Quantum geometry and quantum gravity

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Session 3: Geometric operators

1. Elementary quantum operators.
2. Geometric operators.
3. The area operator.
4. The volume operator.
5. **Epilogue (remaining topics, open problems,...)**
   - Dynamics: the quantum constraints.
   - Geometric observables.
   - Applications: LQC, black hole entropy,...

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Elementary classical operators.

- Classically $\mathcal{A}$ consists of all smooth $SU(2)$ connections on Σ.
- The phase space is a cotangent bundle $T^*(\mathcal{A})$.
- To write the basic Poisson brackets between configuration and momentum variables we put.

$$^3A[v] := \int_\Sigma A_a^i \tilde{v}_i^a, \quad \tilde{v}_i^a : \Sigma \to \mathfrak{su}(2)^*$$

$$^3E[f] := \int_\Sigma \tilde{E}_i^a f_a^i, \quad f_a^i : \Sigma \to \mathfrak{su}(2)$$

$$\{^3A[v],^3 A[v']\} = \{^3E[f],^3 E[f']\} = 0,\ \{^3A[v],^3 E[f]\} = \int_\Sigma \tilde{v}_i^a f_a^i$$

(Usually they are written in a distributional form).

Elementary quantum operators

- This Poisson algebra is not suitable for quantization.
  - These variables are not gauge covariant in the present non-abelian context.
  - If one tries to build cylindrical functions with them (by exactly following the steps of the scalar field construction) there are problems because they are not integrable w.r.t. known diff-invariant measures.
- One has to look for appropriate phase space variables.

- **How?** Using "distributional" smearing fields (with support in lower dimensional submanifolds of Σ) but then one must be *very* careful...

- **Other types of smearings:**

  - 1-dimensional ones for the connection (holonomies).
  - 2-dimensional ones for the triads (fluxes) $E[S,f] := \int_S e_{abi} f^i \mathrm{d}S^{ab}$, with $e^i_{ab} = \eta_{abc}\tilde{E}^{ci}$.

- We must make a choice of some configuration variables and some momentum variables in such a way that:
  - They separate points in phase space.
  - The set of configuration and momentum variables is closed under the action of the Poisson brackets (and, hence, we have a Lie algebra of elementary variables).
- The **choice of configuration variables** is quite straightforward: we will take cylindrical functions (which in particular suffice to separate points in $\mathcal{A}$).
- The **choice of momentum variables** is subtler.
- It is natural to choose momenta as vector fields related to the configuration variables that we have considered ($P(v)$, with $v \in \mathfrak{X}(Cyl)$).
- For example, the Hamiltonian vector fields for the 3-dimensional smearings of the triads introduced above are well defined and, hence, we can compute $\{\Psi_\alpha, {}^3E[f]\}$ (α denotes a graph), however *this is not a cylindrical function*. We must make another choice.

- Let us consider instead a 2-dimensional smearing. This can be defined rigorously by taking certain limits of the 3-dimensional smearings considered at the beginning

$$\lim_{\epsilon\to 0}\{\Psi_\alpha, {}^3E[{}^\epsilon f]\}.$$

Here we use a family of smearing functions ${}^\epsilon f_a^i$ depending on a real parameter ϵ and such that in the limit $\epsilon \to 0$ they tend to a distribution with support on a surface $[{}^\epsilon f_a^i(x,y,z) = h_\epsilon(z)(\nabla_a z)f^i(x,y)]$

- The previous limit is

$$\lim_{\epsilon\to 0}\{\Psi_\alpha, {}^3E[{}^\epsilon f]\} = \frac{1}{2}\sum_p \sum_{I_p} \kappa(I_p) f^i(p) X_{I_p}.\psi \qquad (1)$$

- where:
  - p are the intersection points of the graph and the surface.
  - e_{I_p} are the edges at each intersection point $I_p = 1, \ldots, n_p$.
  - $\kappa(I_p)$ is $+1$ if the edge lies completely "above" S, -1 if it is "below" S and 0 it it is tangent (S is oriented).

- $X_{I_p}.\psi$ is the action of the i-th left (resp. right) invariant vector field on the I_p-th argument of the function $\psi : [SU(2)]^N \to \mathbb{C}$ if e_{I_p} points away from (resp. towards) S.

- Notice that we have obtained a cylindrical function so we are moving in the right direction, hence, if we manage to interpret the previous limit as the Poisson bracket of Ψ_α with something this would be a good candidate for a momentum variable
- The previous choice of ${}^\epsilon f_a^i$ suggests that this variable will be the flux $E[S, f]$ associated to a certain f^i and a choice of a surface S.
- This is indeed the case when some conditions on the f^i's, and the surface S are imposed. These are:
  - The functions f^i must be continuous.
  - The surface S must have the form $\bar{S} - \partial\bar{S}$ with $\bar{S}$ a compact analytic oriented 2-dim submanifold of Σ. In particular it must have no boundary.

Elementary quantum operators

- The analyticity condition (remember that we are working with piece-wise analytic curves to define the holonomies) is used to avoid the appearance of infinite (non-trivial) intersections of S with the edges of the graph α.
- What happens now with the **Poisson bracket between two flux operators**?
  - One would naively expect $\{E[S,f],E[S',f']\}=0$ because "momenta commute". However this cannot be true for 2-dim smeared variables, the reason: this is incompatible with the expression that we have found for $\{\Psi_\alpha,E[S,f]\}$ because the Jacobi identity would be violated.
  - This can be seen by computing for a simple cylindrical function (a Wilson loop W_α)

$$\{\{E[S,f],E[S,g]\},W_\alpha\}+\overbrace{\{\{W_\alpha,E[S,f]\},E[S,g]\}+\{\{E[S,g],W_\alpha\},E[S,f]\}}^{\neq 0}$$

  - The last two terms can be explicitly computed with the help of (1) and, generically, **they are not zero.** Hence the first term cannot be zero if we want the Jacobi identity to be satisfied.

What is going on here?

- For classical finite dimensional systems there is a natural isomorphism between the space of momentum variables and a space of suitably regular vector fields on the configuration space.
- Momenta can be thought of as vector fields in the configuration space when the phase space is a cotangent bundle (a fact that is translated into the action of the momenta operators after the quantization).
- We can write variables of the type $P(v)(q,p) = v^a(q)p_a$ given a fixed vector field v^a on the configuration space. Now if f, f' denote suitable regular functions on the configuration space $\mathcal{C}$ and v, v' are regular vector fields we have

$$\{Q(f), Q(f')\} = 0,\ \{Q(f), P(v)\} = Q(\mathcal{L}_v f),\ \{P(v), P(v')\} = -P(\mathcal{L}_v v')$$

.

- Notice that these operations refer only to structures in the configuration space.
- It is natural then to associate a vector field $X_{S,f}$ with the flux variables $E[S,f]$. We do this by taking

$$X_{S,f}\cdot\Psi_\alpha := \frac{1}{2}\sum_p\sum_{I_p}\kappa(I_p)f^i(p)X_{I_p}.\psi$$

  This is a one to one correspondence.
- **Comments:**
  - $X_{S,f}$ is a vector field in the sense that it is a derivation on the ring Cyl.
  - The commutator of two derivations is a derivation and they form a Lie algebra (no problem with the Jacobi identity.)
  - Only those derivations that can be obtained by taking finite linear combinations and finite number of brackets are considered here.

- The vector fields $X_{S,f}$ do not commute on *Cyl*! In fact:

$$[X_{S,f}, X_{S',f'}](\Psi) =$$

$$\frac{1}{4}\sum_p f^i(\bar{p})f'^i(\bar{p})\epsilon_{ijk}\left(\sum_{I_p^{uu'}} X^k_{I_p^{uu'}} - \sum_{I_p^{ud'}} X^k_{I_p^{ud'}} - \sum_{I_p^{du'}} X^k_{I_p^{du'}} - \sum_{I_p^{dd'}} X^k_{I_p^{dd'}}\right)(\psi)$$

This solves the issue, it is in this precise sense that the momentum functions given by the fluxes do not commute

Comments:

- It is straightforward to see that generic derivations need not correspond to any phase space function. For example, the commutator $[X_{S_1,f_1}, X_{S_2,f_2}]$ with both surfaces intersecting on a curve is a derivation on *Cyl* but its action is only non-trivial on graphs with edges passing trough $S_1 \cap S_2$. The commutator has now 1-dim support and then is not a linear combination of fluxes $E[S, f]$.

  - The set of all derivations would be too large in a definite sense (one would need to incorporate some extra conditions that would make things more complicated).
  - The set formed by the X_{S_1,f_1} is sufficiently small and avoids these difficulties (this is like using ∂_x, ∂_y, and ∂_z in quantum mechanics of a particle.)
  - The $(Cyl, X_{S,f})$ variables suffice to separate points in phase space.

- The reason on the non-commutativity is related to the way the 2-dim smeared things are obtained as limits from the 3-dim ones. If we denote ${}^3X[f]$ the vector fields associated with ${}^3E[f]$ we have

$$[X_{S_2,f_2}, X_{S_1,f_1}].\Psi_\alpha =$$

$$\lim_{\epsilon_2\to 0} {}^3X[{}^{\epsilon_2}f](\lim_{\epsilon_1\to 0} {}^3X[{}^{\epsilon_1}f]\cdot\Psi_\alpha) - \lim_{\epsilon_1\to 0} {}^3X[{}^{\epsilon_1}f](\lim_{\epsilon_2\to 0} {}^3X[{}^{\epsilon_2}f]\cdot\Psi_\alpha)$$

It can be seen that the action of vector fields and taking limits does not commute when acting on cylindrical functions and, hence, the previous expression is not zero.

- A final comment. The variables that we have just introduced are in a sense **hybrid**. Whereas the momentum variables –the fluxes– are linear in the triad field, the configuration variables, obtained from holonomies of the connection, are non-linear (exponential). When we go to the quantum theory the representation that we will be using is something like using q and $\exp(i\beta p)$ as the elementary variables for the quantum particle.

Elementary quantum operators

- **Quantization** is straightforward, configuration variables represented by complex valued cylindrical functions on $\bar{\mathcal{A}}$ act by multiplication on the wave functions

$$(\hat{f}\Psi)[\bar{A}] = f(\bar{A})\Psi[\bar{A}]$$

- The action of momentum operators (fluxes) is given by

$$(\hat{E}[S,f]\Psi)[\bar{A}] = i\{E[S,f],\Psi\}[\bar{A}] = \frac{1}{2}\sum_{v} f^{i}(v)\sum_{e@v}\kappa(e)\hat{J}_{i}^{(v,e)}\Psi[\bar{A}]$$

  The sum is done over the vertices of α where it intersects S. This operator is essentially self adjoint in its domain $Cyl^{(2)}$ and, hence, admits a unique extension to de full $\mathcal{H}$.
- Commutators are represented as $i\hbar$ times the classical Lie bracket between the corresponding classical variables.

Elementary quantum operators

- There are no anomalies in the quantization; commutators exactly mimic the Poisson algebra between classical elementary variables (known as the holonomy algebra or ACZ algebra).
- There is an important uniqueness result related to this:

The LOST theorem

There exist exactly one Yang-Mills gauge invariant and diffeomorphism invariant state on the quantum holonomy-flux $*$-algebra $\mathfrak{U}$.

- The key importance of this result is that it shows that the Ashtekar-Lewandowski measure μ_{AL} is the only diff-invariant measure that supports the representation of the holonomy-flux algebra.

Geometric operators

- I will start now discussing **geometric operators** (quantum Riemannian geometry), we are finally doing the promised quantum geometry!
- Four of them are obviously desirable to understand:
  - The length and angle operators
  - The area operator
  - The volume operator
- General considerations
  - They can all be rigourously defined in the Hilbert space used above.
  - They have **discrete spectra**.
  - The generalized spin network basis introduced before is well adapted to their description.
  - Two of them play an additional important role in applications:
  - The **area operator** is important in the computation of **black hole entropy**.
  - The **volume operator** is a basic ingredient for the quantization of the **scalar constraint**.
- For these last two reasons I will concentrate on **areas** and **volumes**.

- Consider a two-dimensional surface embedded in $S \subset \Sigma$. We will require it to be closed.
- The densitized triad $\tilde{E}_i^a$ encodes the metric information. Hence we can write the area of a surface in terms of it.
- If we use coordinates on the surface σ_1 and σ_2 and the choose a normal n^a to the points of S the area, as a function(al) of $\tilde{E}_i^a$ takes the form

$$A_S[\tilde{E}_i^a] = \int_S (\tilde{E}_i^a \tilde{E}_j^b \delta^{ij} n_a n_b)^{1/2}$$

- We want now to quantize the operator $A_S[\tilde{E}_i^a]$. This means that we have to define its action on the vectors in $\mathcal{H}$. To this end we want to know its action on the elements of the orthonormal basis that we have introduced above.
- A reasonable way to approach this problem is trying to express it in terms of the flux operators $E[S, f]$.

- The idea is to decompose S in N two dimensional cells S_I of "small coordinate" size.
- Use the three Lie algebra vectors τ_i as test fields f^i and consider the flux variables $E[S_I, \tau^i]$ on each cell.
- Consider

$$A_N[S] := \gamma \sum_{I=1}^{N} (E[S_I, \tau_i] E[S_I, \tau_j] \eta^{ij})^{1/2}.$$

  This is an approximate expression for the area ("Riemann sum") in the sense that if the number of cells goes to infinity in such a way that their coordinate size goes to zero uniformly we recover the area in the limit $N \to \infty$.
- To quantize we take advantage of the fact that in each cell $E[S_I, \tau_i] E[S_I, \tau_j] \eta^{ij}$ is a positive self adjoint operator on $\mathcal{H}$ (it then has a well defined square root).

- The action of this operator on an element of $\mathcal{H}_\alpha$ for a fixed graph is straightforward to obtain. The idea is to refine the partition so that every elementary cell has, at most, one transverse intersection with the graph. In this case the only terms contributing come from the S_I that intersect α. Once this point is reacher further refinements do nothing.
- The resulting operator can be written as the following sum over the vertices of α that lie on S

$$\hat{A}_{S,\alpha} = 4\pi\gamma\ell_P^2 \sum_v (-\triangle_{S,v,\alpha})^{1/2}$$

where $\triangle_{S,v,\alpha}$ is an operator given by a quadratic combination of the $\hat{L}_i$ and $\hat{R}_i$ operators associated to each edge leaving or arriving at the v's appearing in the previous sum.

- Comments:
  - The previous expression is defined on $\mathcal{H}_\alpha$ for a fixed graph. In order to see if it is defined on the whole Hilbert space $\mathcal{H}$ one has to check some consistency requirements related to the fact that a function may be cylindrical w.r.t. different graphs. It is possible to prove that this is always possible
  - The previous operator can be extended as a self-adjoint operator to the full Hilbert space $\mathcal{H}$.
  - It is $SU(2)$ invariant and diff-covariant.
  - The eigenvalues of the area operator are given in general by finite sums of the form

$$4\pi\gamma\ell_P^2\sum_{\alpha\cap S}[2j^{(u)}(j^{(u)}+1)+2j^{(d)}(j^{(d)}+1)-j^{(u+d)}(j^{(u+d)+1})]^{1/2}$$

where the $j^{(u)}$, $j^{(d)}$, and $j^{(u+d)}$ are half integers that appear as eigenvalues of the angular momentum operator for the edges that appear in the expression of the area operator (subject to some constraints in the form of inequalities).

Area operator

- It is possible to obtain simple expressions for the area operator if we restrict ourselves to the (internal) **gauge invariant subspace** of $\mathcal{H}$ (this is spanned by the elements of *Cyl* with vanishing eigenvalues for the vertex operators $\hat{J}_i^v$).
- For example, if the intersections of α and S are just 2-degree vertices the spectrum of the area operator takes now the simple form

$$8\pi\gamma\ell_P^2\sum_I (j_I(j_I+1))^{1/2}$$

- Notice that the Immirzi parameter γ appears in all these expressions. This means that it is not an irrelevant arbitrariness in the definition of the canonical transformations leading to the Ashtekar formulation but, rather, a parameter that may show up in (eventually) observable magnitudes such as areas.
- Area operators fail to commute on intersecting surfaces. This implies in particular that it is impossible to diagonalize them simultaneously. This lack of commutativity has no observable consequences at macroscopic scales.

- A final pictorial interpretation of this is the following. The "quantum excitations of geometry" are 1-dimensional and carry a flux or area. Any time a graph pierces a surface it endowes it with a quantum of area. This picture is completely different from the Fock one (quantum excitations as particles).

Volume operator

- The strategy to define it is similar to the one for the area operator.
- The volume of a 3-dim region B (a certain open subset of Σ) is classically given by

$$V_B = \int_B \sqrt{h}$$

- This can be expressed in terms of the triad as

$$V_B = \int_B \left| \frac{1}{3!} \epsilon_{abc} \epsilon^{ijk} \tilde{E}_i^a \tilde{E}_j^b \tilde{E}_k^c \right|^{1/2} .$$

  As before we want to rewrite this expression in terms of flux operators.
- To this end we divide B in cells of a small coordinate volume. In each cell we introduce three surfaces such that each of them splits the cell in two disjoint pieces. This defines the so called "internal regularization" (others are possible)

Volume operator

- An **approximate** expression for the volume in terms of flux operators is then

$$\sum_{cells} \left| \frac{(8\pi\gamma\ell_P^2)^3}{6} \epsilon_{ijk}\eta_{abc} E[S^a,\tau^i]E[S^b,\tau^j]E[S^c,\tau^k] \right|^{1/2}$$

  When the coordinate size of the cells goes to zero this gives the volume of the region B.
- As in the case of the area operator we define a family of operators for each graph α (satisfying similar consistency conditions).
- The resulting operator is given by

$$\hat{V}_{B,\alpha} := \alpha \sum_v \left| \frac{(8\pi\gamma\ell_P^2)^3}{48} \sum_{e_1,e_2,e_3} \epsilon_{ijk}\epsilon(e_1,e_2,e_3) \hat{J}_i^{(v,e_1)} \hat{J}_j^{(v,e_2)} \hat{J}_k^{(v,e_3)} \right|^{1/2}$$

  where α is an undetermined constant and $\epsilon(e_1,e_2,e_3)$ is the orientation factor of the family of edges (e_1,e_2,e_3).

Comments:

- The removal of the regulator (i.e. of the auxiliary partition used to define the volume operator) is non-trivial now because the volume operator obtained by "just taking the limit" keeps some memory of the details of the partition. Nevertheless there is a way to handle this issue.
- The orientation function is zero if the tangent vectors to the edges e_1, e_2, e_3 are linearly dependent at the point where they meet. This means, in particular, that the volume operator is zero when acting on state vectors defined on *planar graphs* i.e. graphs such that at each vertex the tangent vectors are contained in a plane.
- The volume operator is $SU(2)$ gauge invariant and diffeomorphism covariant as the area operator. The total volume operator (i.e. $\hat{V}_\Sigma$) is diff-invariant.

Volume operator

- The volume operator is zero also when acting on gauge invariant states if the vertices are at most of degree 3 (tri-valent vertices).
- The eigenvalues of the volume operator are real and discrete. They are not known in general but can be computed in many interesting cases. In particular when the vertices ar four-valent.
- The volume operator plays a central role in the implementation of the quantum constraints because the quantum version of the scalar constraint can be written in (relatively) simple terms by using Poisson brackets of the total volume operator and the basic canonical variables.
- Other regularizations are possible, for example the so called external regularization obtained by considering the faces of the cells used in the approximation of the volume operator in the process of writing it in terms of the flux operators. The volume operators built by using these different approaches have different properties.

What is left to discuss?

- Dynamics i.e. the implementation of the constraints in the quantum theory to determine the Hilbert space of physical states $\mathcal{H}_{phys}$ *including its scalar product!* (a highly non-trivial step not yet complete at this point).
- The geometric operators on the kinematical Hilbert space that I have described *do not commute* with the constraints so they are not observables in the Dirac sense. Finding geometrical observables is non-trivial. In fact, there are arguments suggesting that their properties may be quite different from the ones discussed above [there is a lively debate about such issues as the persistence of the discreteness of the spectrum for area and volume operators.]

Epilogue

- **Applications:** Some of the most recent advances in the subject have to do with the physical applications of the formalisms for those problems that do not require the solution of the difficult dynamical issues. The most important ones are the study of black hole entropy and Loop Quantum Cosmology (LQC). The results obtained here are very suggestive and give tantalizing glimpses on the physics of the primitive universe and singularities
- **Open problems:** I would like to highlight two of them: the determination of the scalar product in the physical Hilbert space and the issue of the semiclassical limit (how to recover gravitational large scale Physics from the microscopic quantum theory.)

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