The Orbit Space of a Proper Groupoid

Miguel Rodríguez-Olmos

EPFL, Switzerland

Joint work with

Oana Dragulete (EPFL)
Rui Loja Fernandes (IST)
Tudor S. Ratiu (EPFL)

Motivation

Let G be a Lie group, M a smooth manifold and

$$G \times M \longrightarrow M$$

a proper smooth action.

Motivation

Let G be a Lie group, M a smooth manifold and

$$G \times M \to M$$

a proper smooth action.

- If the action is free, M/G is a smooth manifold in the quotient topology.

Motivation

Let G be a Lie group, M a smooth manifold and

$$G \times M \longrightarrow M$$

a proper smooth action.

- If the action is free, M/G is a smooth manifold in the quotient topology.
- If the action is not free, M/G is a locally semi-algebraic space endowed with a canonical Whitney stratification.

Motivation

Let G be a Lie group, M a smooth manifold and

$$G \times M \to M$$

a proper smooth action.

- If the action is free, M/G is a smooth manifold in the quotient topology.
- If the action is not free, M/G is a locally semi-algebraic space endowed with a canonical Whitney stratification.

This correspond to a particular case of a groupoid: The action groupoid:

$$G \times M \rightrightarrows M$$

Motivation

Let G be a Lie group, M a smooth manifold and

$$G \times M \to M$$

a proper smooth action.

- If the action is free, M/G is a smooth manifold in the quotient topology.
- If the action is not free, M/G is a locally semi-algebraic space endowed with a canonical Whitney stratification.

This correspond to a particular case of a groupoid: The action groupoid:

$$G \times M \rightrightarrows M$$

and M/G is precisely the orbit space of this action groupoid.

More generally, let

$$s, t : \mathcal{G} \rightrightarrows M$$

be a groupoid.

More generally, let

$$s, t : \mathcal{G} \rightrightarrows M$$

be a groupoid. Let

$$\mathcal{O}_x = t(s^{-1}(x))$$

be the orbit through $x \in M$,

More generally, let

$$s, t : \mathcal{G} \rightrightarrows M$$

be a groupoid. Let

$$\mathcal{O}_x = t(s^{-1}(x))$$

be the orbit through $x \in M$, and define the equivalence class

$$x \sim y \quad \text{if} \quad \mathcal{O}_x = \mathcal{O}_y \quad \text{for} \quad x, y \in M.$$

More generally, let

$$s, t : \mathcal{G} \rightrightarrows M$$

be a groupoid. Let

$$\mathcal{O}_x = t(s^{-1}(x))$$

be the orbit through $x \in M$, and define the equivalence class

$$x \sim y \quad \text{if} \quad \mathcal{O}_x = \mathcal{O}_y \quad \text{for} \quad x, y \in M.$$

Then the orbit space of $\mathcal{G}$ is $M/\mathcal{G} := M/\sim$.

In the case of the action groupoid,

- $\mathcal{G} = G \times M$

In the case of the action groupoid,

- $\mathcal{G} = G \times M$
- $s(g, x) = x$

In the case of the action groupoid,

- $\mathcal{G} = G \times M$
- $s(g, x) = x$
- $t(g, x) = g \cdot x$

In the case of the action groupoid,

- $\mathcal{G} = G \times M$
- $s(g, x) = x$
- $t(g, x) = g \cdot x$

and then the orbit space $M/\mathcal{G}$ is exactly M/G.

In the case of the action groupoid,

- $\mathcal{G} = G \times M$
- $s(g, x) = x$
- $t(g, x) = g \cdot x$

and then the orbit space $M/\mathcal{G}$ is exactly M/G.

QUESTION: What is the structure of $M/\mathcal{G}$ for a general (not action) groupoid? In particular

In the case of the action groupoid,

- $\mathcal{G} = G \times M$
- $s(g, x) = x$
- $t(g, x) = g \cdot x$

and then the orbit space $M/\mathcal{G}$ is exactly M/G.

QUESTION: What is the structure of $M/\mathcal{G}$ for a general (not action) groupoid? In particular

- Is it a Whitney stratified space?

In the case of the action groupoid,

- $\mathcal{G} = G \times M$
- $s(g, x) = x$
- $t(g, x) = g \cdot x$

and then the orbit space $M/\mathcal{G}$ is exactly M/G.

QUESTION: What is the structure of $M/\mathcal{G}$ for a general (not action) groupoid? In particular

- Is it a Whitney stratified space?
- In that case, what is the global description of the strata?

Outline

1. Proper Lie Group Actions

Outline

Outline

Outline

Outline

1. Proper Lie Group Actions
2. Proper Lie Groupoids
3. Orbit Space of a Proper Groupoid: Local
4. Orbit Space of a Proper Groupoid: Global
5. Applications

Outline

Stratifications

Definition

A topological space $\mathcal{S}$ is a stratified space if for every $x \in \mathcal{S}$ there exists a neighborhood U and a finite family of disjoint locally closed smooth manifolds $U_i \subset U$, $i \in \mathcal{I}$ such that

Stratifications

Definition

A topological space $\mathcal{S}$ is a stratified space if for every $x \in \mathcal{S}$ there exists a neighborhood U and a finite family of disjoint locally closed smooth manifolds $U_i \subset U$, $i \in \mathcal{I}$ such that

- $U = \bigcup_{i \in \mathcal{I}} U_i$,
- If $i \neq j$, $U_i \cap \overline{U_j} \neq \varnothing \Rightarrow U_i \subset \overline{U}_j$.

Stratifications

Definition

A topological space $\mathcal{S}$ is a stratified space if for every $x \in \mathcal{S}$ there exists a neighborhood U and a finite family of disjoint locally closed smooth manifolds $U_i \subset U$, $i \in \mathcal{I}$ such that

- $U = \bigcup_{i \in \mathcal{I}} U_i$,
- If $i \neq j$, $U_i \cap \overline{U_j} \neq \varnothing \Rightarrow U_i \subset \overline{U}_j$.

The stratification is called Whitney if for every pair $\mathcal{U}_i \subset \overline{\mathcal{U}_j}$,

$$U_j \ni \{x_k\}_{k \in \mathbb{N}} \to x \in U_i \Rightarrow T_{x_k} U_j \to V > T_x U_i \quad .$$

(this requires an embedding of $\mathcal{U}$ in $\mathbb{R}^N$).

If $\mathcal{S}$ is a stratified space then there is a family of disjoint locally closed smooth manifolds $\mathcal{S}_k$, $k \in \mathcal{I}_{\mathcal{S}}$ such that for every $k \in \mathcal{I}_{\mathcal{S}}$

$$\mathcal{S}_k \cap U = U_i, \quad \text{for some } i \in \mathcal{I}_U.$$

If $\mathcal{S}$ is a stratified space then there is a family of disjoint locally closed smooth manifolds $\mathcal{S}_k$, $k \in \mathcal{I}_{\mathcal{S}}$ such that for every $k \in \mathcal{I}_{\mathcal{S}}$

$$\mathcal{S}_k \cap U = U_i, \quad \text{for some } i \in \mathcal{I}_U.$$

The manifolds $\mathcal{S}_k$ are called the strata of the stratification.

Stratifications for Lie group actions

Let G be a Lie group, M a smooth manifold and

$$G \times M \to M$$

a proper smooth action. Let $\pi : M \to M/G$ be the projection.

Stratifications for Lie group actions

Let G be a Lie group, M a smooth manifold and

$$G \times M \to M$$

a proper smooth action. Let $\pi : M \to M/G$ be the projection.

Local: A proper Lie group action admits slices and tubes (Palais). Then

Stratifications for Lie group actions

Let G be a Lie group, M a smooth manifold and

$$G \times M \to M$$

a proper smooth action. Let $\pi : M \to M/G$ be the projection.

Local: A proper Lie group action admits slices and tubes (Palais). Then

- If $x \in M$, then there is a neighborhood U of $[x]$ in M/G such that

  $$U \simeq \mathbf{S}/G_x$$

  where $\mathbf{S}$ is a linear slice for the G-action at x and G_x is the stabilizer of x which has a linear representation on $\mathbf{S}$.

Stratifications for Lie group actions

Let G be a Lie group, M a smooth manifold and

$$G \times M \to M$$

a proper smooth action. Let $\pi : M \to M/G$ be the projection.

Local: A proper Lie group action admits slices and tubes (Palais). Then

- If $x \in M$, then there is a neighborhood U of $[x]$ in M/G such that

  $$U \simeq \mathbf{S}/G_x$$

  where $\mathbf{S}$ is a linear slice for the G-action at x and G_x is the stabilizer of x which has a linear representation on $\mathbf{S}$.
- Since the action is proper G_x is compact, therefore using Invariant theory (Hilbert, Schwartz, Tarski-Seidenberg,...) U is a semi-algebraic Whitney stratified space (isotropy stratification)

Global: Let

$$M_{(H)} = \{x \in M \ : \ G_x \text{ is conjugate to } H\} \quad \text{(orbit types)}.$$

Then

Global: Let

$$M_{(H)} = \{x \in M \; : \; G_x \text{ is conjugate to } H\} \quad \text{(orbit types)}.$$

Then

- The connected components of $M_{(H)}$ are submanifolds of M for every $H \subset G$.

Global: Let

$$M_{(H)} = \{x \in M \;:\; G_x \text{ is conjugate to } H\} \quad \text{(orbit types)}.$$

Then

- The connected components of $M_{(H)}$ are submanifolds of M for every $H \subset G$.
- $M = \bigcup_{(H)} M_{(H)}$ is a locally finite disjoint partition.

Global: Let

$$M_{(H)} = \{x \in M \ : \ G_x \text{ is conjugate to } H\} \quad \text{(orbit types)}.$$

Then

- The connected components of $M_{(H)}$ are submanifolds of M for every $H \subset G$.
- $M = \bigcup_{(H)} M_{(H)}$ is a locally finite disjoint partition.
- The connected components of $\pi(M_{(H)})$ are the smooth strata of the isotropy stratification of M/G.

Outline

Let $s, t : \mathcal{G} \rightrightarrows M$ a Lie groupoid.

The analogous construction to the Lie group case cannot be used for studying $M/\mathcal{G}$ since the stabilizers

$$\mathcal{G}_x = s^{-1}(x) \cap t^{-1}(x)$$

cannot be compared by conjugation at points lying in different orbits.

Let $s, t : \mathcal{G} \rightrightarrows M$ a Lie groupoid.

The analogous construction to the Lie group case cannot be used for studying $M/\mathcal{G}$ since the stabilizers

$$\mathcal{G}_x = s^{-1}(x) \cap t^{-1}(x)$$

cannot be compared by conjugation at points lying in different orbits.

Therefore we cannot define orbit types $M_{(H)}$. We need a different approach

Tube theorem + Foliation theory

Tube theorem for proper groupoids

We will assume the following conditions for the Lie groupoid $s, t : \mathcal{G} \rightrightarrows M$:

- $(s, t) : \mathcal{G} \to M \times M$ is a proper map. (proper groupoid)
- s is locally trivial. (source local triviality)
- Every orbit of $\mathcal{G}$ is of finite type.

Tube theorem for proper groupoids

We will assume the following conditions for the Lie groupoid $s, t : \mathcal{G} \rightrightarrows M$:

- $(s, t) : \mathcal{G} \to M \times M$ is a proper map. (proper groupoid)
- s is locally trivial. (source local triviality)
- Every orbit of $\mathcal{G}$ is of finite type.

Theorem (Weinstein, Zung)

Let $\mathcal{G} \rightrightarrows M$ be a source locally trivial proper groupoid and $x \in M$ with orbit $\mathcal{O}$.

Tube theorem for proper groupoids

We will assume the following conditions for the Lie groupoid $s, t : \mathcal{G} \rightrightarrows M$:

- $(s, t) : \mathcal{G} \to M \times M$ is a proper map. (proper groupoid)
- s is locally trivial. (source local triviality)
- Every orbit of $\mathcal{G}$ is of finite type.

Theorem (Weinstein, Zung)

Let $\mathcal{G} \rightrightarrows M$ be a source locally trivial proper groupoid and $x \in M$ with orbit $\mathcal{O}$. Then there is an a action of $\mathcal{G}_{\mathcal{O}}$ on $N\mathcal{O} = T_{\mathcal{O}}M/T\mathcal{O}$, with associated action groupoid

$$\mathcal{G}_{\mathcal{O}} \ltimes N\mathcal{O} \rightrightarrows N\mathcal{O} \quad \text{and}$$

$\mathcal{G}$ is locally isomorphic to $\mathcal{G}_{\mathcal{O}} \ltimes N\mathcal{O}$.

Morita equivalence

Using the Tube theorem we can prove that every proper groupoid is locally Morita equivalent to an action groupoid for a representation of a compact group on a vector space.

Morita equivalence

Using the Tube theorem we can prove that every proper groupoid is locally Morita equivalent to an action groupoid for a representation of a compact group on a vector space.

Theorem

Let $\mathcal{G} \rightrightarrows M$ be a source locally trivial proper groupoid, and $x \in M$.

Morita equivalence

Using the Tube theorem we can prove that every proper groupoid is locally Morita equivalent to an action groupoid for a representation of a compact group on a vector space.

Theorem

Let $\mathcal{G} \rightrightarrows M$ be a source locally trivial proper groupoid, and $x \in M$.

Then the action of $\mathcal{G}_{\mathcal{O}}$ on $N\mathcal{O}$ restricts to a representation of $\mathcal{G}_x$ on $N_x\mathcal{O}$ ($\mathcal{G}_x = s^{-1}(x) \cap t^{-1}(x)$ is the stabilizer, a compact Lie group) with associated action groupoid

$$\mathcal{G}_x \times N_x\mathcal{O} \rightrightarrows N_x\mathcal{O}, \quad \text{and}$$

Morita equivalence

Using the Tube theorem we can prove that every proper groupoid is locally Morita equivalent to an action groupoid for a representation of a compact group on a vector space.

Theorem

Let $\mathcal{G} \rightrightarrows M$ be a source locally trivial proper groupoid, and $x \in M$.

Then the action of $\mathcal{G}_{\mathcal{O}}$ on $N\mathcal{O}$ restricts to a representation of $\mathcal{G}_x$ on $N_x\mathcal{O}$ ($\mathcal{G}_x = s^{-1}(x) \cap t^{-1}(x)$ is the stabilizer, a compact Lie group) with associated action groupoid

$$\mathcal{G}_x \times N_x\mathcal{O} \rightrightarrows N_x\mathcal{O}, \quad \text{and}$$

$\mathcal{G}$ is locally Morita equivalent to $\mathcal{G}_x \times N_x\mathcal{O}$.

Proof.

1. By the Tube theorem, $\mathcal{G}$ is locally isomorphic to $\mathcal{G}_{\mathcal{O}} \ltimes N\mathcal{O}$

Proof.

1. By the Tube theorem, $\mathcal{G}$ is locally isomorphic to $\mathcal{G}_{\mathcal{O}} \ltimes N\mathcal{O}$
2. Form the bi-bundle

$$\begin{tikzcd}
\mathcal{G}_{\mathcal{O}} \ltimes N\mathcal{O} \arrow[d, shift left] \arrow[d, shift right] & \circlearrowleft & s^{-1}(x) \times N_x\mathcal{O} \arrow[dll, "\pi_1"'] \arrow[drr, "\pi_2"] & \circlearrowright & \mathcal{G}_x \times N_x\mathcal{O} \arrow[d, shift left] \arrow[d, shift right] \\
N\mathcal{O} & & & & N_x\mathcal{O}
\end{tikzcd}$$

Proof.

1. By the Tube theorem, $\mathcal{G}$ is locally isomorphic to $\mathcal{G}_{\mathcal{O}} \ltimes N\mathcal{O}$
2. Form the bi-bundle

$$\begin{tikzcd}
\mathcal{G}_{\mathcal{O}} \ltimes N\mathcal{O} \arrow[d, shift left] \arrow[d, shift right] & \circlearrowleft & s^{-1}(x) \times N_x\mathcal{O} \arrow[dll, "\pi_1"'] \arrow[drr, "\pi_2"] & \circlearrowright & \mathcal{G}_x \times N_x\mathcal{O} \arrow[d, shift left] \arrow[d, shift right] \\
N\mathcal{O} & & & & N_x\mathcal{O}
\end{tikzcd}$$

with $\pi_1(g, v) = g \cdot v$ and $\pi_2(g, v) = v$.

$$\begin{aligned}
(\mathcal{G}_{\mathcal{O}} \ltimes N\mathcal{O}) \times (s^{-1}(x) \times N_x\mathcal{O}) &\to (s^{-1}(x) \times N_x\mathcal{O}) \\
(g', v') \cdot (g, v) &\mapsto (g'g, v),
\end{aligned}$$

Proof.

1. By the Tube theorem, $\mathcal{G}$ is locally isomorphic to $\mathcal{G}_{\mathcal{O}} \ltimes N\mathcal{O}$
2. Form the bi-bundle

$$
\begin{tikzcd}
\mathcal{G}_{\mathcal{O}} \ltimes N\mathcal{O} \arrow[d, shift left] \arrow[d, shift right] & \circlearrowleft & s^{-1}(x) \times N_x\mathcal{O} \arrow[dll, "\pi_1"'] \arrow[drr, "\pi_2"] & \circlearrowright & \mathcal{G}_x \times N_x\mathcal{O} \arrow[d, shift left] \arrow[d, shift right] \\
N\mathcal{O} & & & & N_x\mathcal{O}
\end{tikzcd}
$$

$$
\begin{array}{rcl}
(\mathcal{G}_x \times N_x\mathcal{O}) \times (s^{-1}(x) \times N_x\mathcal{O}) & \longrightarrow & (s^{-1}(x) \times N_x\mathcal{O}) \\
(g', v) \cdot (g, v) & \mapsto & (g{g'}^{-1}, g' \cdot v)
\end{array}
$$

Proof.

1. By the Tube theorem, $\mathcal{G}$ is locally isomorphic to $\mathcal{G}_{\mathcal{O}} \ltimes N\mathcal{O}$
2. Form the bi-bundle

$$\begin{tikzcd}
\mathcal{G}_{\mathcal{O}} \ltimes N\mathcal{O} \arrow[d, shift left] \arrow[d, shift right] & \circlearrowleft & s^{-1}(x) \times N_x\mathcal{O} \arrow[dll, "\pi_1"'] \arrow[drr, "\pi_2"] & \circlearrowright & \mathcal{G}_x \times N_x\mathcal{O} \arrow[d, shift left] \arrow[d, shift right] \\
N\mathcal{O} & & & & N_x\mathcal{O}
\end{tikzcd}$$

3. These two actions are free, they commute and the momentum map of one is the orbit map of the other.

Outline

Since $\mathcal{G}$ is locally Morita equivalent to $\mathcal{G}_x \times N_x\mathcal{O}$ then near $[x]$

$$M/\mathcal{G} \sim N_x\mathcal{O}/\mathcal{G}_x \quad \text{(local homeomorphism).}$$

Since $\mathcal{G}$ is locally Morita equivalent to $\mathcal{G}_x \times N_x\mathcal{O}$ then near $[x]$

$$M/\mathcal{G} \sim N_x\mathcal{O}/\mathcal{G}_x \quad \text{(local homeomorphism).}$$

Therefore locally the orbit space of $\mathcal{G}$ is a quotient for a representation of a compact Lie group.

Since $\mathcal{G}$ is locally Morita equivalent to $\mathcal{G}_x \times N_x\mathcal{O}$ then near $[x]$

$$M/\mathcal{G} \sim N_x\mathcal{O}/\mathcal{G}_x \quad \text{(local homeomorphism)}.$$

Therefore locally the orbit space of $\mathcal{G}$ is a quotient for a representation of a compact Lie group.

In particular

Theorem

The orbit space for a source locally trivial proper groupoid is a locally semi-algebraic Whitney stratified space.

Outline

Define the following families of vector fields

- $\mathcal{G}_{\mathrm{bas}}(M) = \{X \in \mathfrak{X}(M) \,:\, Xf \in C^{\mathcal{G}}(M) \forall f \in C^{\mathcal{G}}(M)\}$,

Define the following families of vector fields

- $\mathcal{G}_{\text{bas}}(M) = \{X \in \mathfrak{X}(M) \,:\, Xf \in C^{\mathcal{G}}(M) \,\forall f \in C^{\mathcal{G}}(M)\}$,
- $\mathcal{A}(N_x\mathcal{O}) = \{X + Y \,:\, X \in \mathfrak{X}^{\mathcal{G}_x}(N_x\mathcal{O}),\ Y \text{ tangent to orbits}\}$.

Define the following families of vector fields

- $\mathcal{G}_{\mathrm{bas}}(M) = \{X \in \mathfrak{X}(M) : Xf \in C^{\mathcal{G}}(M)\, \forall f \in C^{\mathcal{G}}(M)\}$,
- $\mathcal{A}(N_x\mathcal{O}) = \{X + Y : X \in \mathfrak{X}^{\mathcal{G}_x}(N_x\mathcal{O}),\ Y \text{ tangent to orbits}\}$.

It is well-known that

Lemma

- *$\mathcal{A}(N_x\mathcal{O})$ defines a singular integrable distribution,*
- *its leaves are $\mathcal{G}_x$-invariant,*
- *the strata of $N_x\mathcal{O}/\mathcal{G}_x$ are the projection of the leaves of $\mathcal{A}(N_x\mathcal{O})$.*

Define the following families of vector fields

- $\mathcal{G}_{\mathrm{bas}}(M) = \{X \in \mathfrak{X}(M) \,:\, Xf \in C^{\mathcal{G}}(M) \forall f \in C^{\mathcal{G}}(M)\}$,
- $\mathcal{A}(N_x\mathcal{O}) = \{X + Y \,:\, X \in \mathfrak{X}^{\mathcal{G}_x}(N_x\mathcal{O}),\ Y \text{ tangent to orbits}\}$.

It is well-known that

Lemma

- *$\mathcal{A}(N_x\mathcal{O})$ defines a singular integrable distribution,*
- *its leaves are $\mathcal{G}_x$-invariant,*
- *the strata of $N_x\mathcal{O}/\mathcal{G}_x$ are the projection of the leaves of $\mathcal{A}(N_x\mathcal{O})$.*

With this Lemma and the Morita equivalence we can prove:

Theorem

Let $\mathcal{G} \rightrightarrows M$ be a source locally trivial proper groupoid. Then

- *$\mathcal{G}_{bas}(M)$ induces a singular integrable distribution,*
- *the strata of $M/\mathcal{G}$ are the projections of the leaves of $\mathcal{G}_{bas}(M)$.*

Proof.

1. For each $x \in M$ there is a saturated neighborhood U of $\mathcal{O}$ Morita equivalent to $\mathcal{G}_x \times N_x\mathcal{O}$

Proof.

1. For each $x \in M$ there is a saturated neighborhood U of $\mathcal{O}$ Morita equivalent to $\mathcal{G}_x \times N_x\mathcal{O}$
2. Morita equivalence $\Rightarrow$ $\mathcal{G}_{\mathrm{bas}}(M)_U \longleftrightarrow \mathcal{A}(N_x\mathcal{O})$ (1:1).

Proof.

1. For each $x \in M$ there is a saturated neighborhood U of $\mathcal{O}$ Morita equivalent to $\mathcal{G}_x \times N_x\mathcal{O}$
2. Morita equivalence $\Rightarrow \mathcal{G}_{\mathrm{bas}}(M)_U \longleftrightarrow \mathcal{A}(N_x\mathcal{O})$ (1:1).
3. by the general properties of stratifications and the Lemma, for each stratum $\mathcal{S}_i$ of $M/\mathcal{G}$,

$$\mathcal{S}_i \cap U/\mathcal{G}_U \quad \text{is the projection of a leaf of} \quad \mathcal{G}_{\mathrm{bas}}(M)_U$$

Proof.

1. For each $x \in M$ there is a saturated neighborhood U of $\mathcal{O}$ Morita equivalent to $\mathcal{G}_x \times N_x\mathcal{O}$
2. Morita equivalence $\Rightarrow$ $\mathcal{G}_{\text{bas}}(M)_U \longleftrightarrow \mathcal{A}(N_x\mathcal{O})$ (1:1).
3. by the general properties of stratifications and the Lemma, for each stratum $\mathcal{S}_i$ of $M/\mathcal{G}$,

$$\mathcal{S}_i \cap U/\mathcal{G}_U \quad \text{is the projection of a leaf of} \quad \mathcal{G}_{\text{bas}}(M)_U$$

4. by the maximality property of the leaves of a foliation, we have globally

$$\mathcal{S}_i \quad \text{is the projection of a leaf of} \quad \mathcal{G}_{\text{bas}}(M)$$

$\square$

Outline

1. Non-complete actions

Let $a : \mathfrak{g} \to \mathfrak{X}(M)$ be a Lie algebra action.

1. Non-complete actions

Let $a : \mathfrak{g} \to \mathfrak{X}(M)$ be a Lie algebra action.

- The action algebroid of this action is $\pi : \mathfrak{g} \times M \to M$ with projection $\pi(\xi, x) = x$ and anchor $\rho_A(\xi, x) = a(\xi)(x)$.

1. Non-complete actions

Let $a : \mathfrak{g} \to \mathfrak{X}(M)$ be a Lie algebra action.

- The action algebroid of this action is $\pi : \mathfrak{g} \times M \to M$ with projection $\pi(\xi, x) = x$ and anchor $\rho_A(\xi, x) = a(\xi)(x)$.
- If the vector fields $a(\xi)$ are not complete, this action does not integrate to a Lie group action. Therefore there is no quotient M/G.

1. Non-complete actions

Let $a : \mathfrak{g} \to \mathfrak{X}(M)$ be a Lie algebra action.

- The action algebroid of this action is $\pi : \mathfrak{g} \times M \to M$ with projection $\pi(\xi, x) = x$ and anchor $\rho_A(\xi, x) = a(\xi)(x)$.
- If the vector fields $a(\xi)$ are not complete, this action does not integrate to a Lie group action. Therefore there is no quotient M/G.
- However the action algebroid is integrable to a Lie groupoid $\mathcal{G} \rightrightarrows M$.

Therefore we can define the quotient of a non-complete action as $M/\mathcal{G}$. By our result this is a Whitney stratified space

Which are the strata of $M/\mathcal{G}$?

2. Hamiltonian groupoid actions

Let (M, ω) be a symplectic manifold, $(\mathcal{P}, \{\cdot, \cdot\})$ an integrable Poisson manifold and $J : M \to \mathcal{P}$ a Poisson map.

2. Hamiltonian groupoid actions

Let (M, ω) be a symplectic manifold, $(\mathcal{P}, \{\cdot, \cdot\})$ an integrable Poisson manifold and $J : M \to \mathcal{P}$ a Poisson map.

- There is a (Hamiltonian) action of the symplectic groupoid of $\Sigma(\mathcal{P})$ on M with momentum map J.

2. Hamiltonian groupoid actions

Let (M,ω) be a symplectic manifold, $(\mathcal{P},\{\cdot,\cdot\})$ an integrable Poisson manifold and $J : M \to \mathcal{P}$ a Poisson map.

- There is a (Hamiltonian) action of the symplectic groupoid of $\Sigma(\mathcal{P})$ on M with momentum map J.
- The orbit space for this action is the same as the orbit space $M/\mathcal{G}$ where $\mathcal{G}$ is the action groupoid

$$\Sigma(\mathcal{P}) \ltimes M \rightrightarrows M$$

Is $M/\mathcal{G}$ a Whitney-Poisson stratified space?, Which are the strata?

THE END