A_i\,\mathrm{d}q^i) = \mathrm{d}q^i \wedge \mathrm{d}p^A_i\,.$$

Being $\varphi\colon Q \to Q$ a diffeomorphism, its *canonical prolongation* to $(T^1_k)^*Q$ is $(T^1_k)^*\varphi\colon (T^1_k)^*Q \to (T^1_k)^*Q$

$$(T^1_k)^*\varphi(\alpha_{1q}, \dots, \alpha_{kq}) = (T^*\varphi(\alpha_{1q}), \dots, T^*\varphi(\alpha_{kq})) \quad , \quad (\alpha_{1q}, \dots, \alpha_{kq}) \in (T^1_k)^*_qQ,\ q \in Q\,.$$

Let $Z \in \mathfrak{X}(Q)$, with local 1-parametric group of transformations $h_s\colon Q \to Q$, the *canonical lift* of Z to $(T^1_k)^*_qQ$ is $Z^{C*} \in \mathfrak{X}((T^1_k)^*Q)$ generated by $(T^1_k)^*(h_s)\colon (T^1_k)^*Q \to (T^1_k)^*Q$. Locally, if $Z = Z^i\dfrac{\partial}{\partial q^i}$ then $Z^{C*} = Z^i\dfrac{\partial}{\partial q^i} - p^A_j\dfrac{\partial Z^j}{\partial q^k}\dfrac{\partial}{\partial p^A_k}$.

Definition 1. *Let M be a differentiable manifold and its k-tangent bundle $T^1_kM = TM\oplus \overset{k}{\dots} \oplus TM$.*

- *A k-vector field on M is a section $\mathbf{X}\colon M \longrightarrow T^1_kM$ of τ.*
  A k-vector field $\mathbf{X}$ defines a family of vector fields $X_1, \dots, X_k \in \mathfrak{X}(M)$ by $X_A = \tau_A \circ \mathbf{X}$, where $\tau_A\colon T^1_kQ \to TQ$ is the projection on the A^{th}-copy TQ of T^1_kQ.
- *An integral section of $\mathbf{X}$ at a point $q \in M$, is a map $\psi\colon U_0 \subset \mathbb{R}^k \to M$, with $0 \in U_0$, such that $\psi(0) = q$, $\psi_*(t)\left(\dfrac{\partial}{\partial t^A}\Big|_t\right) = X_A(\psi(t))$, for every $t \in U_0$.*
  A k-vector field $\mathbf{X}$ on M is integrable if there is an integral section passing through every point of M.

$$\text{Locally,}\qquad \psi^{(1)}(t^1, \dots, t^k) = \left(\psi^i(t^1, \dots, t^k), \frac{\partial\psi^i}{\partial t^A}(t^1, \dots, t^k)\right)\,.$$

Let $H\colon (T^1_k)^*Q \to \mathbb{R}$ be a *Hamiltonian function*. The family $((T^1_k)^*Q, \omega^A, H)$ is a *k-symplectic Hamiltonian system*. The *Hamilton-de Donder-Weyl (HDW) equations* are

$$\frac{\partial H}{\partial q^i}\Big|_{\psi(t)} = -\sum_{A=1}^{k}\frac{\partial\psi^A_i}{\partial t^A}\Big|_t \quad , \quad \frac{\partial H}{\partial p^A_i}\Big|_{\psi(t)} = \frac{\partial\psi^i}{\partial t^A}\Big|_t\,, \tag{1}$$

where $\psi\colon \mathbb{R}^k \to (T^1_k)^*Q$, $\psi(t) = (\psi^i(t), \psi^A_i(t))$, is a solution.

Let $\mathfrak{X}^k_H((T^1_k)^*Q)$ be the set of k-vector fields on $(T^1_k)^*Q$ which are solutions to the equations

$$\sum_{A=1}^{k} i(X_A)\omega^A = \mathrm{d}H\,.$$

If $\mathbf{X} \in \mathfrak{X}^k_H((T^1_k)^*Q)$ is integrable, and $\psi\colon \mathbb{R}^k \to (T^1_k)^*Q$ is an integral section of $\mathbf{X}$, then $\psi(t) = (\psi^i(t), \psi^A_i(t))$ is a solution to the HDW equations (1).

Symmetries and conservation laws

Definition 2. *A conservation law (or a conserved quantity) for the HDW equations (1) is a map $\mathcal{F} = (\mathcal{F}^1, \dots, \mathcal{F}^k)\colon (T^1_k)^*Q \to \mathbb{R}^k$ such that the divergence of $\mathcal{F}\circ\psi = (\mathcal{F}^1\circ\psi, \dots, \mathcal{F}^k\circ\psi)\colon U_0 \subset \mathbb{R}^k \to \mathbb{R}^k$ is zero for every solution ψ to the Hamilton-de Donder-Weyl equations (1); that is $\displaystyle\sum_{A=1}^{k}\frac{\partial(\mathcal{F}^A\circ\psi)}{\partial t^A} = 0$.*

Proposition 1. *If $\mathcal{F} = (\mathcal{F}^1, \dots, \mathcal{F}^k)\colon (T^1_k)^*Q \to \mathbb{R}^k$ is a conservation law, then for every integrable k-vector field $\mathbf{X} = (X_1, \dots, X_k) \in \mathfrak{X}^k_H((T^1_k)^*Q)$ we have $\displaystyle\sum_{A=1}^{k}\mathrm{L}(X_A)\mathcal{F}^A = 0$.*

Definition 3. *Let $((T^1_k)^*Q, \omega^A, H)$ be a k-symplectic Hamiltonian system.*

1 *(a) A symmetry is a diffeomorphism $\Phi\colon (T^1_k)^*Q \to (T^1_k)^*Q$ such that, for every solution ψ to the HDW equations (1), we have $\Phi\circ\psi$ is also a solution to these equations.*
(b) An infinitesimal symmetry is a vector field $Y \in \mathfrak{X}((T^1_k)^*Q)$ whose local flows are local symmetries.

2 *(a) A Cartan or Noether symmetry is a diffeomorphism $\Phi\colon (T^1_k)^*Q \to (T^1_k)^*Q$ such that:*
(i) $\Phi^*\omega^A = \omega^A$, (ii) $\Phi^*H = H$ *(up to a constant).*
(b) An infinitesimal Cartan symmetry is a vector field $Y \in \mathfrak{X}((T^1_k)^*Q)$ such that:
(i) $\mathrm{L}(Y)\omega^A = 0$, (ii) $\mathrm{L}(Y)H = 0$.

If $\Phi = (T^1_k)^*\varphi$ for some $\varphi\colon Q \to Q$, the (Cartan) symmetry Φ is said to be natural.
If $Y = Z^{C*}$ for some $Z \in \mathfrak{X}(Q)$, the infinitesimal (Cartan) symmetry Y is said to be natural.

Remarks: ★ If $\Phi\colon (T^1_k)^*Q \to (T^1_k)^*Q$ is a Cartan symmetry, then it is a symmetry.
★ If $\mathbf{X} = (X_1, \dots, X_k) \in \mathfrak{X}^k_H((T^1_k)^*Q)$, then $\Phi_*\mathbf{X} = (\Phi_*X_1, \dots, \Phi_*X_k) \in \mathfrak{X}^k_H((T^1_k)^*Q)$.

Proposition 2. *Let $Y \in \mathfrak{X}((T^1_k)^*Q)$ be an infinitesimal Cartan symmetry. Then, for every $p \in (T^1_k)^*Q$, there is an open neighbourhood $U_p \ni p$, such that:*

1. *There exist $f^A \in \mathrm{C}^\infty(U_p)$, unique up to constant functions, such that $i(Y)\omega^A = \mathrm{d}f^A$ (on U_p).*
2. *There exist $\zeta^A \in \mathrm{C}^\infty(U_p)$, verifying that $\mathrm{L}(Y)\theta^A = \mathrm{d}\zeta^A$, on U_p; and then $f^A = i(Y)\theta^A - \zeta^A$ (up to constant functions on U_p).*

Theorem 1. (Noether's theorem)*: Let $Y \in \mathfrak{X}((T^1_k)^*Q)$ be an infinitesimal Cartan symmetry.*

1. *For every $p \in (T^1_k)^*Q$, there is an open neighborhood U_p such that the functions $f^A = i(Y)\theta^A - \zeta^A$ define a conservation law $f = (f^1, \dots, f^k)$ on U_p.*
2. *For every $\mathbf{X} = (X_1, \dots, X_k) \in \mathfrak{X}^k_H((T^1_k)^*Q)$, we have $\sum_{A=1}^k \mathrm{L}(X_A)f^A = 0$ (on U_p).*

Lagrangian k-symplectic case

Geometric elements

Let Q be a n-dimensional differentiable manifold, and $T^1_kQ = TQ\oplus \overset{k}{\dots} \oplus TQ$ its *k-tangent bundle* with natural projection $\tau\colon T^1_kQ \to Q$. Natural coordinates on T^1_kQ are (q^i, v^i_A).

For $Z_q \in T_qQ$, the *vertical A-lift* at $(v_{1q}, \dots, v_{kq}) \in T^1_kQ$ is the vector $(Z_q)^{V_A}$ tangent to $\tau^{-1}(q) \subset T^1_kQ$,

$$(Z_q)^{V_A}(v_{1q}, \dots, v_A) = \frac{d}{ds}(v_{1q}, \dots, v_{A-1q}, v_{A_q} + sZ_q, v_{A+1q}, \dots, v_{kq})|_{s=0}\,.$$

Locally, if $X_q = a^i\dfrac{\partial}{\partial q^i}\Big|_q$, then $(Z_q)^{V_A}(v_{1q}, \dots, v_{kq}) = a^i\dfrac{\partial}{\partial v^i_A}\Big|_{(v_{1q},\dots,v_{kq})}$.

The *canonical k-tangent structure* on T^1_kQ is the set $(S^1, \dots, S^k)$ of $(1,1)$-tensor fields defined by

$$S^A(w_q)(Z_{w_q}) = (\tau_*(w_q)(Z_{w_q}))^{V_A}(w_q) \quad , \quad \text{for} \quad w_q \in T^1_kQ, \quad Z_{w_q} \in T_{w_q}(T^1_kQ)\,.$$

The *Liouville vector field* $\Delta \in \mathfrak{X}(T^1_kQ)$, is the infinitesimal generator of the flow

$$\psi\colon \mathbb{R}\times T^1_kQ \longrightarrow T^1_kQ \quad , \quad \psi(s, v_{1_q}, \dots, v_{k_q}) = (e^s v_{1_q}, \dots, e^s v_{k_q})\,,$$

Locally, $\quad S^A = \frac{\partial}{\partial v^i_A}\otimes \mathrm{d}q^i$, $\quad$ and $\quad \Delta = \sum_{A=1}^k \Delta_A = \sum_{A=1}^k v^i_A\frac{\partial}{\partial v^i_A}$.

Being $\varphi\colon Q \to Q$ a diffeomorphism, its *canonical prolongation* to T^1_kQ is $T^1_k\varphi : T^1_kQ \to T^1_kQ$ given by

$$T^1_k\varphi(v_{1q}, \dots, v_{kq}) = (\varphi_*(q)v_{1q}, \dots, \varphi_*(q)v_{kq}) \quad , \quad (v_{1q}, \dots, v_{kq}) \in (T^1_k)_qQ\ ,\ q \in Q\,.$$

Let $Z \in \mathfrak{X}(Q)$, with local 1-parametric group $h_s\colon Q \to Q$, the *canonical lift* of Z to $(T^1_k)_qQ$ is $Z^C \in \mathfrak{X}(T^1_kQ)$ generated by $T^1_kh_s\colon T^1_kQ \to T^1_kQ$. Locally, if $Z = Z^i\dfrac{\partial}{\partial q^i}$, then $Z^C = Z^i\dfrac{\partial}{\partial q^i} + v^j_A\dfrac{\partial Z^k}{\partial q^j}\dfrac{\partial}{\partial v^k_A}$.

Definition 4. *A second order partial differential equation (SOPDE) is a k-vector field $\mathbf{\Gamma}$ in T^1_kQ which is a section of the projection $T^1_k\tau\colon T^1_k(T^1_kQ) \to T^1_kQ$; that is, $T^1_k\tau\circ\mathbf{\Gamma} = \mathrm{Id}_{T^1_kQ}$.*

Locally, a SOPDE $\mathbf{\Gamma} = (\Gamma_1, \dots, \Gamma_k)$ is given by $\Gamma_A(q^i, v^i_A) = v^i_A\dfrac{\partial}{\partial q^i} + (\Gamma_A)^i_B\dfrac{\partial}{\partial v^i_B}$, $\quad (\Gamma_A)^i_B \in \mathrm{C}^\infty(T^1_kQ)$.

Proposition 3. *If ψ is an integral section of an integrable SOPDE $\mathbf{\Gamma}$, then $\psi = \phi^{(1)}$, being $\phi^{(1)}$ the first prolongation of $\phi = \tau\circ\psi$, and ϕ is a solution to the system $\dfrac{\partial^2\phi^i}{\partial t^A\partial t^B}(t) = (\Gamma_A)^i_B\left(\phi^i(t), \dfrac{\partial\phi^i}{\partial t^C}(t)\right)$.*
Conversely, if $\phi\colon \mathbb{R}^k \to Q$ is a solution to this system, then $\phi^{(1)}$ is an integral section of $\mathbf{\Gamma}$.

k-symplectic Lagrangian systems

Let $L\colon T^1_kQ \to \mathbb{R}$ be a Lagrangian. The *generalized Euler-Lagrange equations* for L are:

$$\sum_{A=1}^{k}\frac{\partial}{\partial t^A}\Big|_t\left(\frac{\partial L}{\partial v^i_A}\Big|_{\psi(t)}\right) = \frac{\partial L}{\partial q^i}\Big|_{\psi(t)} \quad , \quad v^i_A(\psi(t)) = \frac{\partial\psi^i}{\partial t^A} \tag{2}$$

whose solutions are maps $\psi\colon \mathbb{R}^k \to T^1_kQ$. Observe that $\psi(t) = \phi^{(1)}(t)$, for some $\phi = \tau\circ\psi$.

We introduce the forms $\quad \theta^A_L = dL\circ S^A \in \Omega^1(T^1_kQ)$, $\omega^A_L = -\mathrm{d}\theta^A_L \in \Omega^2(T^1_kQ)$, and the *Energy Lagrangian function* $\quad E_L = \Delta(L) - L \in \mathrm{C}^\infty(T^1_kQ)$. Locally

$$\theta^A_L = \frac{\partial L}{\partial v^i_A}\mathrm{d}q^i \quad , \quad \omega^A_L = \frac{\partial^2 L}{\partial q^j\partial v^i_A}\mathrm{d}q^i\wedge\mathrm{d}q^j + \frac{\partial^2 L}{\partial v^j_B\partial v^i_A}\mathrm{d}q^i\wedge\mathrm{d}v^j_B \quad , \quad E_L = v^i_A\frac{\partial L}{\partial v^i_A} - L\,.$$

The Lagrangian $L\colon T^1_kQ \longrightarrow \mathbb{R}$ is *regular* if the matrix $\left(\frac{\partial^2 L}{\partial v^i_A\partial v^j_B}\right)$ is not singular at every point of T^1_kQ.

This condition is equivalent to say that $(\omega^1_L, \dots, \omega^k_L; V)$ is a k-symplectic structure, where $V = \ker\tau_*$.

The family $(T^1_kQ, \omega^A_L, E_L)$ is called a *k-symplectic Lagrangian system*.

Let $\mathfrak{X}^k_L(T^1_kQ)$ the set of k-vector fields $\mathbf{\Gamma} = (\Gamma_1, \dots, \Gamma_k)$ in T^1_kQ, which are solutions to the equation

$$\sum_{A=1}^{k} i(\Gamma_A)\omega^A_L = \mathrm{d}E_L\,. \tag{3}$$

Locally, if $\Gamma_A = (\Gamma_A)^i\dfrac{\partial}{\partial q^i} + (\Gamma_A)^i_B\dfrac{\partial}{\partial v^i_B}$ and L is regular, then $\mathbf{\Gamma} = (\Gamma_1, \dots, \Gamma_k)$ is a solution to (3) iff

$$\frac{\partial^2 L}{\partial q^j\partial v^i_A}v^j_A + \frac{\partial^2 L}{\partial v^i_A\partial v^j_B}(\Gamma_A)^j_B = \frac{\partial L}{\partial q^i} \quad , \quad (\Gamma_A)^i = v^i_A\,.$$

Thus, if $\mathbf{\Gamma} \in \mathfrak{X}^k_L(T^1_kQ)$ then it is a SOPDE and, if it is integrable, its integral sections are first prolongations of maps $\phi\colon \mathbb{R}^k \to Q$ which are solutions to the Euler-Lagrange equations (2).

Symmetries and conservation laws

Definitions 2, 3, and Propositions 1, 2 are also applied to the Lagrangian case, just considering $(T^1_kQ, \omega^A_L, E_L)$ as a Hamiltonian system with Hamiltonian function E_L. Furthermore:

Theorem 2. (Lagrangian Noether's theorem)*: Let $Y \in \mathfrak{X}(T^1_kQ)$ be an infinitesimal Cartan symmetry.*

1. *For every $p \in T^1_kQ$, there is an open neighborhood U_p such that the functions $f^A = i(Y)\theta^A_L - \zeta^A$ define a conservation law $f = (f^1, \dots, f^k)$ on U_p.*
   In particular, if $Y = Z^C \in \mathfrak{X}(T^1_kQ)$ is an infinitesimal natural Cartan symmetry then the functions $f^A = Z^{V_A}(L) - \zeta^A$ define a conservation law on U_p.
2. *For every $\mathbf{\Gamma} = (\Gamma_1, \dots, \Gamma_k) \in \mathfrak{X}^k_L(T^1_kQ)$, we have $\sum_{A=1}^k \mathrm{L}(\Gamma_A)f^A = 0$ (on U_p).*

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