Flat deformations of a semi-Riemannian metric admitting a symmetry group

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Flat deformation theorem (2005)

Given a semi-Riemannian analytic metric, γ, on a manifold $\mathcal{M}$, it exists a 2-form F and a scalar function c such that:

1. *an arbitrary scalar constraint $\Psi(c, F, x) = 0$, $x \in \mathcal{M}$ is fulfilled and*
2. *the deformed metric* $\quad \eta := c\gamma + \epsilon M$, *is semi-Riemannian and flat*

$$\text{where} \quad |\epsilon| = 1\,, \quad M(V, W) = \gamma(F_V, F_W)\,, \quad F_V := i_V F.$$

We shall limit ourselves to the case of a 4-dimensional spacetime $(\mathcal{M}, g)$.

Some algebraic remarks Given $F \in \Lambda^2\mathcal{M}$, it exists a null tetrad $\{\mu,\ \nu,\ \kappa\,,\lambda\}$ such that either:

Singular case: $\underline{F = \kappa \wedge \mu}\,, \quad M = -\kappa \otimes \kappa$ $\quad \boxed{\gamma = \frac{1}{c}\,\eta - \frac{\epsilon}{c}\,\kappa \otimes \kappa}$

with κ isotropic and η flat (conformal Kerr-Schild metric) or

Non-singular case: $\underline{F = -B\,\mu \wedge \nu + E\,\kappa \wedge \lambda}\,,$

$$M = -B^2\,(\mu \otimes \mu + \nu \otimes \nu) - E^2\,(\kappa \otimes \lambda + \lambda \otimes \kappa)$$

$$\eta = \left(c - \epsilon B^2\right)\,\gamma - \epsilon\left(B^2 + E^2\right)\,(\kappa \otimes \lambda + \lambda \otimes \kappa)$$

$\boxed{\eta = a\gamma + bH}$

Prop. 1 *Given a Lorentzian analytic metric γ, there exist two scalar functions, a and b, and a hyperbolic 2-plane H such that the metric*

$$\underline{\eta := a\gamma + bH} \qquad \textit{is Lorentzian and flat.}$$

$\eta := a\gamma - b\,(\kappa \otimes \lambda + \lambda \otimes \kappa)$ (Recalls a conformal Kerr-Schild transformation)

Let k and l be two vectors such that $\kappa = \gamma(k, -)$ and $\lambda = \gamma(l, -)$

The endomorphism $\mathbb{H} = -k \otimes \lambda - l \otimes \kappa$ associated to H is a 2-dimensional projector on a hyperbolic 2-plane $\mathcal{H}_x \subset T_x\mathcal{M}$

$$\mathbb{H}^2 = \mathbb{H} \qquad \text{and} \qquad \mathrm{trace}\, \mathbb{H} = 2$$

$K := \gamma - H$ (complementary 2-plane)

$$\gamma = H + K \qquad \text{and} \qquad \eta = a\, K + (a + b)\, H$$

The *almost-product structure* defined by H is compatible with both, γ and η.

Assume that γ admits a Killing vector X, $\quad \mathcal{L}_X \gamma = 0.$

Does it exist a flat deformation law $\quad \eta := a\gamma + bH$ such that $\quad \mathcal{L}_X \eta = 0$?

(i.e., such that X is also a Killing vector for η)

This is equivalent to $\mathcal{L}_X a = \mathcal{L}_X b = 0\,, \qquad \mathcal{L}_X H = 0$

1-parameter symmetry group

$(\mathcal{M}, \eta)$ is a semi-Riemannian 4-manifold and G is a connected 1-parameter Lie group acting smoothly on $\mathcal{M}$

$$\psi : G \times \mathcal{M} \longrightarrow \mathcal{M}\,, \qquad (g, x) \longrightarrow gx\,.$$

leaving η invariant: $\underline{g^*\eta_{gx} = \eta_x\,.}$

$\mathcal{S} = \{Gx,\ x \in \mathcal{M}\}$ is the class of all orbits and $\pi : x \in \mathcal{M} \to Gx \in \mathcal{S}$ is the canonical projection.

We assume that $\mathcal{S}$ is a manifold, π is smooth and $\pi_* : T\mathcal{M} \to T\mathcal{S}$ jacobian map.

$\forall x \in \mathcal{M}$, $\quad \psi_x : G \to \mathcal{M}$ $\quad$ is smooth and $\psi_{x*} : TG \to T\mathcal{M}$

$\qquad\qquad g \to gx \qquad\qquad\qquad\qquad T_eG \to \mathcal{G}_x \subset T_x\mathcal{M}$

If X is an infinitesimal generator of the action of G, $\mathcal{G} = \text{span}\,[X]$ and

$$g_* X_x = X_{gx}, \quad \forall g \in G \quad \text{and} \quad x \in \mathcal{M}$$

[2] Projectable vectors and tensors

$\boxed{\xi := \eta(X, \;\;) \in \Lambda^1\mathcal{M}}$ Assume $l := \eta(X, X) = \langle \xi, X \rangle \neq 0$

$g^*\xi_{gx} = \xi_x \,, \forall g \in G \,,$ or, locally $\mathcal{L}_X \xi = 0$

Any $V \in T_x\mathcal{M}$ can be separated in two components that are, respectively, transverse to ξ and parallel to X:

$$T\mathcal{M} = \xi^\perp \oplus \mathrm{span}\,[X] \,, \qquad V = V^\perp + \frac{\langle \xi, V \rangle}{l}\, X$$

Killing equation

$\mathcal{L}_X \eta = 0 \quad \Leftrightarrow \quad \nabla\xi$ skewsymmetric $\quad \boxed{\nabla\xi = \frac{1}{2}\, d\xi}$

$\mathcal{L}_X \xi = 0 \quad \Rightarrow \quad i_X\,(d\xi) = -d\,(i_X\xi) = -dl$

$\boxed{d\xi = df \wedge \xi + \Theta}$ where $f := \log|l|$ and $i_X\Theta = 0$

Def. 1: $Y \in \mathcal{X}(\mathcal{M})$ *is* projectable *if*

$$x, y \in \mathcal{M}, \quad \pi x = \pi y \;\Rightarrow\; \pi_* Y_x = \pi_* Y_y$$

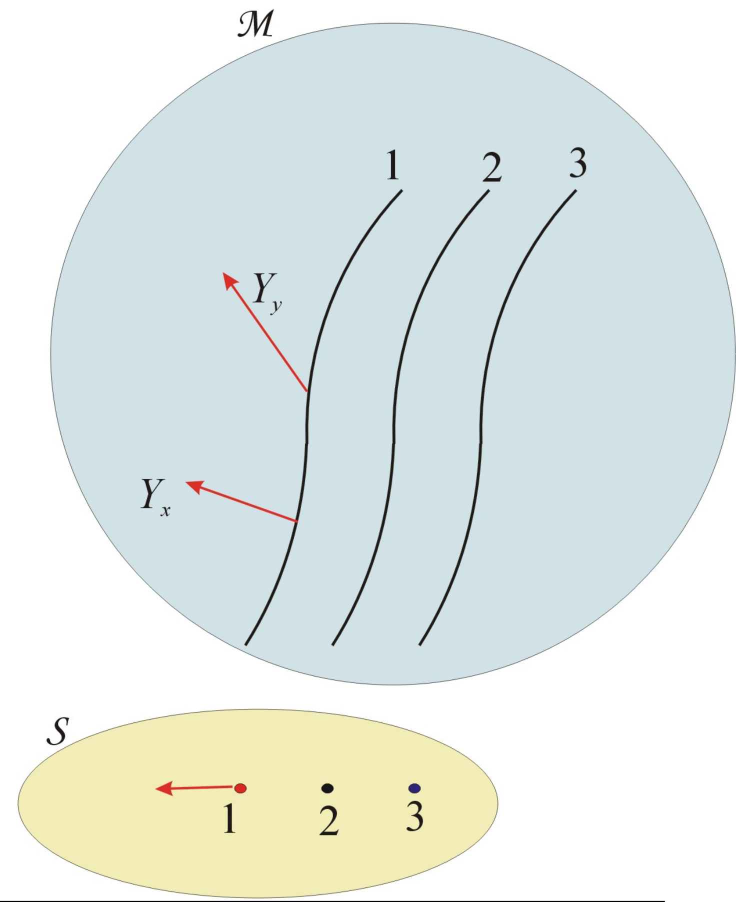
Prop. 2: $Y \in \mathcal{X}(\mathcal{M})$ *is projectable if, and only if,* $\pi_* \left(\mathcal{L}_X Y\right) = 0$

For transverse vector fields

$$\pi_* \left(\mathcal{L}_X Y\right) = 0 \qquad \Leftrightarrow \qquad \mathcal{L}_X Y = 0$$

Class of projectable transverse vector fields

$$\mathcal{X}_\pi(\mathcal{M}) = \{Y \in \mathcal{X}(\mathcal{M}) |\, \mathcal{L}_X Y = 0,\, \langle \xi, Y \rangle = 0\}$$



Prop. 3: *Let* $\vec{w} \in \mathcal{X}(\mathcal{S})$, *then:*

(a) *it exists a unique vector field* $W \in \mathcal{X}_\pi(\mathcal{M})$ *such that* $\pi_* W = \vec{w}$,

(b) $\pi_* : \mathcal{X}_\pi(\mathcal{M}) :\longrightarrow \mathcal{X}(\mathcal{S})$ *is bijective and we write* $W = \pi_*^{-1} \vec{w}$.

$\pi^* : T^*_{\pi x}\mathcal{S} \longrightarrow T^*_x\mathcal{M}$ is the pull-back map $\quad \langle \pi^*\lambda, Z\rangle = \langle \lambda, \pi_* Z\rangle$.

$\pi^*\lambda$ is transverse: $\quad \langle \pi^*\lambda, X\rangle = 0 \qquad$ and $\qquad \mathcal{L}_X\,(\pi^*\lambda) = 0$

$\alpha \in T^*\mathcal{M}$ can be separated in two components, transverse to X and parallel to ξ

$$\alpha = \alpha^\perp + \frac{\langle \alpha, X\rangle}{l}\,\xi\,, \qquad\qquad T^*\mathcal{M} = X^\perp \oplus \operatorname{span}[\xi]$$

$$\Lambda^1_\pi\mathcal{M} := \{\alpha \in \Lambda^1\mathcal{M} \,|\, \mathcal{L}_X\alpha = 0,\; \langle \alpha, X\rangle = 0\}\,.$$

Prop. 4: $\quad \pi^*(\Lambda^1\mathcal{S}) = \Lambda^1_\pi\mathcal{M}$ *and* $\pi^* : \Lambda^1\mathcal{S} \longrightarrow \Lambda^1_\pi\mathcal{M}$ *is bijective.*

Transverse G-preserved covariant tensors:

$$\mathcal{T}_{n\,\pi}\mathcal{M} := \{T \in \mathcal{T}_n\mathcal{M} \,|\, T(X,\ \ ) = T(\ ,X,\ \ ) = \ldots = 0,\; \mathcal{L}_X T = 0\}$$

$\boxed{h := \eta - \frac{1}{l}\,\xi \otimes \xi}$ is symmetric, transverse $\quad \mathrm{Rad}\, h = \mathrm{Ker}\, \pi_* = \mathrm{span}\,[X]$
and preserved by the group action, $\qquad g^* h_{gx} = h_x.$

(Because the action of G preserves both η, ξ and l.)

It can be easily proved that it exists $\underline{h} \in \mathcal{T}_2\mathcal{S}$, non-degenerate, such that $\boxed{\pi^*\underline{h} = h}$

Riemannian connection for $\underline{h}$

Let $\vec{v}$, $\vec{w} \in \mathcal{X}(\mathcal{S})$ and $V = \pi_*^{-1}\vec{v}$, $W = \pi_*^{-1}\vec{w} \in \mathcal{X}_\pi(\mathcal{M})$.

Although $\nabla_V W \notin \mathcal{X}_\pi(\mathcal{M})$, we have that $\mathcal{L}_X(\nabla_V W) = 0$ because,
$\mathcal{L}_X V = \mathcal{L}_X W = 0$ and, as the G-action preserves η, then $\mathcal{L}_X \nabla = 0$

Define: $\qquad \boxed{D_{\vec{v}}\vec{w} := \pi_*\left(\nabla_V W\right)}$

It is a connection on $\mathcal{S}$, which is symmetric and $D_{\vec{v}}\,\underline{h} = 0$, hence Riemannian.

Some useful relations:

$$\nabla_V W = \pi_*^{-1}\,(D_{\vec{v}}\vec{w}) + \frac{1}{2l}\,d\xi(V, W)\,X$$

$$\underline{h}(\vec{y}, \pi_* \nabla_V X) = \tfrac{1}{2}\,d\xi(V, Y)\,, \qquad \langle \xi, \nabla_V X \rangle = frac12\,\langle dl, V \rangle$$

Riemannian-Christoffel tensors ♠

$K(Y,Z;V,W) = \eta\left(Y, \nabla_V \nabla_W Z - \nabla_W \nabla_V Z - \nabla_{[V,W]} Z\right)$

$\underline{K}(\vec{y},\vec{z};\vec{v},\vec{w}) = \underline{h}\left(\vec{y}, D_{\vec{v}} D_{\vec{w}} \vec{z} - D_{\vec{w}} D_{\vec{v}} \vec{z} - D_{[\vec{v},\vec{w}]} \vec{z}\right)$

(a) For transverse Y, Z, V, W, $\vec{y} = \pi_* Y$ and so on.

$$\begin{aligned} K(Y,Z;V,W) \;=\; & \underline{K}(\vec{y},\vec{z};\vec{v},\vec{w}) + \frac{1}{4l}\,(\Theta(V,Z)\,\Theta(W,Y) - \\ & \Theta(V,Y)\,\Theta(W,Z) - 2\,\Theta(V,W)\,\Theta(Y,Z)) \end{aligned}$$

(b) $\quad K(X,Z;V,W) = -\frac{1}{2}\,\nabla_V d\xi(W,Z) + \frac{1}{2}\,\nabla_W d\xi(V,Z) = \frac{1}{2}\,\nabla d\xi(Z,V,W)$

(c) $\quad K(X,V;X,W) = -\frac{1}{2}\,\nabla dl(W,V) + \frac{1}{4}\,l\,(Vf)\,(Wf) + \frac{1}{4}\,h(\Theta_V,\Theta_W)$

where $\Theta_V := i_V \Theta$.

$$\mathrm{Ric}\,(Z,W) = \underline{\mathrm{Ric}}\,(\vec{z},\vec{w}) - \frac{1}{2l}\,h(\Theta_Z,\Theta_W) - \frac{1}{2}\,\nabla df(Z,W) - \frac{1}{4}\,(Zf)\,(Wf)$$

Let G be a 1-parameter Lie group acting on $\mathcal{M}$ and let the quotient $\mathcal{S} = \mathcal{M}/G$ be a manifold with a semi-Riemannian metric $\underline{h}$.

Is there a non-degenerate metric η on $\mathcal{M}$ such that $\mathcal{L}_X \eta = 0$ and having $\underline{h}$ as the quotient metric?

It depends on the choice of $\xi \in \Lambda^1 \mathcal{M}$ such that $\langle \xi, X \rangle \neq 0$, with constant sign on $\mathcal{M}$, and $\mathcal{L}_X \xi = 0$. Then we take

$$\eta = \pi^* \underline{h} + \frac{1}{\langle \xi, X \rangle} \, \xi \otimes \xi$$

How to choose ξ?

Since $\mathcal{L}_X \xi = 0$, $\quad d\xi = df \wedge \xi + \Theta$, with $\; f := \log |\langle \xi, X \rangle| \;$ and $\; i_X \Theta = 0$

$$d^2 = 0 \qquad \Rightarrow \qquad \xi = e^f \; (du + \beta)$$

$$f \in \pi^* \Lambda^0 \mathcal{S} \, , \quad \beta \in \pi^* \Lambda^1 \mathcal{S} \, , \qquad u \in \Lambda^0 \mathcal{M} \, , \quad |\langle du, X \rangle| = 1$$

Equations above merely give the values of Riemann-Christoffel tensor K on $\mathcal{M}$. ♠

However, if the output K is prescribed, then they become conditions on $\underline{h}$ and ξ (alternatively, on $\underline{h}$, β and f).

These equations are solved in $\mathcal{S}$. Then the solutions are pulled-back to $\mathcal{M}$.

The flat deformation law as a PDS

γ is G-invariant, <u>find $\eta = a\,\gamma + b\,H$, flat and G-invariant</u>

Find a , b and H such that $K(\eta) = 0$ on $\mathcal{U} \subset \mathcal{S}$ and then π^* pulls them back to $\pi^{-1}\mathcal{U} \subset \mathcal{M}$

X is a Killing vector for both γ and η:

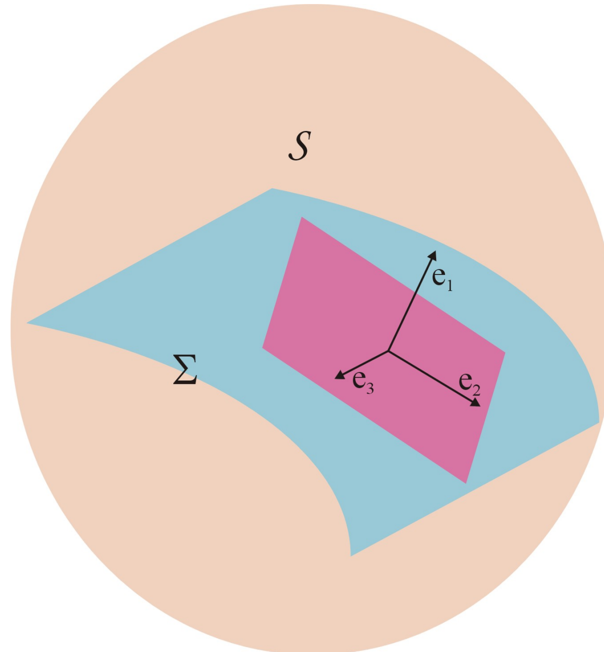
$$\gamma = \pi^* p + \frac{1}{\bar{l}}\,\bar{\xi} \otimes \bar{\xi}\,, \qquad \bar{\xi} := i_X\gamma\,, \qquad \bar{l} := \langle\bar{\xi}, X\rangle = \gamma(X, X)\,, \qquad p \in \mathcal{T}_2\mathcal{S}$$

$$\eta = \pi^* \underline{h} + \frac{1}{l}\,\xi \otimes \xi\,, \qquad \xi := i_X\eta\,, \qquad l := \langle\xi, X\rangle\,, \qquad \underline{h} \in \mathcal{T}_2\mathcal{S}$$

$K_{\alpha\beta\mu\nu} = 0$ are 20 independent equations for 6 unknowns. (Overdetermined.)

$$K_{\alpha\beta\mu\nu} \equiv L_{\alpha\beta\mu\nu} + \frac{2}{l}\,\left(L_{\alpha\beta[\mu}\xi_{\nu]} + L_{\mu\nu[\alpha}\xi_{\beta]}\right) + \frac{4}{l^2}\,\xi_{[\beta}L_{\alpha][\mu}\xi_{\nu]}$$

$e_0 = X$ and e_1, e_2, e_3 natural base for Gaussian normal coordinates (x^1, x^2, x^3), $x^1 = 0$ on Σ.



$$L_{abcd} := \underline{K}_{abcd} - \frac{1}{2l}\left(\Theta_{ab}\Theta_{cd} + \Theta_{[ac}\Theta_{b]d}\right)$$

$$L_{bcd} := \frac{1}{2} D_b\Theta_{cd} + \frac{1}{2}\Theta_{b[d}f_{c]}$$

$$L_{bd} := -\frac{l}{2}\left(D_b f_d + \frac{1}{2} f_b f_d\right) - \frac{1}{4}\Theta_b{}^a\Theta_{ad}$$

$$L_{abcd} = 0\,, \qquad L_{bcd} = 0\,, \qquad L_{bd} = 0$$

Reduced PDS: $L_{11} = 0\,,\qquad L_{11j} = 0\,,\qquad \rho_{ij} := L^a{}_{iaj} - \frac{1}{2} L^{ac}{}_{ac}h_{ij} = 0$

Constraints (on Σ that extend to a neigbourhood by the 2nd Bianchi identity):

$$L_{aj} = 0\,, \qquad \rho_{ij} := L^a{}_{1ab} - \frac{1}{2} L^{ac}{}_{ac}h_{1b} = 0\,, \qquad \epsilon^{cda}L_{jcd} = 0$$

6 unknowns: a, b, H $\qquad\qquad \gamma = \pi^* p + \frac{1}{\bar{l}}\,\bar{\xi} \otimes \bar{\xi}$

It exists a triad $\{\omega, \tau, \zeta\}$ such that $\quad p = -s\omega \otimes \omega + \tau \otimes \tau + \zeta \otimes \zeta$ and

$$H = s\,(\beta \otimes \beta - \omega \otimes \omega)\;, \qquad \beta = \frac{m}{\bar{l}}\,\bar{\xi} - s''\sqrt{\frac{\bar{l} - sm^2}{|\bar{l}|}}\,\tau$$

$m = +\sqrt{H(X,X)}\,, \quad s = \operatorname{sign} H(X,X)$ and $s'' = \operatorname{sign} \bar{l}$.

<u>New unknowns:</u> $\quad a$, b, m and the p-orthonormal triad $\{\omega, \tau, \zeta\}$

$$\partial_1^2 \omega \cong \Omega_1 \tau + \Omega_2 \zeta\,, \qquad \partial_1^2 \tau \cong s\Omega_1 \omega + \Omega_3 \zeta\,, \qquad \partial_1^2 \zeta \cong s\Omega_2 \omega - \Omega_3 \tau$$

$$n := \frac{ss''\,b\,(\bar{l} - sm^2)a}{\bar{l}a + sbm^2}\,, \qquad y := ss''bm\sqrt{\frac{\bar{l} - sm^2}{|\bar{l}|}}$$

$$
\begin{aligned}
L_{11} = 0 \;&\longrightarrow\; l\,\partial_1^2 f \cong 0 \\
L_{11i} = 0 \;&\longrightarrow\; y\,\left(\partial_1^2\tau_i - \tau_i\partial_1^2 f + \tau_i\partial_1^2 \log y\right) \cong 0 \\
\rho_{ij} = 0 \;&\longrightarrow\; -\frac{1}{2}\left(\overline{h}^{11}\delta_i^k\delta_j^l + \overline{h}^{1k}\overline{h}^{1l}\underline{h}_{ij} - \overline{h}^{11}\overline{h}^{kl}\underline{h}_{ij}\right)\,\partial_1^2\underline{h}_{kl} \cong 0 \\
\text{with} \qquad & \partial_1^2\underline{h}_{ij} \cong \partial_1^2(a+b)\,\omega_i\omega_j + \partial_1^2(a+n)\,\tau_i\tau_j + \partial_1^2 a\,\zeta_i\zeta_j \\
& +2s(n-b)\Omega_1\,\omega_{(i}\tau_{j)} - 2sb\Omega_2\,\omega_{(i}\zeta_{j)} + 2n\Omega_3\,\zeta_{(i}\tau_{j)})
\end{aligned}
$$

To be solved for $\partial_1^2 a$, $\partial_1^2 b$, $\partial_1^2 m$ and Ω_c, $c = 1, 2, 3$.

Characteristic determinant:

$$
\begin{aligned}
\chi \;=\; & \frac{4ss''}{|\bar{l}|}\,b^4\,m\tau_1\zeta_1^2\,\left(-s'\bar{l} + \bar{l}\zeta_1^2 + sm^2\tau_1^2\right)\,\left(\bar{l}a + sbm^2\right) \\
& \left([\bar{l} - sm^2]\zeta_1^2 + s''m^2[\tau_1^2 + \zeta_1^2]\right)\,\left(\overline{h}^{11}\right)^3
\end{aligned}
$$

Geometrical meaning of the constraints

Let η be a solution of the reduced PDS.

$$\left.\begin{array}{lll} K_{ijkl} = -2\rho^1_1\, h_{j[l}\, h_{ik]} & K_{1jkl} = 2h_{j[l}\, \rho_{1k]} - 2\rho^1_1\, h_{j[l}\, h_{1k]} & K_{0ijk} = -2\mu^1{}_i\, \epsilon_{1jk} \\ K_{01jk} = 2\mu^l{}_l\, \epsilon_{1jk} & K_{0j1k} = -\mu^l{}_j\, \epsilon_{l1k} & K_{0ajk} = L_{aj} \end{array}\right\}$$

The fulfilling of the constraints is equivalent to (for $\alpha, \beta, \mu, \nu \neq 1$)

$$K_{\alpha\beta\mu\nu} = 0 \qquad \text{and} \qquad K_{1\beta\mu\nu} = 0$$

$\mathcal{N} := \pi^{-1}\Sigma$ is a hypersurface of $\mathcal{M}$. $\qquad J : \mathcal{N} \to \mathcal{M}$

$\bar{\nu} = \pi^* dx^1 \in \Lambda^1\mathcal{M}$ (recall Gaussian γ-normal coordinates) is orthogonal to $\mathcal{N}$.

$(\mathcal{M}, \eta)$ and $(\mathcal{M}, \gamma)$ are two Riemannian structures. Let n and $\bar{n}$ be the respective unit vectors normal to $\mathcal{N}$, and ν and $\bar{\nu}$ the corresponding covectors, $\quad \bar{\nu} \propto \nu$

$K_{\alpha\beta\mu\nu} = 0$ and $K_{1\beta\mu\nu} = 0$ whenever $\alpha,\ \beta,\ \mu,\ \nu \neq 1$ is equivalent to

$$\underline{J^* K = 0} \quad \text{and} \quad \underline{J^*\left(i_n K\right) = 0}$$

$J^* K$ is connected to $K(\vartheta)$ and Φ (Gauss) and $J^*\left(i_n K\right)$ is connected to $\nabla\Phi$ (Codazzi-Mainardi).

A particular solution is $\Phi = 0$ and $K^{(\vartheta)} = 0$

$\mathcal{N}$ has two couples of fundamental forms: $(\vartheta = J^*\eta,\ \Phi)$ and $(\varphi = J^*\gamma,\ \phi)$

$$\Phi(v,w) = \langle \nu, \nabla_v w\rangle \quad \text{and} \quad \phi(v,w) = \langle \bar{\nu}, \bar{\nabla}_v w\rangle\,, \qquad v, w \in T\mathcal{N}$$

If $\underline{\Phi = 0}$, then $\phi(v,w) = -\langle \bar{\nu}, B(v,w)\rangle\,, \qquad B := \nabla - \bar{\nabla}$

which results in a condition on the normal derivatives of the unknowns:

$$\bar{\nabla}_1 \eta_{\mu\nu} = \phi_{\mu\nu} + \bar{\nabla}_{(\mu}\eta_{\nu)1}\,, \qquad \mu, \nu \neq 1$$