Geometrical Structures of Space-Time in General Relativity (Poster)*

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Abstract

Space-Time in general relativity is a dynamical entity because it is subject to the Einstein field equations.

From the point of view of differential geometry, the space-time is a manifold with a Lorentzian metric. The space-time metric provides different geometrical structures: conformal, volume, projective ...

A deep understanding of the geometrical structures has consequences on the dynamical role played by geometry. We explain these geometrical structures, establishing relationships among them and clarifying the meaning of associated geometric magnitudes.

Recently, some of my research [1,2] have been taken into consideration for one of the lines of thought about quantum gravity [3,4]. This poster is a set of my latest reflections and conclusions about the applications of my work to physics.

References

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1 Introduction

The space-time of general relativity (GR) is a 4-dimensional manifold M, with a C^∞ atlas $\mathcal{A}$. The atlas is the *differential structure* of our space-time.

The *equivalence principle* of GR establishes the invariance by diffeomorphisms. This let us to think that a *physical event* is not a point, but a geometrical structure on a neighborhood.

The *fundamental geometrical structures* that we consider defined in the space-time are:

- Volume (4-form)
- Conformal structure (Lorentzian)
- Metric (Lorentzian)
- Linear connection (symmetric)
- Projective structure

They are defined in terms of the *most primitive* differential structure, via the concept of G-structure. Volume, conformal and metric are *first order G-structures*. But linear connection and projective are *second order G-structures*.

For certain G's, *classified in* [8], every first order G-structure lead to a unique second order structure, named its *prolongation*. This is the case for the volume, metric and conformal structures.

2 Frame bundles

The *r-th order frame bundle* $\mathcal{F}^r M$ is a quotient space in a subset of $\mathcal{A}$ ([12, p.38]). An *r-frame*, $j^r\varphi \in \mathcal{F}^r M$ is an r-jet at 0, where $x = \varphi^{-1}$ is a chart with 0 as a target.

The first order frame bundle $\mathcal{F}^1 M$ is usually identified with the *linear frame bundle* LM.

Let LLM be the linear bundle of LM. There is a *canonical inclusion* $\mathcal{F}^2 M \hookrightarrow LLM$, $j^2\varphi \mapsto j^1\tilde{\varphi}$, where $\tilde{\varphi}$ is the diffeomorphism induced by φ, between neighborhoods of $0 \in \mathbf{R}^{n+n^2}$ and $j^1\varphi \in LM$ ([10], [12, p.50])).

Let J^1LM be the bundle of 1-*jets of (local) sections* of LM and s be a section of LM. Each $j^1_p s$ is characterized by the *transversal n-subspace* $H_l = s_*(T_pM) \subset T_l LM$.

Then, there is also a *canonical inclusion* $J^1LM \hookrightarrow LLM$, $j^1_p s \mapsto z$, where z is the basis of $T_l LM$, whose first n vectors span H_l and correspond to the usual basis of $\mathbf{R}^n$, via the *canonical form of LM* (see [6]).

By the previous canonical maps, it happens that $\mathcal{F}^2M$ is mapped one to one into the subset of J^1LM, corresponding with the *torsion-free transversal n-subspaces* in TLM.

Theorem 1. *We have the canonical embeddings:*

$$\mathcal{F}^2M \hookrightarrow J^1LM \hookrightarrow LLM$$

3 Structural groups

Each $\mathcal{F}^rM$ is a principal bundle respect to the group G^r_n of *r-jets at 0 of diffeomorphisms of* $\mathbf{R}^n$, $j^r_0\phi$, with $\phi(0) = 0$.

The group G^1_n is identified with $\mathrm{GL}(n, \mathbf{R})$. Then, there is a canonical inclusion of G^1_n into G^r_n, if we take the r-jet at 0 of every linear map of $\mathbf{R}^n$. Furthermore, G^r_n is the *semidirect product* of G^1_n with a nilpotent normal subgroup (see [13] for details).

Let's see *the structure for* G^2_n. We consider the underlying additive group of the vector space S^2_n of symmetric bilinear maps of $\mathbf{R}^n \times \mathbf{R}^n$ into $\mathbf{R}^n$. There is a monomorphism $\imath\colon \mathrm{S}^2_n \to \mathrm{G}^2_n$ defined by $\imath(s) = j^2_0\phi$ with $s = (s^i_{jk})$ and $\phi(u^i) := (u^i + \frac{1}{2}s^i_{jk}u^j u^k)$.

Theorem 2. *We obtain the split exact sequence of groups:*

$$0 \to \mathrm{S}^2_n \xrightarrow{\imath} \mathrm{G}^2_n \rightleftarrows \mathrm{G}^1_n \to 1$$

It makes G^2_n *isomorphic to the semidirect product* $\mathrm{G}^1_n \rtimes \mathrm{S}^2_n$, whose multiplication rule is $(a, s)(b, t) := (ab, b^{-1}s(b, b) + t)$. The isomorphism is given by $j^2_0\phi \mapsto (D\phi|_0\,,\, D\phi|_0^{-1}D^2\phi|_0)$.

For a linear group G, let $\mathfrak{g}$ denote the Lie algebra of G. The *first prolongation of* $\mathfrak{g}$ is defined by $\mathfrak{g}_1 := \mathrm{S}^2_n \cap L(\mathbf{R}^n, \mathfrak{g})$. We obtain that $G \rtimes \mathfrak{g}_1$ *is a subgroup* of $\mathrm{G}^1_n \rtimes \mathrm{S}^2_n$, and hence, a subgroup of G^2_n (see more details in [1]).

4 G-structures

We define an *r-th order G-structure* on M as a *reduction* of $\mathcal{F}^rM$ to a subgroup $G \subset \mathrm{G}^r_n$ ([10]). The idea of *geometrical structure* on M concerns the classification of charts in $\mathcal{A}$, *when choosing the meaningful classes* guided by an structural group.

We exemplify the concept of a G-structure studying a volume on a manifold, which rarely is treated this way (see [3]).

Let's define *volume on M* as a first order G-structure V, with $G = \mathrm{SL}_n^{\pm} := \{a \in \mathrm{GL}(n, \mathbf{R}) \colon |det(a)| = 1\}$. For an orientable M, V has two components for two $\mathrm{SL}(n, \mathbf{R})$-structures, for two equal, except sign, *volume n-forms*. For a general M, volume corresponds to *odd type n-form*, as in [4, pp. 21-27].

From *principal bundle theory* ([9]), $\mathrm{SL}_n^{\pm}$-structures are the sections of the *bundle associated* with LM and the left action of G_n^1 on $\mathrm{G}_n^1/\mathrm{SL}_n^{\pm}$. This is the *volume bundle, $\mathcal{V}M$*. Furthermore, the sections of $\mathcal{V}M$ correspond to *G_n^1-equivariant functions* f of LM to $\mathrm{G}_n^1/\mathrm{SL}_n^{\pm}$. The equivariance condition is $f(la) = |\det a|^{\frac{-1}{n}} I_n \cdot f(l)$, $\forall a \in \mathrm{G}_n^1$. We have bijections:

$$\text{Volumes on } M \longleftrightarrow \operatorname{Sec} \mathcal{V}M \longleftrightarrow C^{\infty}_{\text{eq}}(LM, \mathrm{G}_n^1/\mathrm{SL}_n^{\pm})$$

The isomorphisms $\mathrm{G}_n^1/\mathrm{SL}_n^{\pm} \simeq \mathrm{H}_n$, with $\mathrm{H}_n := \{kI_n \colon k > 0\}$ and $\mathrm{H}_n \simeq \mathbf{R}^+$, the multiplicative group of positive numbers, allow to represent a volume as an *(odd) scalar density* on M.

5 Second order structures

We can view a *symmetric linear connection (SLC) on M* as a G_n^1-structure of second order. An SLC is also the image of an *injective homomorphism* of LM to $\mathcal{F}^2M$ ([10]).

From the *principal bundle theory*, SLC's on M are sections of the *SLC bundle, $\mathcal{D}M$*, associated with $\mathcal{F}^2M$ and the action of G_n^2 on $\mathrm{G}_n^2/\mathrm{G}_n^1 \simeq \mathrm{S}_n^2$. Furthermore, each SLC, ∇, corresponds to a *G_n^2-equivariant function* $f^{\nabla} \colon \mathcal{F}^2M \to \mathrm{S}_n^2$, verifying $f^{\nabla}(z(a,s)) = a^{-1} f^{\nabla}(z)(a,a) + s$. We have the bijections:

$$\text{SLC's on } M \quad \longleftrightarrow \quad \operatorname{Sec} \mathcal{D}M \quad \longleftrightarrow \quad C^{\infty}_{\text{eq}}(\mathcal{F}^2M, \mathrm{S}_n^2)$$

Given two SLC's, ∇ and $\widehat{\nabla}$, *the difference* $f^{\nabla} - f^{\widehat{\nabla}}$ verifies $z(a,s) \mapsto a^{-1}(f^{\nabla}(z) - f^{\widehat{\nabla}}(z))(a,a)$. Then, it is projectable to a function $f \colon LM \to \mathrm{S}_n^2$ verifying $f(la) = a^{-1} f(l)(a,a)$, which corresponds to *a symmetric $\binom{1}{2}$-tensor* $\rho = (\rho^i_{jk})$ on M.

A *projective structure (PS)* is a set of SLC's which have the same family of *pregeodesics*. This is the cornerstone to understand the *freely falling bodies* in GR ([5]).

We define a PS on M as a *second order $\mathrm{G}_n^1 \rtimes \mathfrak{p}$-structure*, Q, with $\mathfrak{p} := \{s \in \mathrm{S}_n^2 \colon s^i_{jk} = \delta^i_j \mu_k + \mu_j \delta^i_k,\ \mu = (\mu_i) \in \mathbf{R}^{n*}\}$.

Now, for two SLC *included* in the same PS, i.e. $\nabla, \widehat{\nabla} \subset Q$, the tensor ρ, expressing *their difference*, is determined by the contraction $C(\rho) = (\rho^s_{si})$, which is *an 1-form* on M.

6 Prolongations

Let B a *first order G-structure*. A connection in B is a distribution H of transversal n-subspaces, $H_l \subset T_l B$. If the subspaces are *free-torsion*, these determine a *second order G-structure*, whose G^1_n-*extension* ([7, p.206]) is a SLC on M. Then, we say that B *admits an SLC*. Let us give two examples:

- An SLC and a parallel volume is an *equiaffine structure* on M ([11]); hence, it is a second order $\mathrm{SL}^{\pm}_n$-structure.
- An SLC compatible with a conformal structure is a *Weyl structure*; hence, it is a second order $\mathrm{CO}(n)$-structure ([2]).

Theorem 3. *Let $B \subset LM$ a G-structure, admitting an SLC. Then, the set of 2-frames, corresponding with torsion-free transversal n-subspaces which are included in TB, is a reduction of $\mathcal{F}^2M$ to $G \rtimes \mathfrak{g}_1$. It is named the prolongation of B and denoted by B^2.* (For a proof, see [12, p.150-155]).

Let us give a well known example: if B is an $O(n)$-structure, B^2 is isomorphic to B on account of $\mathfrak{o}(n)_1 = \{0\}$; this explain the uniqueness of Levi-Civita connection.

There is an important theorem ([8]) *classifying the groups G* such that *every G-structure admits an SLC*: only the groups of *volume, metric and conformal* structures, and a class of groups preserving an 1-dimensional distribution, have this property.

7 Concluding remarks

We have done a unified description of the geometrical structures that have been used by GR to define intrinsic properties of the space-time. The unifying criterion we used for it, *not only is natural in the sense that geometric objects are sections of bundles associated with the $\mathcal{F}^rM$ frame bundles* ([13]), *but that the objects themselves are reductions of $\mathcal{F}^rM$*. We have not considered a *linear connection with torsion* because it is a section of an associated bundle of $\mathcal{F}^2M$, but not a reduction.

We have tried to clarify the relationships between the structures involved. Only simple relations, such as intersection, inclusion, reduction and extension, have been used for it, on account of the previous *prolongation* of G-structures admitting SLC. For instance, it follows readily from the last section that the classical *equiaffine* or *Weyl structures* can be defined as the intersection of an SLC with the prolongation of a volume or a conformal structure, respectively.

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