QUANTUM GEOMETRY AND QUANTUM GRAVITY

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Session 1: Quantum gravity and Ashtekar variables

1. Why quantum gravity?
2. The case for diffeomorphism invariance.
3. A new point of view.
4. A prelude to quantization: Hamiltonian formulation of GR.
5. The SO(3)-ADM formulation.
6. GR in Yang-Mills phase space: Ashtekar variables.
7. Real or complex formulations?: The self-dual and Holst actions.

Session 2: Quantization with the new variables

1. General aspects of quantization.
2. Quantum configuration space for field theories. A simple example: The scalar field.
3. Quantum configuration space for connection field theories.
4. The Hilbert space.
5. The Ashtekar-Lewandowski measure.
6. A useful orthonormal basis.

Session 3: Geometric operators

1. Elementary quantum operators.
2. The area operator.
3. The volume operator.
4. Other geometric operators.

Epilogue (remaining topics, open problems,...)

- Dynamics: the quantum constraints.
- Geometric observables.
- Applications: LQC, black hole entropy,...

Fundamental interactions

- At the beginning of the XX century two long range forces were known: gravity and EM. All the other ways in which matter interacts (contact forces) were interpreted in terms of them after making some hypotheses about the structure of matter.
- The detailed understanding of the stability of matter required the introduction of a new mechanics (Eduardo's talk). This also led to the necessity of quantizing fields (emission of radiation, atom deexcitation, the physics of black bodies,...)
- Two other fundamental interactions (the weak and strong forces) were later recognized in the nuclear realm. Given the scale at which they are manifest they are intrinsically quantum.

Fundamental interactions and QFT's

- Weak, strong and EM forces are described by QFT's. QED was the first successful QFT. It contains classical EM in the certain limit.
- It gives the detailed behavior of EM fields, their interactions with charged particles, and the interaction of charged particles themselves.
- Particles are described as "quantum excitations" of certain fields, i.e. certain types of vectors in the Hilbert space used to quantize a field theory.
- This point of view was adopted after efforts to construct relativistic quantum mechanics of particles failed. It has far reaching consequences because it allows to describe processes in which particles are created and destroyed (weak interactions).
- The weak and strong forces can also be described by QFT's such as QCD. The resulting framework is known as the **standard model of particles and interactions.**

Key ingredients of QFT's in standard formulations

$$\mathcal{L}_{QED} = \underbrace{\bar{\psi}(i\partial\!\!\!/ - m)\psi + \mathrm{d}A \wedge *\mathrm{d}A}_{free} + \underbrace{e\bar{\psi}A\!\!\!/\psi}_{int}$$

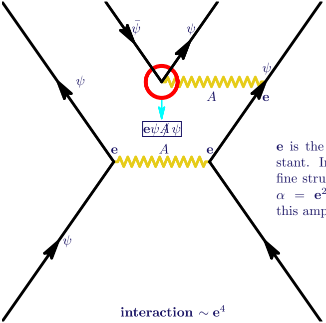
Fock spaces

- Hilbert space (Fock space) obtained from the free, non-interacting theory. This free theory can be understood as a theory of an infinite number of uncoupled harmonic oscillators.
- The Hilbert space for the field theory is not a tensor product of the Hilbert spaces for each harmonic oscillator (non-separability, reducibility of the representation for quantum commutators,...).
- A different construction is use to build a Fock Hilbert space, [this IS **separable** (more on this later)].

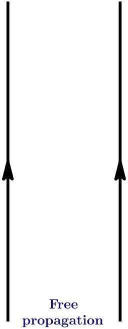
$$\mathcal{F}_s(\mathcal{H}) = \bigoplus_{n=0}^{\infty}\left(\otimes_s^n \mathcal{H}\right) = \mathbb{C} \oplus \mathcal{H} \oplus \cdots \;; \quad \Psi = (\psi, \psi^{a_1}, \psi^{a_1 a_2}, \ldots)$$

- Interpretation of states:
  - 1-particle: $|k\rangle \in \mathcal{H}$. (the label k is the four-momentum)
  - n-particle: $|k_1, \ldots, k_n\rangle \in \otimes_s^n \mathcal{H}$, but also $|\underbrace{k, \ldots, k}_{n}\rangle$.
  - vacuum $|\Omega\rangle \in \mathbb{C}$.
- Scalar product $\langle \Psi | \chi \rangle = \bar{\psi}\chi + \bar{\psi}_a \chi^a + \bar{\psi}_{a_1 a_2} \chi^{a_1 a_2} + \cdots$.

Interactions: Higher order terms in the action (Feynman diagrams).



e is the coupling constant. In terms of the fine structure constant $\alpha = \mathbf{e}^2/\hbar c \simeq 1/137$ this amplitude is $\sim \alpha^2$



- Relevant physical observables ("in a straightforward experimental sense", i.e. cross sections, the S-matrix) are obtained as power series in the relevant coupling constant (α for QED). **It is in this sense that the framework is perturbative.**
- An important background geometric object: The Minkowski metric (Poincaré invariance). It appears, for example, in the Maxwell Lagrangian to define the Hodge $*$.
- This symmetry is very important to select a quantization (there is a huge non-uniqueness when one quantizes field theories due to the existence of non-unitary equivalent representations of the algebra of elementary variables). The vacuum is Poincaré invariant.
- For free theories the construction can be generalized to other non-Minkowskian (curved) backgrounds. Some work also exists for interacting field theories in curved backgrounds. This allows to extend some methods and results of QFT to weak, but nontrivial, gravitational fields.

The fourth (first?) interaction: gravity

- What about gravity? (the first interaction that was successfully described in a mathematical sense).
- The "true" nature of gravity as **spacetime geometry** was recognized by Einstein in his attempts to find a relativistic description of gravitation (avoiding instant propagation and incorporating such pillars of relativity as the existence of a fundamental velocity).
- A very important first step towards this was Minkowski's realization of the fact that special relativity could be understood as geometry in a spacetime endowed with a particular metric of signature $(-+++)$ [the Minkowski metric].

Is gravity described by a QFT at the fundamental level?

Two points of view

- The particle physicist point of view.
- The relativist point of view.

The questions

1. Should gravity be quantized?
2. How do we do it?
3. Can gravity be quantized?

Particle physicist quantum gravity

- Questions that a particle physicist would love to ask and answer with a quantum theory of gravity:
- What happens in collision experiments at arbitrary high energies [build the DLC (Dream Linear Collider) and collide beams of say, $\mathrm{e}^-, \mathrm{e}^+$ all the way to the Planck energy $E_p = (\hbar c^5 G_N^{-1})^{1/2} \simeq 10^{19} GeV$...]

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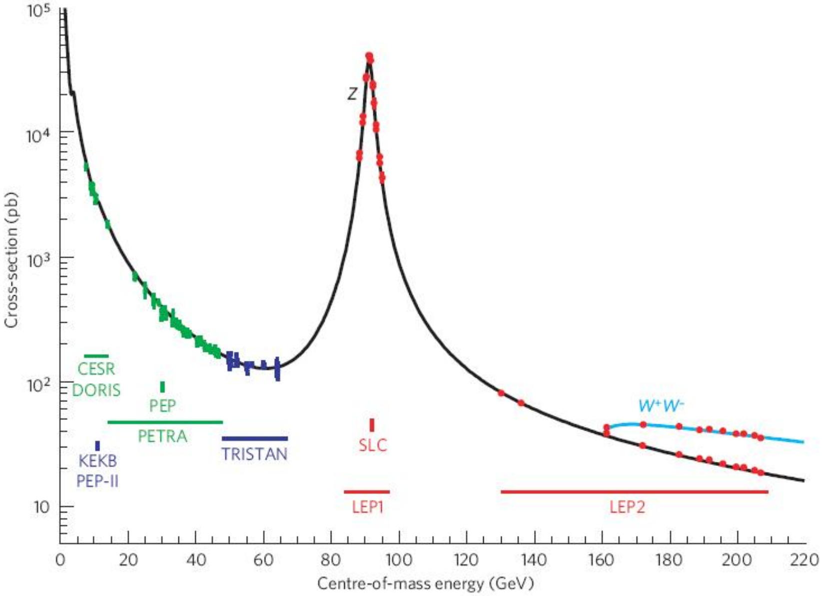
WHY QUANTUM GRAVITY? SHOULD IT...



- One expects that at a very high energy regime gravity should be important ("feeble" interactions manifest themselves at very small distances-high energies).
- The energy of a photon such that its Schwarzschild radius $R_S = 2G_N E/c^4$ (remember $E = Mc^2$) is of the order of its wavelength $\lambda = hc/E$ is $E \simeq (hc^5/G_N)^{1/2}$ i.e. the Planck energy!
- Even if the gravitational quanta (whatever they may be) are difficult to detect at least they should manifest themselves in the interactions (something like this happens with Higgs, ...).
- Once the interaction at this regime is understood one can do astroparticle physics and study the early universe.
- Unification.
- String theory belongs in spirit to this side.

Perturbative quantum gravity

Particle physicists tried to use the same approach that worked for the other interactions to quantize gravity but failed.

- The goal is to obtain S matrix elements (containing the physics of particle interactions, giving predictions for accelerator experiments, and providing necessary information for cosmology).
- The use of perturbation theory for this means that one wants to obtain cross sections as power series in G_N.
- For this to be at all possible there are some "consistency" requirements (that can be met for the other interactions) referred to as **renormalizability**.
- One must be able to absorb infinities appearing in physical amplitudes in a redefinition of the coupling constants. This works for the other interactions and, in fact, fantastically accurate predictions, confirmed by experiments, are obtained.

Can it be done?

- **NO:** perturbative quantum gravity is non-renormalizable
- **Maybe:** String theory

Relativist's quantum gravity

What kind of phenomena would a relativist love to understand in a quantum theory of gravity?

- **Singularities** (in a sense QFT solves some problems associated with singularities of the classical em field, Coulomb potential, infinite energy of a single charge... without introducing extended objects but rather by changing the description of the interactions, particles are still points but...).
- Understand the **high curvature regime of general relativity**, the nature of gravitational singularities, and, in particular, black holes and the Big-Bang. The perception is that the appearance of singularities in a physical theory is a clear indication that there is a better theory (that reduces to the old one in a certain low energy, large distance scale but supersedes it outside these regimes) where these are not present. This includes astrophysical situations (black holes), and the big bang (ambitious, huh!)

Why quantum gravity? Should it...

- What is the origin of **black hole entropy** and its detailed **microscopical description**? Understand the process of **black hole evaporation**, in particular those issues related to unitarity and information loss.
- Define a **mathematically rigorous** quantum theory of the gravitational field, in particular of general relativity.
- Find useful quantum **gravitational observables** (hopefully leading to experimental verification).
- Solve the **problem of time** and, in general, the problem of **spacetime covariance** in the canonical approach (time does not exist as an external object in general relativity).

Why quantum gravity? Should it...

- As the gravitational field is a manifestation of space-time geometry the quantization of general relativity will require us to understand the fate of geometry after quantization or, in other words, the **meaning of quantized geometry**.
- Alternatively, one should understand the **emergence of classical geometry** at large scales from the purely quantum gravity theory (semi-classical and classical regime).

- Use **canonical quantization**, in particular **Dirac's method** to quantize constrained systems.
- The starting point is a **Hamiltonian formulation** for gravity. This was first obtained in the early sixties by Arnowitt, Deser, and Misner (the ADM formalism) after some pioneering work by Dirac.
- Attempt then to write a quantum version of the constraints [apparently referred to by one of his authors (Wheeler and DeWitt) as *that damned equation!*].
- Pioneering work on this provided lots of interesting insights but did not produce a useful theory of QG and the lack of mathematical rigor was at the same level as with perturbative approaches.

Can it be done?

- **NO:** Failure of quantum geometrodynamics
- **MAYBE!** Loop quantum gravity (the subject of my talks).

The background field method in perturbative QFT's

- A standard way to discuss renormalizability in standard QFT in an efficient way is to use the so called **background field method** that requires us to write the basic dynamical objects (i.e. fields) as

$$\phi = \phi_{back} + \varphi$$

  with a **FIXED** background field ϕ_{back} and a dynamical φ.
- This is most useful when gauge symmetries are present (as in QED, EW model and QCD) because it allows us to find gauge invariant counterterms in a very effective way (by using what are known as **covariant gauges**).
- Though φ is usually referred to as a "small perturbation" this is not the case. In fact, this way of rewriting ϕ is a change of variables in the Lagrangian. Notice that the usual actions for Yang-Mills theories are polynomial so there are no convergence issues.

The case for diff-invariance

- For polynomial actions (basically all but the Einstein-Hilbert action for gravity) it is not strictly necessary to use this, however in the gravitational case the splitting of the metric as $g_{ab} = g_{ab}^{back} + h_{ab}$ is strictly necessary to use the standard approaches because
  - Interactions in the perturbative approach are described by degree$>$2 monomials in the Lagrangian density. We have then to generate them by a power series expansion in h_{ab} (this is where the "smallness" of h_{ab} could play a role).
  - A **background spacetime metric** is a necessary to apply the usual perturbative techniques. It provides the **causal structure** which is a key ingredient for such important items as the spin-statistics theorem, microcausality (commutation of fields at spacially separated points)...
  - The Minkowski metric plays a crucial if, somewhat backstage role, because its presence allows these theories to wear the tag qualifying them as "relativistic".
  - The representation theory of the Poincaré group (symmetries of the Minkowski metric) ultimately allows us to talk about particle excitations, mass and spin (gravitons).

The use of the background field method in quantum gravity is very problematic

- What is the right background? For particle physics experiments it is possibly OK to use Minkowski but if we are (as we do) interested in the universe as a whole what is the right choice?
- There is a clear conflict with some of the things that we expect to understand from a quantum theory of gravity. Does it make any sense to use a singular background? if not how would the quantized theory describe the very high curvature regimes?

The question: can we quantize field theories without a background?

- **Problem 1:** What is *physically* a diff-invariant field theory?
- **Problem 2:** Why the absence of a background leads us to diff-invariant models?

A new point of view

- We must learn how to **quantize diff-invariant theories** of gravity i.e. without the help of a background metric.
- A surprising development [Ashtekar 1986]. The best way to do this for gravity is to describe it as **a theory of** $SU(2)$ **connections**. For historical reasons this approach is known as **Loop Quantum Gravity** (**LQG** in short) and is the main subject of my lectures.
- The label **non-perturbative** (which has become a trade mark for **LQG**) refers to the fact that **no splitting of a metric** is used.
- Notice, however, that some approximation scheme (perturbation theory) may well have to be developed to obtain sensible and testable (at least in principle) physical predictions.
- **Problem 3:** $\displaystyle\int_{\mathbb{R}_+}\frac{1}{r}\exp\left[-\frac{(\log r+s^2/2)^2}{2s^2}-Nr\right]\mathrm{d}r,\ (N\to\infty,\ s\to\infty).$

A prelude to quantization: GR Hamiltonian

The starting point for quantization of a system is to find a Hamiltonian describing its evolution. We will consider pure GR (no matter by now).

Can we give a Hamiltonian formulation for GR?

YES

Derive it from the Einstein-Hilbert action $S = \frac{1}{2\kappa}\int_{\mathcal{M}} \mathbf{e}\sqrt{\sigma g}R$; $\kappa = \frac{8\pi G_N}{c^3}$

- Here $\mathcal{M}$ is a 4-dim manifold $\mathcal{M} = \mathbb{R}\times\Sigma$ (global hyperbolicity [Geroch]). I will choose Σ to be a smooth, orientable, closed (i.e. compact and without boundary) manifold, and σ is the space-time signature ($\sigma = -1$ Lorentzian, $\sigma = +1$ Riemannian)
- Introduce a "time function" t defining a foliation of $\mathcal{M}$ by smooth 3-dim hypersurfaces Σ_t diffeomorphic to Σ and a "time flow direction t^a" (a globally defined smooth vector field such that $t^a\nabla_a t = 1$). Alternatively one can introduce a congruence of spacetime filling curves.

- Given a metric of signature $(-+++)$ it defines a unit time-like normal n^a on the points of each Σ_t.
- **Notation:** I use Penrose notation (no coordinate charts!) $(t^a \in \mathfrak{X}(\mathcal{M}),\ t_a \in \mathfrak{X}^*(\mathcal{M}),\ t^{ab} \in \mathfrak{X}(\mathcal{M}) \otimes \mathfrak{X}(\mathcal{M}) \dots)$
- Let us define:
  - The induced metric $h_{ab} = g_{ab} + n_a n_b$ (on vectors X^a tangent to each Σ_t we have $h_{ab}X^b = g_{ab}X^b := X_b$, and $h_{ab}n^a = 0$).
  - The lapse $N := -g_{ab}t^a n^b = (n^a \nabla_a t)^{-1}$, ∇_a denotes a torsion-free connection on $\mathcal{M}$.
  - The shift $N^a := h^a{}_b t^b = t^a - N n^a$.
- Let us define also:
  - The unique, torsion-free, derivative operator D_a on each Σ_t compatible with h_{ab}.
  - This is given by $D_a T_{b_1 \dots b_\ell}{}^{a_1,\dots a_k} := h^{a_1}_{d_1} \cdots h_{b_\ell}^{e_\ell} h_a{}^f \nabla_f T_{e_1 \dots e_\ell}{}^{d_1,\dots d_k}$.
  - The extrinsic curvature $K_{ab} := h_a{}^c \nabla_c n_b = \frac{1}{2N}(\mathcal{L}_t h_{ab} - 2D_{(a}N_{b)})$.

- The information contained in g_{ab} is the same as the one present in (N, N^a, h_{ab}) but these are "3-dimensional objects". Let us rewrite the E-H action in terms of these (The fiducial, non-dynamical volume form $\mathbf{e} := e_{abcd}$ is chosen so that it satisfies $\mathcal{L}_t e_{abcd} = 0$). Remember that K_{ab} can be written as a "time derivative" of h_{ab} and $D_a N_b$.

$$S = \int_{\mathcal{M}} \mathbf{e}\sqrt{h} N[\sigma^{(3)}R + K_{ab}K^{ab} - K^2] := \int_{\mathbb{R}} \int_{\Sigma_t} {}^{(3)}\mathbf{e}\mathcal{L}_G := \int_{\mathbb{R}} L_G$$

Here ${}^{(3)}e_{abc} := t^d e_{dabc}$.

The Hamiltonian density

- Canonical conjugate momenta

$$p^{ab} := \frac{\partial \mathcal{L}_G}{\partial \dot{h}_{ab}} = \sqrt{h}(K^{ab} - Kh^{ab})$$

- Hamiltonian density

$$S = \int_{\mathbb{R}} \int_{\Sigma_t} {}^{(3)}\mathbf{e}\mathcal{H}_G := \int_{\mathbb{R}} H_G$$

$$\mathcal{H}_G = p^{ab}\dot{h}_{ab} - \mathcal{L}_G = N\left[\sigma\sqrt{h}^{(3)}R + \frac{1}{\sqrt{h}}(p^{ab}p_{ab} - \frac{1}{2}p^2)\right]$$
$$-2N_b D_a(h^{-1/2}p^{ab})$$

- No time derivatives of N and N_a appear. They behave as Lagrange multipliers enforcing the **constraints**:
  - $\sigma\sqrt{h}\,{}^{(3)}R + \frac{1}{\sqrt{h}}(p^{ab}p_{ab} - \frac{1}{2}p^2) = 0$ (Scalar constraint)
  - $D_a(h^{-1/2}p^{ab}) = 0$ (Vector constraint)
- The evolution of h_{ab} and K_{ab} is given by the **Hamilton equations** (H_G denotes the Hamiltonian)
  - $\dot{h}_{ab} = \frac{\delta H_G}{\delta p^{ab}}, \qquad \dot{p}^{ab} = -\frac{\delta H_G}{\delta h_{ab}}$
  - $\dot{h}_{ab} = \{h_{ab}, H_G\}, \quad \dot{p}^{ab} = \{p^{ab}, H_G\}$
- Notice that we have only the Lagrange multiplier terms (N and N^a are essentially arbitrary).

The final description (disclaimer)

- Phase space $\Gamma(h_{ab}, p^{ab})$
- Symplectic structure: $\Omega = \int_\Sigma {}^{(3)}\mathbf{e}\,\delta h_{ab} \wedge \delta p^{ab}$. Alternatively the Poisson brackets are [these should be understood as brackets between weighted versions of the configuration and momentum variables]

$$\{h_{ab}(x), h_{cd}(y)\} = 0$$
$$\{p^{ab}(x), h_{cd}(y)\} = \delta^a_{(c}\delta^b_{d)}\delta^3(x,y)$$
$$\{p^{ab}(x), p^{cd}(y)\} = 0$$

- Constraints (first class)
  - $\sigma\sqrt{h}\,{}^{(3)}R + \frac{1}{\sqrt{h}}(p^{ab}p_{ab} - \frac{1}{2}p^2) = 0$ (Scalar constraint)
  - $D_a(h^{-1/2}p^{ab}) = 0$ (Vector constraint)
- They generate gauge transformations.

The $SO(3)-ADM$ formulation

- Let us perform now a simple change of variables.
- Introduce a triad i.e. three 1-forms e_a^i, $i = 1, 2, 3$ defining a frame at each point of Σ ($\det e \neq 0$)
- Write the metric as $h_{ab} = e_a^i e_b^j \delta_{ij}$
- Introduce $\tilde{E}_i^a = (\det e) e_i^a$ with $e_i^a e_{aj} = \delta_{ij}$ (densitized inverse triad).
- Define $K_a^i = \dfrac{1}{\det e} K_{ab} \tilde{E}_j^b \delta^{ij}$.
- $p^{ab} \dot{h}_{ab} \rightarrow \dot{\tilde{E}}_i^a K_a^i$ so the new variables are canonical.
- We can rewrite the constraints in terms of these variables. Before we do that it is important to realize that we have now **local** $SO(3)$ **rotations** of e_a^i and K_a^i that do not change neither h_{ab} nor K_{ab} so there must be extra constraints to generate them.
- These can easily be found from the condition $K_{[ab]} = 0$ (the 2^{nd} fundamental form is symmetric).

- Phase space $\Gamma(K_a^i, \tilde{E}_i^a)$
- Symplectic structure (Poisson brackets):

$$\begin{aligned} \{K_a^i(x), K_b^j(y)\} &= \{\tilde{E}_i^a(x), \tilde{E}_j^b(y)\} = 0 \\ \{\tilde{E}_i^a(x), K_b^j(y)\} &= \delta_j^i \delta_b^a \delta^3(x,y) \end{aligned}$$

- Constraints (first class). R is the scalar curvature of $h_{ab} := e_a^i e_{bi}$.

$$\begin{aligned} &\epsilon_{ijk} K_a^j \tilde{E}^{ak} = 0 \\ &\mathcal{D}_a \left[\tilde{E}_k^a K_b^k - \delta_b^a \tilde{E}_k^c K_c^k \right] = 0 \\ &-\sigma\sqrt{h}R + \frac{2}{\sqrt{h}} \tilde{E}_k^{[c} \tilde{E}_l^{d]} K_c^k K_d^l = 0 \end{aligned} \tag{1}$$

- Consider the transformation

$$
{}_{\gamma}\tilde{E}_i^a = -\frac{1}{\gamma}\tilde{E}_i^a
$$

$$
{}_{\gamma}A_a^i = \Gamma_a^i + \gamma K_a^i
$$

  - Γ_a^i is a $SO(3)$ connection that defines a covariant derivative compatible with the triad.
  - $\partial_{[a}e^i_{b]} + \epsilon^i{}_{jk}\Gamma^j_{[a}e^k_{b]} = 0$ can be inverted to get Γ_a^i is a $SO(3)$.
  - $\gamma \in \mathbb{C}$ is known as the Immirzi parameter (γ).
- The Poisson brackets between the new variables ${}_{\gamma}A_a^i$ and ${}_{\gamma}\tilde{E}_i^a$ are

$$
\{{}_{\gamma}A_a^i(x), {}_{\gamma}A_b^j(y)\} = \{{}_{\gamma}\tilde{E}_i^a(x), {}_{\gamma}\tilde{E}_j^b(y)\} = 0
$$

$$
\{{}_{\gamma}A_a^i(x), {}_{\gamma}\tilde{E}_j^b(y)\} = \delta_j^i\delta_a^b\delta^3(x,y)
$$

So **this is a canonical transformation**!

- Phase space $\Gamma({}_\gamma A^i_a, {}_\gamma \tilde{E}^a_i)$ [smooth $SO(3)$ connections and triads on Σ, i.e. a Yang-Mills phase space]
- Symplectic structure (Poisson brackets):
  The variables ${}_\gamma A_{ai}(x)$ and ${}_\gamma \tilde{E}^b_j(y)$ are canonical;
- Constraints (first class)

$$\begin{aligned}
&\mathcal{D}_a \tilde{E}^a_i = 0 && \text{Gauss law}\\
&F^i_{ab} \tilde{E}^b_i = 0 && \text{Vector c.}\\
&\tilde{E}^{[a}_i \tilde{E}^{b]}_j \left[\epsilon_{ijk} F^k_{ab} + \frac{2(\sigma - \gamma^2)}{\gamma^2} (A^i_a - \Gamma^i_a)(A^i_a - \Gamma^j_b) \right] = 0, && \text{Scalar c.}
\end{aligned}$$

- Where $\mathcal{D}_a \tilde{E}^a_i = \partial_a \tilde{E}^a_i + \epsilon_{ijk} A^j_a \tilde{E}^{ak}$, and $F^i_{ab} = 2\partial_{[a} A^i_{b]} + \epsilon^{ijk} A_{aj} A_{bk}$ is the $SO(3)$ curvature.

Comments

- GR in these new variables is a background independent relative of $SO(3)$ [or $SU(2)$] Yang-Mills theory.
- The fact that the configuration variable is a connection is a cornerstone of the formalism.

What happens with γ?

- If $\sigma = +1$ (Riemannian signature) we can cancel the last term by choosing $\gamma = \pm 1$. In this case the variables A_a^i and $\tilde{E}_i^a$ are **real** and the scalar constraint takes a very simple form.
- If $\sigma = -1$ (Lorentzian signatures, i.e. the real thing) we face two choices:
  - If we want to remove the ugly term we have to take $\gamma = \pm i$.
  - If we want to have real variables we have to live with complicated constraints.

- Once it was understood that even the complicated form of the constraints could be handled (more or less...) the emphasis was placed on the geometric meaning of the new variables.
- The fact that the "internal" symmetry group is compact is very important in the construction of the Hilbert spaces used to quantize the theory (a good reason to use real variables).
- This parameter shows up in the definition of the area and volume observables that are an essential ingredient of the formalism. In particular the entropy of a black hole is proportional to γ. By choosing its value to reproduce the Beckenstein formula one can correctly obtain the right entropy for any realistic ("astrophysical") black hole.
- It also appears in LQC.

The new formulation can be derived from an action principle

The Holst action

$$S = \int_{\mathcal{M}} e^I \wedge e^J \wedge (\epsilon_{IJKL}\Omega^{KL} - \frac{2}{\gamma}\Omega_{IJ})$$

- e^I takes values in a 4-dimensional $\mathbb{R}$-vector space ($I = 0, 1, 2, 3$).
- ${\omega^I}_J$ takes values in the Lie algebra of $SO(1,3)$ (Lorentzian signature) or $SO(4)$ (Riemannian signatures).
- ϵ_{IJKL} is the alternatong tensor in V.
- ${\Omega^I}_J = \mathrm{d}{\omega^I}_J + {\omega^I}_K \wedge \omega^K_J$ is the curvature 2-form of ${\omega^I}_J$.
- If $\gamma = i$ (Lor. case) or $\gamma = 1$ (Riem. case) the action can be written in terms of the self-dual curvature of a self-dual connection ${\omega^I}_J^+$.
- This action is invariant under diffeomorphisms of $\mathcal{M}$ and "internal" gauge transformations ($SO(1,3)$ or $SO(4)$ depending on the signature).