

OPTIMAL CONTROL PROBLEMS FOR AFFINE CONNECTION CONTROL SYSTEMS: CHARACTERIZATION OF EXTREMALS

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OUTLINE

- ① OPTIMAL CONTROL PROBLEM FOR AFFINE CONNECTION CONTROL SYSTEMS
- ② PRESYMPLECTIC CONSTRAINT ALGORITHM FOR ACCS
- ③ APPLICATION: TIME-OPTIMAL CONTROL PROBLEM, $F = 1$

- ① OPTIMAL CONTROL PROBLEM FOR AFFINE CONNECTION CONTROL SYSTEMS
- ② Presymplectic Constraint Algorithm for ACCS
- ③ Application: Time-Optimal Control Problem, $F = 1$

AFFINE CONNECTION CONTROL SYSTEM (ACCS)

Let Q be a smooth manifold, $\dim Q = n$.

Let ∇ be an affine connection on Q .

Consider the control system

$$\nabla_{\dot{\gamma}(t)} \dot{\gamma}(t) = u^k(t) Y_k(\gamma(t)),$$

where

- $\gamma: I \subset \mathbb{R} \rightarrow Q$ is a curve,
- $u: I \rightarrow U \subset \mathbb{R}^m$ are locally integrable *controls*,
- U is an open set,
- Y_k are *input vector fields* on Q .

An *Affine Connection Control System* is $\Sigma = (Q, \nabla, \mathcal{Y}, U)$,

where $\mathcal{Y} = \{Y_1, \dots, Y_m\}$.

The above second-order equation is rewritten on TQ ,

$$\dot{\Upsilon}(t) = Z(\Upsilon(t)) + u^k(t) Y_k^V(\Upsilon(t)), \quad X = Z + u^k Y_k^V,$$

where

- $\Upsilon: I \rightarrow TQ$ is a curve such that $\Upsilon = \dot{\gamma}$,
- Z is the *geodesic spray* associated to ∇ , a vector field on TQ . In natural coordinates (x, v) for TQ ,

$$Z = v^j \frac{\partial}{\partial x^i} - \Gamma_{jl}^i(x) v^j v^l \frac{\partial}{\partial v^i}, \quad \Gamma_{jl}^i \text{ Christoffel symbols for } \nabla.$$

- Y_k^V denotes the vertical lift of the vector field Y_k .

FREE-TIME OPTIMAL CONTROL PROBLEM FOR ACCS (OCP)

Let $F: TQ \times U \rightarrow \mathbb{R}$ be a *cost function*.

Given $\Sigma = (Q, \nabla, \mathcal{Y}, U)$, F .

Find $I = [a, b] \subset \mathbb{R}$ and $(\gamma, u): I \rightarrow Q \times U$

such that there exists $\Upsilon: I \rightarrow TQ$ along γ satisfying

$$(1) \quad \Upsilon(a) = v_{x_a}, \quad \Upsilon(b) = v_{x_b}, \quad \text{given } v_{x_a} \in T_{x_a}Q, \quad v_{x_b} \in T_{x_b}Q,$$

$$(2) \quad \dot{\Upsilon}(t) = (Z + u^k Y_k^V)(\Upsilon(t)) \quad (\Rightarrow \Upsilon = \dot{\gamma}),$$

$$(3) \quad \mathcal{S}[\Upsilon, u] = \int_I F(\Upsilon(t), u(t)) dt \text{ is minimum over all curves on } TQ \times U \text{ satisfying (1) and (2).}$$

PRESYMPLECTIC FORMALISM IN OCP

Let M be a smooth manifold and $\pi_1: T^*M \times U \rightarrow T^*M$.

Let $(T^*M \times U, \Omega)$ be the *presymplectic manifold*, where

Ω is the π_1 -pullback of the natural 2-form in T^*M .

In natural coordinates (x, p, u) for $T^*M \times U$,

$$\Omega = dp_i \wedge dx^i, \quad \ker \Omega = \left\{ \frac{\partial}{\partial u^k} \right\}_{k=1, \dots, m}.$$

PRESYMPLECTIC FORMALISM IN OCP

Let X be a vector field along $\pi: M \times U \rightarrow M$,
the cost function $F: M \times U \rightarrow \mathbb{R}$ and $p_0 \in \{-1, 0\}$,

we define the *Hamiltonian* $H: T^*M \times U \rightarrow \mathbb{R}$,

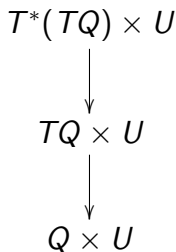
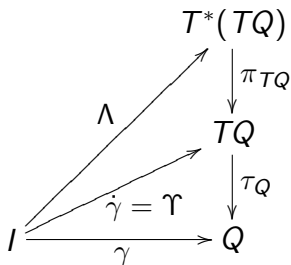
$$H(p, u) = (H_X + p_0 F)(p, u) = \langle p, X(x, u) \rangle + p_0 F(x, u), \quad p \in T_x^*M.$$

Then $(T^*M \times U, \Omega, H)$ is a *presymplectic Hamiltonian system*

and $i_{X_H} \Omega = dH$ is the *presymplectic equation*.

Now

- $M = TQ$,
- $X = Z + u^k Y_k^V \in \mathfrak{X}(TQ)$,
- $H: T^*(TQ) \times U \rightarrow \mathbb{R}$, $H = H_Z + u^k H_{Y_k^V} + p_0 F$,
- $(T^*(TQ) \times U, \Omega, H)$ is the presymplectic Hamiltonian system in OCP for ACCS.



WEAK PONTRYAGIN'S MAXIMUM PRINCIPLE (PMP)

THEOREM

Let $(\Upsilon, u): [a, b] \rightarrow TQ \times U$ be a solution of OCP with initial conditions v_{x_a}, v_{x_b} . Then there exist $\Lambda: [a, b] \rightarrow T^*(TQ)$ along Υ , and a constant $p_0 \in \{-1, 0\}$ such that:

- ① (Λ, u) is an integral curve of the Hamiltonian vector field X_H , $i_{X_H}\Omega = dH$;
- ② $\Upsilon = \pi_{TQ} \circ \Lambda$, where $\pi_{TQ}: T^*(TQ) \rightarrow TQ$;
- ③ Υ satisfies the initial conditions in TQ ;
- ④ (A) $\max_{\tilde{u} \in U} H(\Lambda(t), \tilde{u}) = 0$ for $t \in [a, b]$;
 (B) $(p_0, \Lambda(t)) \neq 0$ for each $t \in [a, b]$.

DIFFERENT KINDS OF EXTREMALS

DEFINITION

A curve $(\Upsilon, u): [a, b] \rightarrow TQ \times U$ for OCP is

- 1 an **extremal** if there exist $\Lambda: [a, b] \rightarrow T^*(TQ)$ and a constant $p_0 \in \{-1, 0\}$ such that $\Upsilon = \pi_{TQ} \circ \Lambda$ and (Λ, u) satisfies the necessary conditions of PMP;
- 2 a **normal extremal** if it is an extremal and $p_0 = -1$;
- 3 an **abnormal extremal** if it is an extremal and $p_0 = 0$;
- 4 a **strictly abnormal extremal** if it is not a normal extremal, but it is an abnormal extremal.

The curve $(\Lambda, u): [a, b] \rightarrow T^*(TQ) \times U$ along Υ is called **biextremal for OCP**.

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PRESYMPLECTIC CONSTRAINT ALGORITHM (GOTAY-NESTER)

Given (M, Ω, H) and $i_X \Omega = dH$, find (N, X) such that

- (A) N is a submanifold of M ,
- (B) X is a vector field tangent to N ,
- (C) N is maximal among all the submanifolds satisfying A, B.

$$\begin{aligned}
 \text{Primary} \quad N_0 &= \{x \in M \mid \exists v_x \in T_x M, i_{v_x} \Omega = d_x H\} \\
 \text{constraint} \quad &= \{x \in M \mid Z(H)(x) = 0, \forall Z \in \ker \Omega\} \\
 \text{submanifold} \quad X^{N_0} &= X^0 + \ker \Omega, X^0 \text{ is a solution of } i_X \Omega = dH
 \end{aligned}$$

Stabilization: $N_1 = \{x \in N_0 \mid \exists X \in X^{N_0}, X(x) \in T_x N_0\}$.

$$(N_i, X^{N_i}), \quad N_{i+1} = \{x \in N_i \mid \exists X \in X^{N_i}, X(x) \in T_x N_i\}.$$

If $\exists i \in \mathbb{N}$ such that $N_i = N_{i-1}$,

$N_f = N_{i-1}$ is the **final constraint submanifold**.

NOW IN OCP FOR ACCS

- $M = T^*(TQ) \times U$,
- $H: T^*(TQ) \times U \rightarrow \mathbb{R}$, $H = H_Z + u^k H_{Y_k^V} + p_0 F$,
- $(T^*(TQ) \times U, \Omega, H)$ is the presymplectic Hamiltonian system in OCP for ACCS,
- $i_{X_H} \Omega = dH$ and locally

$$X_H = \frac{\partial H}{\partial p_j} \frac{\partial}{\partial x^j} - \frac{\partial H}{\partial x^i} \frac{\partial}{\partial p_i} + C^k \frac{\partial}{\partial u^k}.$$

CONSTRAINT ALGORITHM IN OCP FOR ACCS (FREE-TIME)

Primary submanifold

$$N_0 = \left\{ (\Lambda, u) \in T^*(TQ) \times U \mid \overbrace{H_{Y_k^V} + p_0 \frac{\partial F}{\partial u^k}}^{\frac{\partial H}{\partial u^k} =} = 0, k = 1, \dots, m \right. \\ \left. H = 0. \right\}$$

First stabilization step:

$$N_1 = \{(\Lambda, u) \in N_0 \mid X_H(\Lambda, u) \in T_{(\Lambda, u)} N_0\}.$$

Tangency conditions:

$$X_H(H_{Y_k^V} + p_0 \frac{\partial F}{\partial u^k}) = 0,$$

$$X_H(H) = 0 \quad \text{Trivially.}$$

Normality	Abnormality
$p_0 = -1$	$p_0 = 0$
$\{H_{Y_k^V} = \frac{\partial F}{\partial u^k}, H = 0\} (= N_0^{[-1]})$	$\{H_{Y_k^V} = 0, H = 0\} (= N_0^{[0]})$
$N_1^{[-1]}$	$N_0^{[0]} \cap \{H_{[Z, Y_k^V]} = 0\} (= N_1^{[0]})$
\vdots	\vdots
$(N_f^{[-1]}, X_f^{[-1]})$	$(N_f^{[0]}, X_f^{[0]})$ Delete zero covector

STRICT ABNORMALITY

Let $\rho: T^*(TQ) \times U \rightarrow TQ \times U$ and $\mathbf{P} = \rho(N_f^{[0]}) \cap \rho(N_f^{[-1]})$.

$\mathbf{P} = \emptyset$	$\rho(N_f^{[0]}) \neq \emptyset$	all the abnormal extremals are strict .
	$\rho(N_f^{[-1]}) \neq \emptyset$	all the normal extremals are strict normal .
$\mathbf{P} \neq \emptyset$	$\mathbf{P} = \rho(N_f^{[0]})$	no strict abnormal extremals.
	$\mathbf{P} \neq \rho(N_f^{[0]})$	local strict abnormal extremals.
	$\mathbf{P} = \rho(N_f^{[0]})$ $= \rho(N_f^{[-1]})$	all the abnormal extremals are also normal and viceversa.

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CONSTRAINT ALGORITHM FOR TIME-OPTIMAL PROBLEM, $F = 1$

Pontryagin's Hamiltonian $H = H_Z + u^k H_{Y_k^V} + p_0$.

On the submanifold $H = 0$, we obtain $N_f^{[-1]}$ and $N_f^{[0]}$.

PUT CONDITION $H = 0$ ASIDE and apply the algorithm:

$$N_0 = N_0^{[0]} = N_0^{[-1]} = \{(\Lambda, u) \in T^*(TQ) \times U \mid H_{Y_k^V} = 0\},$$

$$N_1 = \{(\Lambda, u) \in N_0 \mid H_{[Z, Y_k^V]} = 0\},$$

for $k = 1, \dots, m$, and so on until N_f , if it exists.

The actual final constraint submanifolds are

$$N_f^{[0]} = N_f \cap \{(\Lambda, u) \in T^*(TQ) \times U \mid H_Z + u^k H_{Y_k^V} = 0\},$$

$$N_f^{[-1]} = N_f \cap \{(\Lambda, u) \in T^*(TQ) \times U \mid H_Z + u^k H_{Y_k^V} = 1\}.$$




RESULTS FOR TIME-OPTIMAL CONTROL PROBLEM, $F = 1$

PROPOSITION




Let Σ be an ACCS. Given a time-optimal control problem:

- 1 If $N_f^{[0]}$ only has zero covectors, there are *no abnormal extremals*.
- 2 If $N_f^{[0]}$ has nonzero covectors and $N_f \subset \{(\Lambda, u) \in T^*(TQ) \times U \mid (H_Z + u^j H_{Y_j^v}) = 0\}$, then *every abnormal extremal is strict* and there are *no normal extremals*.

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