# Vakonomic Mechanics on Lie Affgebroids

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## Abstract

*We develop a constraint algorithm for pre-cosymplectic Lie algebroids which is a generalization of the constraint algorithms discussed in [2, 4]. We use our algorithm to present a geometric description of vakonomic mechanics on Lie affgebroids. In fact, we obtain the dynamical equations for a vakonomic system on a Lie affgebroid $\mathcal{A}$. Moreover, in the particular case when $\mathcal{A}$ is the standard Lie affgebroid, we recover the equations obtained in [7] (see also [1]).*

## Lie affgebroids

Let $\tau_{\mathcal{A}} : \mathcal{A} \to Q$ be an affine bundle with associated vector bundle $\tau_V : V \to Q$. Denote by $\tau_{\mathcal{A}^+} : \mathcal{A}^+ = Aff(\mathcal{A}, \mathbb{R}) \to Q$ the dual bundle which has a distinguished section $1_{\mathcal{A}} \in \Gamma(\tau_{\mathcal{A}^+})$ corresponding to the constant function $1$ on $\mathcal{A}$ and by $\tau_{\widetilde{\mathcal{A}}} : \widetilde{\mathcal{A}} = (\mathcal{A}^+)^* \to Q$ the bidual bundle.

**Definition 1.** *A Lie affgebroid structure on $\mathcal{A}$ consists of a Lie algebra structure $[\![\cdot,\cdot]\!]_V$ on $\Gamma(\tau_V)$, a $\mathbb{R}$-linear action $D : \Gamma(\tau_{\mathcal{A}}) \times \Gamma(\tau_V) \to \Gamma(\tau_V)$ and an affine map $\rho_{\mathcal{A}} : \mathcal{A} \to TQ$, the anchor map, satisfying:*

- $D_X[\![\bar{Y}, \bar{Z}]\!]_V = [\![D_X\bar{Y}, \bar{Z}]\!]_V + [\![\bar{Y}, D_X\bar{Z}]\!]_V$,
- $D_X(f\bar{Y}) = fD_X\bar{Y} + \rho_{\mathcal{A}}(X)(f)\bar{Y}$,
- $D_{X+\bar{Y}}\bar{Z} = D_X\bar{Z} + [\![\bar{Y}, \bar{Z}]\!]_V$,

*for $X \in \Gamma(\tau_{\mathcal{A}})$, $\bar{Y}, \bar{Z} \in \Gamma(\tau_V)$ and $f \in C^\infty(Q)$.*

A Lie affgebroid structure on $\tau_{\mathcal{A}} : \mathcal{A} \to Q$ induces a Lie algebroid structure $([\![\cdot,\cdot]\!]_{\widetilde{\mathcal{A}}}, \rho_{\widetilde{\mathcal{A}}})$ on the bidual bundle $\widetilde{\mathcal{A}}$ s.t. $1_{\mathcal{A}} \in \Gamma(\tau_{\mathcal{A}^+})$ is a $1$-cocycle (i.e. $d^{\widetilde{\mathcal{A}}}1_{\mathcal{A}} = 0$). Conversely, if $(U, [\![\cdot,\cdot]\!]_U, \rho_U)$ is a Lie algebroid over $Q$ and $\phi : U \to \mathbb{R}$ is a $1$-cocycle s.t. $\phi_{|U_x} \neq 0$, $\forall x \in Q$, then $\tau_{\mathcal{A}} : \mathcal{A} = \phi^{-1}\{1\} \to Q$ admits a Lie affgebroid structure s.t. $(\widetilde{\mathcal{A}}, [\![\cdot,\cdot]\!]_{\widetilde{\mathcal{A}}}, \rho_{\widetilde{\mathcal{A}}}) \cong (U, [\![\cdot,\cdot]\!]_U, \rho_U)$, the $1$-cocycle $1_{\mathcal{A}} \equiv \phi$ and $\tau_{\mathcal{A}} : \mathcal{A} \to Q$ is modelled on the vector bundle $\tau_V : V = \phi^{-1}\{0\} \to Q$ (for more details, see [6]).

**Example 1.** Let $\tau : Q \to \mathbb{R}$ be a fibration. The $1$-jet bundle $\tau_{1,0} : J^1\tau \to Q$ of local sections of $\tau : Q \to \mathbb{R}$ is an affine bundle modelled on the vector bundle $\pi = (\pi_Q)_{|V\tau} : V\tau \to Q$. Moreover, if $t$ is the usual coordinate on $\mathbb{R}$ and $\eta = \tau^*(dt) \in T^*Q$, then $J^1\tau \cong \{v \in TQ \,|\, \eta(v) = 1\}$. Note that $V\tau = \{v \in TQ \,|\, \eta(v) = 0\}$. Thus, the bidual bundle $\widetilde{J^1\tau} \cong TQ$. Therefore, the affine bundle $\tau_{1,0} : J^1\tau \to Q$ admits a Lie affgebroid structure. In fact, the Lie algebroid structure on $\pi_Q : TQ \to Q$ is the standard Lie algebroid structure and the $1$-cocycle $1_{J^1\tau}$ is just the closed $1$-form $\eta$. $\triangle$

Now, let $(x^i)$ be local coordinates on $Q$ and $\{e_0, e_\alpha\}$ be a local basis of $\Gamma(\tau_{\widetilde{\mathcal{A}}})$ adapted to $1_{\mathcal{A}}$, i.e., $1_{\mathcal{A}}(e_0) = 1$ and $1_{\mathcal{A}}(e_\alpha) = 0$, $\forall\alpha$. Denote by $(x^i, y^0, y^\alpha)$ the corresponding local coordinates on $\widetilde{\mathcal{A}}$. Then, the local equation defining the affine subbundle $\mathcal{A}$ (resp., the vector subbundle $V$) of $\widetilde{\mathcal{A}}$ is $y^0 = 1$ (resp., $y^0 = 0$).

## Lagrangian mechanics on Lie affgebroids

We consider the Lie algebroid prolongation $(\tau^{\tau_{\mathcal{A}}}_{\widetilde{\mathcal{A}}} : \mathcal{T}^{\widetilde{\mathcal{A}}}\mathcal{A} \to \mathcal{A}, [\![\cdot,\cdot]\!]^{\tau_{\mathcal{A}}}_{\widetilde{\mathcal{A}}}, \rho^{\tau_{\mathcal{A}}}_{\widetilde{\mathcal{A}}})$ of the Lie algebroid $(\widetilde{\mathcal{A}}, [\![\cdot,\cdot]\!]_{\widetilde{\mathcal{A}}}, \rho_{\widetilde{\mathcal{A}}})$ over the fibration $\tau_{\mathcal{A}} : \mathcal{A} \to Q$ (see [6]). We define a local basis $\{\mathcal{X}_0, \mathcal{X}_\alpha, \mathcal{V}_\alpha\}$ of $\Gamma(\tau^{\tau_{\mathcal{A}}}_{\widetilde{\mathcal{A}}})$:

$$\mathcal{X}_0(\mathrm{a}) = \Big(e_0(\tau_{\mathcal{A}}(\mathrm{a})), \rho_0^i\frac{\partial}{\partial x^i}_{|\mathrm{a}}\Big), \quad \mathcal{X}_\alpha(\mathrm{a}) = \Big(e_\alpha(\tau_{\mathcal{A}}(\mathrm{a})), \rho_\alpha^i\frac{\partial}{\partial x^i}_{|\mathrm{a}}\Big), \quad \mathcal{V}_\alpha(\mathrm{a}) = \Big(0, \frac{\partial}{\partial y^\alpha}_{|\mathrm{a}}\Big),$$

where $\rho_0^i, \rho_\alpha^i$ are the components of the anchor map $\rho_{\widetilde{\mathcal{A}}}$. If $\{\mathcal{X}^0, \mathcal{X}^\alpha, \mathcal{V}^\alpha\}$ is the dual basis of $\{\mathcal{X}_0, \mathcal{X}_\alpha, \mathcal{V}_\alpha\}$, then $\mathcal{X}^0$ is globally defined and it is a $1$-cocycle. We will denote by $\phi_0$ the $1$-cocycle $\mathcal{X}^0$.

One may also consider *the vertical endomorphism* $S : \mathcal{T}^{\widetilde{\mathcal{A}}}\mathcal{A} \to \mathcal{T}^{\widetilde{\mathcal{A}}}\mathcal{A}$, as a section of the vector bundle $\mathcal{T}^{\widetilde{\mathcal{A}}}\mathcal{A} \otimes (\mathcal{T}^{\widetilde{\mathcal{A}}}\mathcal{A})^* \to \mathcal{A}$, whose local expression is $S = (\mathcal{X}^\alpha - y^\alpha\phi_0) \otimes \mathcal{V}_\alpha$.

A section $\xi \in \Gamma(\tau^{\tau_{\mathcal{A}}}_{\widetilde{\mathcal{A}}})$ is said to be a *second order differential equation (SODE)* on $\mathcal{A}$ if $\phi_0(\xi) = 1$ and $S\xi = 0$. In such a case, $\xi = \mathcal{X}_0 + y^\alpha\mathcal{X}_\alpha + \xi^\alpha\mathcal{V}_\alpha$, where $\xi^\alpha$ are local functions on $\mathcal{A}$.

Now, a curve $\gamma : I \subseteq \mathbb{R} \to \mathcal{A}$ in $\mathcal{A}$ is said to be admissible if $(i_{\mathcal{A}}(\gamma(t)), \dot\gamma(t)) \in \mathcal{T}^{\widetilde{\mathcal{A}}}_{\gamma(t)}\mathcal{A}$, for all $t \in I$. Here, $i_{\mathcal{A}} : \mathcal{A} \to \widetilde{\mathcal{A}}$ is the canonical inclusion.

It is clear that if $\xi$ is a SODE then the integral curves of the vector field $\rho^{\tau_{\mathcal{A}}}_{\widetilde{\mathcal{A}}}(\xi)$ are admissible.

On the other hand, let $L : \mathcal{A} \to \mathbb{R}$ be a Lagrangian function. We introduce *the Poincaré-Cartan $1$-section* $\Theta_L \in \Gamma((\tau^{\tau_{\mathcal{A}}}_{\widetilde{\mathcal{A}}})^*)$ and *the Poincaré-Cartan $2$-section* $\Omega_L \in \Gamma(\wedge^2(\tau^{\tau_{\mathcal{A}}}_{\widetilde{\mathcal{A}}})^*)$ associated with $L$ defined by

$$\Theta_L = L\phi_0 + (d^{\mathcal{T}^{\widetilde{\mathcal{A}}}\mathcal{A}}L)\circ S, \quad \Omega_L = -d^{\mathcal{T}^{\widetilde{\mathcal{A}}}\mathcal{A}}\Theta_L.$$

A curve $\gamma : I = (-\epsilon, \epsilon) \subseteq \mathbb{R} \to \mathcal{A}$ on $\mathcal{A}$ is *a solution of the Euler-Lagrange equations associated with $L$* iff $\gamma$ is admissible and $i_{(i_{\mathcal{A}}(\gamma(t)),\dot\gamma(t))}\Omega_L(\gamma(t)) = 0$, for all $t$ (see [6]).
If $\gamma(t) = (x^i(t), y^\alpha(t))$ then $\gamma$ is a solution of the Euler-Lagrange equations iff

$$\frac{dx^i}{dt} = \rho_0^i + \rho_\alpha^i y^\alpha, \quad \frac{d}{dt}\Big(\frac{\partial L}{\partial y^\alpha}\Big) = \rho_\alpha^i\frac{\partial L}{\partial x^i} + (C_{0\alpha}^\gamma + C_{\beta\alpha}^\gamma y^\beta)\frac{\partial L}{\partial y^\gamma},$$

$C_{0\alpha}^\gamma, C_{\beta\alpha}^\gamma$ being the (local) structure functions of $[\![\cdot,\cdot]\!]_{\widetilde{\mathcal{A}}}$ with respect to the basis $\{e_0, e_\alpha\}$.

The Lagrangian $L$ is regular iff the matrix $(W_{\alpha\beta}) = \left(\dfrac{\partial^2 L}{\partial y^\alpha\partial y^\beta}\right)$ is regular or, in other words, the pair $(\Omega_L, \phi_0)$ is a cosymplectic structure on $\mathcal{T}^{\widetilde{\mathcal{A}}}\mathcal{A}$. In such a case, the Reeb section $R_L$ of $(\Omega_L, \phi_0)$ is the unique Lagrangian SODE associated with $L$ and, thus, the integral curves of $\rho^{\tau_{\mathcal{A}}}_{\widetilde{\mathcal{A}}}(R_L)$ are solutions of the Euler-Lagrange equations associated with $L$. $R_L$ is called *the Euler-Lagrange section associated with $L$.*

## Constraint Algorithm for precosymplectic Lie algebroids

Let $\tau_E : E \to Q$ be a Lie algebroid, $\Omega \in \Gamma(\wedge^2\tau_E^*)$ be a presymplectic 2-section ($d^E\Omega = 0$) and $\eta \in \Gamma(\tau_E^*)$ be a closed $1$-section ($d^E\eta = 0$).

The dynamics of the precosymplectic system defined by $(\Omega, \eta)$ is given by a section $X \in \Gamma(\tau_E)$ satisfying the dynamical equations

$$i_X\Omega = 0 \;\text{ and }\; i_X\eta = 1\,. \tag{1}$$

Now, we take an arbitrary section $Y \in \Gamma(\tau_E)$ s.t. $i_Y\eta = 1$. Then, $\Omega = \Omega^V + \eta \wedge i_Y\Omega$. Thus, for $e \in E_x$:

$$i_e\Omega(x) = 0 \;\text{ and }\; i_e\eta(x) = 1 \Leftrightarrow i_e\Omega^V(x) = -i_{Y_x}\Omega(x) \;\text{ and }\; i_e\eta(x) = 1.$$

We define the vector bundle morphism $\flat : E \to E^*$ (over the identity of $Q$) as follows

$$\flat(e) = i(e)\Omega^V(x) + \eta(x)(e)\eta(x), \quad \text{for } e \in E_x.$$

If $x \in Q$ and $F_x$ is a subspace of $E_x$, we may introduce the vector subspace $F_x^\perp$ of $E_x$ given by

$$F_x^\perp = (\flat(F_x))^\circ = \{e \in E_x \,|\, (-i_e\Omega^V(x) + \eta(x)(e)\eta(x))(f) = 0, \forall f \in F_x\}.$$

In general, a section $X$ satisfying (1) cannot be found in all points of $Q$. We define

$$Q_1 = \{x \in Q \,|\, \exists e \in E_x : i_e\Omega(x) = 0 \text{ and } i_e\eta(x) = 1\} = \{x \in Q \,|\, \eta(x) - i_{Y_x}\Omega(x) \in \flat(E_x) = (E_x^\perp)^\circ\}.$$

If $Q_1$ is an embedded submanifold of $Q$, then there exists $X : Q_1 \to E$ a section of $\tau : E \to Q$ along $Q_1$ s.t. (1) holds. But $\rho(X)$ is not, in general, tangent to $Q_1$ ($\rho : E \to TQ$ is the anchor map of the Lie algebroid $E$). Thus, we have that to restrict to $E_1 = \rho^{-1}(TQ_1)$. If $E_1$ is a manifold and $\tau_1 = \tau_{|E_1} : E_1 \to Q_1$ is a vector bundle, then $\tau_1 : E_1 \to Q_1$ is a Lie subalgebroid of $E \to Q$.

Now, we must consider

$$Q_2 = \{x \in Q_1 \,|\, \eta(x) - i_{Y_x}\Omega(x) \in \flat((E_1)_x)\} = \{x \in Q_1 \,|\, (\eta(x) - i_{Y_x}\Omega(x))(e) = 0, \forall e \in (E_1)_x^\perp\}.$$

If $Q_2$ is an embedded submanifold of $Q_1$, then there exists $X : Q_2 \to E_1$ a section of $\tau_1 : E_1 \to Q_1$ along $Q_2$ such that (1) holds. However, $\rho(X)$ is not, in general, tangent to $Q_2$. Therefore, we have that to restrict to $E_2 = \rho^{-1}(TQ_2)$. As above, if $E_2$ is a manifold and $\tau_2 = \tau_{|E_2} : E_2 \to Q_2$ is a vector bundle, it follows that $\tau_2 : E_2 \to Q_2$ is a Lie subalgebroid of $\tau_1 : E_1 \to Q_1$.

Consequently, if we repeat the process, we obtain a sequence of Lie subalgebroids (by assumption):

$$\begin{tikzcd}
\ldots \arrow[r, hook] & Q_{k+1} \arrow[r, hook] & Q_k \arrow[r, hook] & \ldots \arrow[r, hook] & Q_2 \arrow[r, hook] & Q_1 \arrow[r, hook] & Q_0 = Q \\
\ldots \arrow[r, hook] & E_{k+1} \arrow[r, hook] \arrow[u, "\tau_{k+1}"'] & E_k \arrow[r, hook] \arrow[u, "\tau_k"'] & \ldots \arrow[r, hook] & E_2 \arrow[r, hook] \arrow[u, "\tau_2"'] & E_1 \arrow[r, hook] \arrow[u, "\tau_1"'] & E_0 = E \arrow[u, "\tau_0 = \tau"']
\end{tikzcd}$$

where $Q_{k+1} = \{x \in Q_k \,|\, (\eta(x) - i_{Y_x}\Omega(x))(e) = 0, \forall e \in (E_k)_x^\perp\}$ and $E_{k+1} = \rho^{-1}(TQ_{k+1})$. If $\exists k \in \mathbb{N}$ s.t. $Q_k = Q_{k+1}$, then we say that *the sequence stabilizes*. In such a case, $\tau_f = \tau_k : E_f = E_k \to Q_f = Q_k$ is a Lie subalgebroid of $\tau : E \to Q$ and $\exists X \in \Gamma(\tau_f)$ (but non necessarily unique), satisfying (1).

## Vakonomic equations

Let $\tau_{\mathcal{A}} : \mathcal{A} \to Q$ be a Lie affgebroid of rank $n$ over a manifold $Q$ of dimension $m$, $L : \mathcal{A} \to \mathbb{R}$ be a Lagrangian function and $\mathcal{M} \subseteq \mathcal{A}$ be an embedded submanifold of dimension $m + (n - \bar{m})$, *the constraint submanifold*, such that $\tau_{\mathcal{M}} = \tau_{\mathcal{A}|\mathcal{M}} : \mathcal{M} \to Q$ is a surjective submersion. Denote by $\tilde{L}$ the restriction of $L$ to $\mathcal{M}$.

Then, we can choose local coordinates $(x^i, y^\alpha) = (x^i, y^A, y^a)$ on $\mathcal{A}$, with $1 \le \alpha \le n$, $1 \le A \le \bar{m}$ and $\bar{m} + 1 \le a \le n$ such that $\mathcal{M} \equiv \{(x^i, y^\alpha) \,|\, y^A = \Psi^A(x^i, y^a),\ A = 1, \ldots, \bar{m}\}$. Thus, $(x^i, y^a)$ are local coordinates on $\mathcal{M}$.

We consider the following diagrams

$$\begin{tikzcd}
 & \mathcal{A}^+ \oplus_Q \mathcal{A} \arrow[dl, "\mathrm{pr}_1"'] \arrow[dr, "\mathrm{pr}_2"] & \\
\mathcal{A}^+ & & \mathcal{A}
\end{tikzcd}
\qquad
\begin{tikzcd}
 & W_0 = pr_2^{-1}(\mathcal{M}) = \mathcal{A}^+ \oplus_Q \mathcal{M} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
\mathcal{A}^+ & & \mathcal{M}
\end{tikzcd}$$

Denote by $\nu : W_0 \to Q$ the canonical projection whose local expression is $\nu(x^i, y_0, y_\alpha, y^a) = (x^i)$.

We prolong $\pi_1 : W_0 \to \mathcal{A}^+$ to a Lie algebroid morphism $\mathcal{T}\pi_1 : \mathcal{T}^{\widetilde{\mathcal{A}}}W_0 \to \mathcal{T}^{\widetilde{\mathcal{A}}}\mathcal{A}^+$ defined by $\mathcal{T}\pi_1 = (Id, T\pi_1)$. Then, $\Omega = (\mathcal{T}\pi_1, \pi_1)^*\Omega_{\widetilde{\mathcal{A}}}$ is a presymplectic 2-section on $\mathcal{T}^{\widetilde{\mathcal{A}}}W_0$, $\Omega_{\widetilde{\mathcal{A}}}$ being the canonical symplectic section associated with the Lie algebroid $\widetilde{\mathcal{A}}$ (see [5]).

*The Pontryagin Hamiltonian* is the function $H_{W_0} : W_0 \to \mathbb{R}$ given by $H_{W_0}(\varphi, \mathrm{a}) = \varphi(\mathrm{a}) - \tilde{L}(\mathrm{a})$.

We define the $1$-cocycle $\eta \in \Gamma((\tau^{\nu}_{\widetilde{\mathcal{A}}})^*)$ given by $\eta(\tilde{\mathrm{a}}, X) = 1_{\mathcal{A}}(\tilde{\mathrm{a}})$, for $(\tilde{\mathrm{a}}, X) \in \mathcal{T}^{\widetilde{\mathcal{A}}}W_0$ and we consider the presymplectic 2-section $\Omega_0 = \Omega + d^{\mathcal{T}^{\widetilde{\mathcal{A}}}W_0}H_{W_0} \wedge \eta$ on $\mathcal{T}^{\widetilde{\mathcal{A}}}W_0$.

The vakonomic problem $(L, \mathcal{M})$ on $\mathcal{A}$ is find the solutions for the equations

$$i_X\Omega_0 = 0 \;\text{ and }\; i_X\eta = 1, \; X \in \Gamma(\tau^{\nu}_{\widetilde{\mathcal{A}}}), \tag{2}$$

i.e., to solve the constraint algorithm for $(\mathcal{T}^{\widetilde{\mathcal{A}}}W_0, \Omega_0, \eta)$.

Note that in the free case, i.e., $\mathcal{M} = \mathcal{A}$, we obtain a Skinner and Rusk formulation for the Lagrangian function $L$ on the Lie affgebroid $\mathcal{A}$. It is a extension of the results obtained in [3].

The equations (2) only have sense in the points of $W_0$ satisfying that $y_a = \dfrac{\partial\tilde{L}}{\partial y^a} - y_A\dfrac{\partial\Psi^A}{\partial y^a}$, $\forall a$. If $\{\mathcal{Y}_0, \mathcal{Y}_\alpha, \mathcal{P}_0, \mathcal{P}_\alpha, \mathcal{V}_a\}$ is the local basis of $\Gamma(\tau^{\nu}_{\widetilde{\mathcal{A}}})$ given by

$$\mathcal{Y}_0(\varphi, \mathrm{a}) = \Big(e_0(\nu(\varphi, \mathrm{a})), \rho_0^i\frac{\partial}{\partial x^i}_{|\varphi}, 0\Big), \quad \mathcal{Y}_\alpha(\varphi, \mathrm{a}) = \Big(e_\alpha(\nu(\varphi, \mathrm{a})), \rho_\alpha^i\frac{\partial}{\partial x^i}_{|\varphi}, 0\Big),$$

$$\mathcal{P}_0(\varphi, \mathrm{a}) = \Big(0, \frac{\partial}{\partial y_0}_{|\varphi}, 0\Big), \quad \mathcal{P}_\alpha(\varphi, \mathrm{a}) = \Big(0, \frac{\partial}{\partial y_\alpha}_{|\varphi}, 0\Big), \quad \mathcal{V}^a(\varphi, \mathrm{a}) = \Big(0, 0, \frac{\partial}{\partial y^a}_{|\mathrm{a}}\Big),$$

then, a solution of (2) is of this form

$$X = \mathcal{Y}_0 + \Psi^A\mathcal{Y}_A + y^a\mathcal{Y}_a + X^0\mathcal{P}_0 + \Big[\rho_\alpha^i\Big(\frac{\partial\tilde{L}}{\partial x^i} - y_B\frac{\partial\Psi^B}{\partial x^i}\Big) + y_\gamma(C_{0\alpha}^\gamma + \Psi^B C_{B\alpha}^\gamma + y^b C_{b\alpha}^\gamma)\Big]\mathcal{P}_\alpha + X_a\mathcal{V}^a.$$

A curve $\sigma : t \mapsto \sigma(t) = (x^i(t), y_0(t), y_\alpha(t), y^a(t))$ on $W_0$ is *a solution of the vakonomics equations associated with $(L, \mathcal{M})$* if

$$\begin{cases} \dot{x}^i = \rho_0^i + \Psi^A\rho_A^i + y^a\rho_a^i, \\ \dot{y}_A = \Big(\dfrac{\partial\tilde{L}}{\partial x^i} - y_B\dfrac{\partial\Psi^B}{\partial x^i}\Big)\rho_A^i - y_\gamma(C_{A0}^\gamma + \Psi^B C_{AB}^\gamma + y^a C_{Aa}^\gamma), \\ \dfrac{d}{dt}\left(\dfrac{\partial\tilde{L}}{\partial y^a} - y_A\dfrac{\partial\Psi^A}{\partial y^a}\right) = \Big(\dfrac{\partial\tilde{L}}{\partial x^i} - y_A\dfrac{\partial\Psi^A}{\partial x^i}\Big)\rho_a^i - y_\gamma(C_{a0}^\gamma + \Psi^B C_{aB}^\gamma + y^b C_{ab}^\gamma). \end{cases} \tag{3}$$

**Example 2.** *Let $\tau_{1,0} : J^1\tau \to Q$ be the Lie affgebroid associated with the fibration $\tau : Q \to \mathbb{R}$. Thus, we can consider a constrained system $(L, \mathcal{M})$ on $J^1\tau$. In this case, the vakonomic equations (3) are just the equations obtained in [7] (see also [1]). Note that if $(t, q^\alpha)$ are local coordinates on $Q$ which are adapted to the fibration $\tau$, then $\{e_0 = \frac{\partial}{\partial t}, e_\alpha = \frac{\partial}{\partial q^\alpha}\}$ is a local basis of $\widetilde{\mathcal{A}} \equiv TQ$ adapted to $1_{\mathcal{A}} = \eta$. Furthermore, we have that*

$$\rho_{\widetilde{\mathcal{A}}}(e_0) = \frac{\partial}{\partial t}, \quad \rho_{\widetilde{\mathcal{A}}}(e_\alpha) = \frac{\partial}{\partial q^\alpha}, \quad [\![e_0, e_\alpha]\!]_{\widetilde{\mathcal{A}}} = [\![e_\alpha, e_\beta]\!]_{\widetilde{\mathcal{A}}} = 0.$$

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