The principle of general tovariance

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Motto

'At a certain point in its history, the fundamental problems of physics have to do with the way in which fundamental concepts are defined. In those circumstances, the pursuit of physics in accord with those concepts evidently has not resolved the underlying problems. These are the times at which philosophical analysis has become an unavoidable task of physics itself.'

R. DiSalle, *Understanding space-time: The philosophical development of physics from Newton to Einstein.* Cambridge University Press, 2006

Einstein's road to general relativity:

- **Principle of general covariance**
- **Equivalence principle**

Bohr's interpretation of quantum mechanics:

- **Doctrine of classical concepts**
- **Principle of complementarity**

Fundamental problems of physics

- Logical vs probabilistic structure of quantum theory
- Three roads to quantum gravity (Smolin):
    1. Start with quantum mechanics (string theory)
    2. Start with general relativity (loop quantum gravity)
    3. Discard quantum mechanics *and* general relativity

Recent trend in quantum theory and third road to quantum gravity:

topos theory

- Rethinks the logical foundations of physics (and mathematics!)
- Releases the tension between the noncommutativity of quantum theory and the 'commutativity' of general relativity

Butterfield & Isham (1998–2000), Isham & Döring (2007)
Mulvey et al. (1974–2006), Heunen & Spitters (2007),
Markopoulou (2000), Corbett et al. (2007),...

What is a topos?

'A startling aspect of topos theory is that it unifies two seemingly wholly distinct mathematical subjects: on the one hand, topology and algebraic geometry [Grothendieck] and on the other hand, logic and set theory [Lawvere].'

S. Mac Lane & I. Moerdijk, *Sheaves in geometry and logic: A first introduction to topos theory.* Springer, 1994.

A topos is a generalization of the category *Sets* of sets, in which "everything" you can do with sets can still be done, e.g. define and compose functions, form subsets, cartesian and fibered products, . . . except classical logic

- **A category has objects X, Y and arrows $X \xrightarrow{f} Y$ with associative composition $X \xrightarrow{f} Y \xrightarrow{g} Z = X \xrightarrow{g \circ f} Z$ and identity $X \xrightarrow{\mathrm{id}} X$**
- **The objects of *Sets* are sets and the arrows are functions**
- **Maps between categories are called functors**

Definition of a topos

A topos is a category with:

1. **Terminal object**
2. **Pullbacks**
3. **Exponentials**
4. **Subobject classifier**

In *Sets* these things are as follows:

1. **Singleton $1 = \{\emptyset\}$, with unique arrow $X \to 1$ for any set X; vice versa, this condition** *defines* **terminal objects in a category**
2. **The pullback of functions $B \xrightarrow{f} A$ and $C \xrightarrow{g} A$ is the fibered product $B \times_A C = \{(b, c) \in B \times C \mid f(b) = g(c)\}$ with the obvious projections $B \times_A C \to B$ and $B \times_A C \to C$; N.B. $B \times C = B \times_1 C$**
3. **$Y^X = \{X \xrightarrow{f} Y\}$ is itself an object in *Sets*, with $Y^X \times X \xrightarrow{\mathrm{ev}} Y$**
4. **$\Omega = \{0, 1\}$ with logical structure as $\Omega \equiv \{\perp, \top\} = \{\mathrm{false}, \mathrm{true}\}$**

Pullback

Given f and g, there is an object $B \times_A C$ with arrows p, q such that the square below commutes, and if P accomplishes same as $B \times_A C$ there must be a unique arrow $P \to B \times_A C$

$$\begin{tikzcd}
P \arrow[dr, "\exists!"] \arrow[drr] \arrow[ddr] & & \\
& B \times_A C \arrow[r, "p"'] \arrow[d, "q"] & C \arrow[d, "g"] \\
& B \arrow[r, "f"'] & A
\end{tikzcd}$$

Subobject classifier

Arrow $X \xhookrightarrow{f} Y$ **defines subobject** $X \subset Y$ **if** f **is "injective" (monic)**

There exists an object Ω **and an arrow** $1 \xrightarrow{\top} \Omega$**, such that for every monic** f **there is a** *unique* **arrow** χ_f **defining a pullback**

$$\begin{tikzcd} X \arrow[r] \arrow[d, hook, "f"'] & 1 \arrow[d, "\top"] \\ Y \arrow[r, "\chi_f"'] & \Omega \end{tikzcd}$$

In ***Sets*****:** $\Omega = \{0, 1\}$ **with** $\top(1) = 1$**, and** $\chi_f = \chi_X$ **(for** $X \subset Y$**)**

Ω **has logical structure (in** ***Sets*****:** $\Omega \equiv \{\perp, \top\} = \{\text{false}, \text{true}\}$**):**

true $1 \xrightarrow{\top} \Omega$**, negation** $\neg : \Omega \to \Omega$**, conjunction** $\wedge : \Omega \times \Omega \to \Omega$**,**

false $1 \xrightarrow{\perp} \Omega$**, disjunction** $\vee : \Omega \times \Omega \to \Omega$**, implication** $\Rightarrow : \Omega \times \Omega \to \Omega$

Famous example

Topological space X defines lattice $\mathcal{O}(X)$ of open sets

Partial ordering turns $\mathcal{O}(X)$ into category, $\exists!\, U \to V$ iff $U \supseteq V$

Sheaf on X is functor $F : \mathcal{O}(X) \to$ *Sets* that is determined locally

Example: $F(U) = C(U)$, **and if** $U \supseteq V$, **map** $C(U) \xrightarrow{\text{restriction}} C(V)$

The sheaves on X form a topos $\mathrm{Sh}(X)$, with subobject classifier

$\Omega : \mathcal{O}(X) \to$ ***Sets*** **given by** $\Omega(U) = \mathcal{O}(U)$, **with logical structure:**

true $= U$, **negation** $\neg V = \mathrm{int}(U \backslash V)$, **conjunction** $V \wedge W = V \cap W$,

false $= \emptyset$, **disjunction** $V \vee W = V \cup W$, **implication** $V \Rightarrow W = \neg V \cup W$

Note that $V \vee \neg V = U \backslash \partial V \neq U = \top$: **This logic is intuitionistic!!!**

Geometric (observational) logic

In general topoi one does *not* have:

- Aristotle's principle of the excluded third
- Zermelo's Axiom of Choice (any "surjection" has right inverse)

Further restrictions come from natural maps between topoi:

A geometric morphism $F \to E$ between topoi is a *pair* of functors $\phi^* : E \to F$ and $\phi_* : F \to E$ such that ϕ^* is left adjoint to ϕ_* and left exact (i.e. preserves limits)

E.g., map $f : X \to Y$ induces geometric morphism $\mathrm{Sh}(X) \to \mathrm{Sh}(Y)$

A mathematical structure in a topos is preserved under geometric morphisms if it is defined "geometrically":

- Predicate logic involving *arbitrary* $\vee$, *finitely* many $\wedge$, $\top$, $\bot$, $\exists$
- Axioms of the form $\forall x : \varphi(x) \to \psi(x)$

C^*-algebras can be defined "geometrically" (Mulvey)

The principle of general tovariance

Principle of general covariance: laws of physics must be covariant under arbitrary coordinate transformations (Einstein)

Originally intended to express the general relativity of motion

It actually says that "general relativity" uses differential geometry

Principle of general tovariance: any mathematical structure appearing in the laws of physics must be definable in an arbitrary topos (with natural numbers object) and must be preserved under geometric morphisms

General tovariance has no immediate physical content; it identifies the mathematical language of physics as geometric logic

("physics is information"? physics is logic!)

Since C^*-algebras satisfy general tovariance, we can state our second principle: the equivalence principle

The equivalence principle

- Einstein's equivalence principle: free fall in a gravitational field is locally indistinguishable from rest or uniform motion in Minkowski space-time without gravitational forces
- Bohr's equivalence principle (doctrine of classical concepts): quantum theory is empirically accessible through classical physics
- New equivalence principle: any C^*-algebra of observables is empirically equivalent to a commutative one

Observer locally avoids gravitational force by moving along a geodesic

(Einstein saw this as a special choice of coordinates)

Observer in quantum theory avoids noncommutativity by special choice of topos in which C^*-algebra "becomes commutative"

What follows is just a first step in this direction (Heunen&Spitters)

Abelianization of a C^*-algebra

1. **Start from a C^*-algebra $\mathfrak{A}$, e.g. in topos *Sets***
2. **Form set $\mathsf{C}(\mathfrak{A})$ of all commutative C^*-subalgebras of $\mathfrak{A}$**
3. **Partially order $\mathsf{C}(\mathfrak{A})$ by inclusion and see it as a category**
4. **Define topos $T(\mathfrak{A}) = \mathbf{Sets}^{\mathsf{C}(\mathfrak{A})}$ of functors $F : \mathsf{C}(\mathfrak{A}) \to \mathbf{Sets}$**
5. **Take "tautological functor" $A(C) = C$, $A(C \to D) = C \hookrightarrow D$**

A is a commutative C^*-algebra in the topos $T(\mathfrak{A})$

Interpretation (principle of complementarity):

A is what a (classical) observer can extract from $\mathfrak{A}$

Mutually exclusive "classical snapshots" $C \subset \mathfrak{A}$ form picture of $\mathfrak{A}$

Note: cannot reconstruct $\mathfrak{A}$ from A

Gelfand theory in a topos (Mulvey)

A commutative C^*-algebra in ***Sets***: $A \cong C(X, \mathbb{C})$, $X \cong P(A)$

Arbitrary topos: locales instead of topological spaces

Locale: sup-complete distributive lattice s.t. $x \wedge \vee_\lambda y_\lambda = \vee_\lambda x \wedge y_\lambda$

Example: $\mathcal{O}(X)$ **with** $U \leq V$ **if** $U \subseteq V$ **is a locale (cf. topos** $\mathrm{Sh}(X)$**)**

A **commutative** C^***-algebra in topos** T**: there is a locale** $\Sigma \equiv \Sigma(A)$ **in** T **s.t.** $A \cong C(\Sigma, \mathcal{O}(\mathbb{C}))$ **("internal complex numers** $\mathbb{C}$**")**

Locale Σ **is defined as "geometric theory" built from propositions** "$a \in U$" ($a \in A$, $U \in \mathcal{O}(\mathbb{C})$) **subject to axioms motivated by** ***Sets***: $\Sigma \cong \mathcal{O}(P(A))$ **and** "$a \in U$" $= \{\varphi \in P(A) \mid \varphi(a) \in U\}$

Elements of spectrum Σ are (equivalence classes of) propositions, of precisely the type you want in quantum mechanics!

Application to topos $T(\mathfrak{A}) = \mathbf{Sets}^{\mathsf{C}(\mathfrak{A})}$ **and** C^***-algebra** $A(C) = C$**:**

Functor $\Sigma : \mathsf{C}(\mathfrak{A}) \to \mathbf{Sets}$ **is given by** $\Sigma(C) = \mathcal{O}(P(C))$

States and propositions

Standard physics (in *Sets*) is based on pairing

$\langle\psi, P\rangle \mapsto [0,1]$, ψ **state**, P **proposition of type** $P =$ "$a \in U$" **(**$U \subset \mathbb{R}$**)**

$\langle\psi, P\rangle$ **gives probability that proposition** P **is true in state** ψ

- **Classical mechanics:**

Observable is function $a : M \to \mathbb{R}$, **pure state is point** $\psi \in M$

Proposition $P =$ "$a \in U$" **is true if** $a(\psi) \in U$ **and false if** $a(\psi) \notin U$

$\Rightarrow \langle\psi, P\rangle = 1$ **if** $\psi \in a^{-1}(U)$ **and** $\langle\psi, P\rangle = 0$ **if** $\psi \notin a^{-1}(U)$

- **Quantum mechanics:**

Observable a **is selfadjoint operator, pure state** ψ **is unit vector**

Proposition $P =$ "$a \in U$" **is spectral projection** $E_a(U)$ **of** a **on** U

$\Rightarrow \langle\psi, P\rangle = \|E_a(U)\Psi\|^2$ **(Born rule)**

Truth

In classical physics propositions have a naive truth value
$\{0,1\} = \{\perp, \top\} = \{\mathrm{false}, \mathrm{true}\}$ **in any pure state**

Recall that $\{\perp, \top\} = \Omega$ **is the subobject classifier in the topos *Sets***

Recall that in any topos Ω **carries an intrinsic logical structure**

- **Goal: The pairing** $\langle \psi, P \rangle$ **in quantum physics should not** *a priori* **be probabilistic but should take values in the subobject classifier** Ω **in an appropriate topos; then** *derive* **Born rule**
- **Slogan: Truth is prior to probability**

Program:

1. **Reformulate pairing in classical physics in topos terms**
2. **Adapt this to quantum physics**

Classical pairing as arrows

In *Sets* have phase space M and subobject classifier $\Omega = \{0,1\}$

- **Pure state is arrow $1 \xrightarrow{\psi} M$ (i.e. point of M)**
- **Proposition $P =$ "$a \in U$" is subobject $a^{-1}(U) \overset{P}{\hookrightarrow} M$ with classifying arrow $M \xrightarrow{\chi_P} \Omega$ $(\chi_P \equiv \chi_{a^{-1}(U)})$**
- **Pairing $\langle \psi, P \rangle = \chi_P \circ \psi$ yields point $1 \xrightarrow{\langle \psi, P \rangle} \Omega$ and indeed:**

$\langle \psi, P \rangle = 1 = \top$ **if** $\psi \in a^{-1}(U)$ **and** $\langle \psi, P \rangle = 0 = \perp$ **if** $\psi \notin a^{-1}(U)$

"Localic" reformulation: now replace space M by locale $\mathcal{O}(M)$

- **Pure state is subobject $\Psi \equiv \{U \in \mathcal{O}(M) \mid \delta_\psi(U) = 1\} \overset{\psi}{\hookrightarrow} \mathcal{O}(M)$ with classifying arrow $\mathcal{O}(M) \xrightarrow{\chi_\psi} \Omega$**
- **Proposition $P =$ "$a \in U$" is open $1 \overset{a^{-1}(U)}{\hookrightarrow} \mathcal{O}(M) \equiv 1 \xrightarrow{P} \mathcal{O}(M)$**
- **Pairing $\langle \psi, P \rangle = \chi_\psi \circ P$ yields same point $1 \xrightarrow{\langle \psi, P \rangle} \Omega$**

Quantum pairing as arrows

Now work in topos $T(\mathfrak{A}) = \mathbf{Sets}^{\mathsf{C}(\mathfrak{A})}$ of functors $F : \mathsf{C}(\mathfrak{A}) \to \mathbf{Sets}$

($\mathsf{C}(\mathfrak{A})$ is set of all commutative C^*-subalgebras of C^*-algebra $\mathfrak{A}$)

Commutative C^*-algebra A defined by $A(C) = C$ has spectrum $\Sigma : \mathsf{C}(\mathfrak{A}) \to \mathbf{Sets}$, given by $\Sigma(C) = \mathcal{O}(P(C))$

- **Pure state is subobject $\Psi \overset{\psi}{\hookrightarrow} \Sigma$ with $\Psi : \mathsf{C}(\mathfrak{A}) \to \mathbf{Sets}$ given by**

$$\Psi(C) = \{U \in \mathcal{O}(P(C)) \mid \mu_\psi^C(U) = 1\}$$

(μ_ψ^C is probability measure on $P(C)$ induced by state ψ on $\mathfrak{A}$)

- **Proposition $P =$ "$a \in U$" should be open $1 \overset{a^{-1}(U)}{\hookrightarrow} \Sigma \equiv 1 \xrightarrow{P} \Sigma$**

- **Pairing $\langle \psi, P \rangle = \chi_\psi \circ P$ should yield point $1 \xrightarrow{\langle \psi, P\rangle} \Omega$**

Problem: observable $a \in \mathfrak{A}$ does not naturally define element of A (although state on $\mathfrak{A}$ *does* define state on A). Various attempts...

Summary

- **Topos theory provides a new famework for all of physics**
- **It readdresses the logical structure of physical theories**
- **It implies intuitionistic logic (no middle third, no Choice)**
- **Principle of general tovariance even implies "geometric logic"**
- **It contains a new multi-valued notion of truth**
- **It carries the hope of deriving the probabilisitic structure of quantum mechanics from its logical structure (von Neumann)**
- **It softens the noncommutativity of C^*-algebras, which in a suitable topos "become commutative"**
- **Eases tension between quantum theory & general relativity?**
- **Guiding principles are needed (we have proposed a few)**