The Orbit Space of a Proper Groupoid

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Joint work with

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and M/G is precisely the orbit space of this action groupoid.

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$$x \sim y \quad \text{if} \quad \mathcal{O}_x = \mathcal{O}_y \quad \text{for} \quad x, y \in M.$$

Then the orbit space of $\mathcal{G}$ is $M/\mathcal{G} := M/\sim$.

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- In that case, what is the global description of the strata?

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The stratification is called Whitney if for every pair $\mathcal{U}_i \subset \overline{\mathcal{U}_j}$,

$$U_j \ni \{x_k\}_{k \in \mathbb{N}} \to x \in U_i \Rightarrow T_{x_k} U_j \to V > T_x U_i \quad .$$

(this requires an embedding of $\mathcal{U}$ in $\mathbb{R}^N$).

If $\mathcal{S}$ is a stratified space then there is a family of disjoint locally closed smooth manifolds $\mathcal{S}_k$, $k \in \mathcal{I}_{\mathcal{S}}$ such that for every $k \in \mathcal{I}_{\mathcal{S}}$

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The manifolds $\mathcal{S}_k$ are called the strata of the stratification.

Stratifications for Lie group actions

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- If $x \in M$, then there is a neighborhood U of $[x]$ in M/G such that

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  where $\mathbf{S}$ is a linear slice for the G-action at x and G_x is the stabilizer of x which has a linear representation on $\mathbf{S}$.

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- Since the action is proper G_x is compact, therefore using Invariant theory (Hilbert, Schwartz, Tarski-Seidenberg,...) U is a semi-algebraic Whitney stratified space (isotropy stratification)

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- The connected components of $M_{(H)}$ are submanifolds of M for every $H \subset G$.
- $M = \bigcup_{(H)} M_{(H)}$ is a locally finite disjoint partition.
- The connected components of $\pi(M_{(H)})$ are the smooth strata of the isotropy stratification of M/G.

Outline

Let $s, t : \mathcal{G} \rightrightarrows M$ a Lie groupoid.

The analogous construction to the Lie group case cannot be used for studying $M/\mathcal{G}$ since the stabilizers

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Therefore we cannot define orbit types $M_{(H)}$. We need a different approach

Tube theorem + Foliation theory

Tube theorem for proper groupoids

We will assume the following conditions for the Lie groupoid $s, t : \mathcal{G} \rightrightarrows M$:

- $(s, t) : \mathcal{G} \to M \times M$ is a proper map. (proper groupoid)
- s is locally trivial. (source local triviality)
- Every orbit of $\mathcal{G}$ is of finite type.

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Let $\mathcal{G} \rightrightarrows M$ be a source locally trivial proper groupoid and $x \in M$ with orbit $\mathcal{O}$. Then there is an a action of $\mathcal{G}_{\mathcal{O}}$ on $N\mathcal{O} = T_{\mathcal{O}}M/T\mathcal{O}$, with associated action groupoid

$$\mathcal{G}_{\mathcal{O}} \ltimes N\mathcal{O} \rightrightarrows N\mathcal{O} \quad \text{and}$$

$\mathcal{G}$ is locally isomorphic to $\mathcal{G}_{\mathcal{O}} \ltimes N\mathcal{O}$.

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Then the action of $\mathcal{G}_{\mathcal{O}}$ on $N\mathcal{O}$ restricts to a representation of $\mathcal{G}_x$ on $N_x\mathcal{O}$ ($\mathcal{G}_x = s^{-1}(x) \cap t^{-1}(x)$ is the stabilizer, a compact Lie group) with associated action groupoid

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with $\pi_1(g, v) = g \cdot v$ and $\pi_2(g, v) = v$.

$$\begin{array}{rcl}
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3. These two actions are free, they commute and the momentum map of one is the orbit map of the other.

Outline

Since $\mathcal{G}$ is locally Morita equivalent to $\mathcal{G}_x \times N_x\mathcal{O}$ then near $[x]$

$$M/\mathcal{G} \sim N_x\mathcal{O}/\mathcal{G}_x \quad \text{(local homeomorphism).}$$

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In particular

Theorem

The orbit space for a source locally trivial proper groupoid is a locally semi-algebraic Whitney stratified space.

Outline

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It is well-known that

Lemma

- *$\mathcal{A}(N_x\mathcal{O})$ defines a singular integrable distribution,*
- *its leaves are $\mathcal{G}_x$-invariant,*
- *the strata of $N_x\mathcal{O}/\mathcal{G}_x$ are the projection of the leaves of $\mathcal{A}(N_x\mathcal{O})$.*

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With this Lemma and the Morita equivalence we can prove:

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- *$\mathcal{G}_{bas}(M)$ induces a singular integrable distribution,*
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3. by the general properties of stratifications and the Lemma, for each stratum $\mathcal{S}_i$ of $M/\mathcal{G}$,

$$\mathcal{S}_i \cap U/\mathcal{G}_U \quad \text{is the projection of a leaf of} \quad \mathcal{G}_{\mathrm{bas}}(M)_U$$

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4. by the maximality property of the leaves of a foliation, we have globally

$$\mathcal{S}_i \quad \text{is the projection of a leaf of} \quad \mathcal{G}_{\mathrm{bas}}(M)$$

□

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- If the vector fields $a(\xi)$ are not complete, this action does not integrate to a Lie group action. Therefore there is no quotient M/G.
- However the action algebroid is integrable to a Lie groupoid $\mathcal{G} \rightrightarrows M$.

Therefore we can define the quotient of a non-complete action as $M/\mathcal{G}$. By our result this is a Whitney stratified space

Which are the strata of $M/\mathcal{G}$?

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Let (M, ω) be a symplectic manifold, $(\mathcal{P}, \{\cdot,\cdot\})$ an integrable Poisson manifold and $J : M \to \mathcal{P}$ a Poisson map.

- There is a (Hamiltonian) action of the symplectic groupoid of $\Sigma(\mathcal{P})$ on M with momentum map J.
- The orbit space for this action is the same as the orbit space $M/\mathcal{G}$ where $\mathcal{G}$ is the action groupoid

$$\Sigma(\mathcal{P}) \ltimes M \rightrightarrows M$$

Is $M/\mathcal{G}$ a Whitney-Poisson stratified space?, Which are the strata?

THE END