Functionally Graded Media

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Abstract

We introduce the classical concept of uniform material through groupoids and frame bundles. We also give an equivalent framework for Functionally Graded Media describing some properties relative to their homogeneity. Later, we focus on elastic materials, solids and fluids. This poster is based on the preprint [1].

Part I
Geometric Structures

1. Groupoids

Definition 1.1. Given two sets Ω and M, a *groupoid* Ω *over* M, the *base*, consists of these two sets together with two maps $\alpha, \beta : \Omega \to M$, called the *source* and the *target projections*, and a composition law satisfying the following conditions:

1. The composition law is defined only for those $\eta, \xi \in \Omega$ such that $\alpha(\eta) = \beta(\xi)$ and, in this case, $\alpha(\eta\xi) = \alpha(\xi)$ and $\beta(\eta\xi) = \beta(\eta)$. We will denote Ω_Δ the set of such pairs of elements.
2. The composition law is associative, that is $\zeta(\eta\xi) = (\zeta\eta)\xi$ for those $\zeta, \eta, \xi \in \Omega$ such that one of the members of the previous equality is well defined.
3. For each $x \in M$ there exists an element $1_x \in \Omega$, called the *unity over* x, such that
   (a) $\alpha(1_x) = \beta(1_x) = x$;
   (b) $\eta \cdot 1_x = \eta$, whenever $\alpha(\eta) = x$;
   (c) $1_x \cdot \xi = \xi$, whenever $\beta(\xi) = x$.
4. For each $\xi \in \Omega$ there exists an element $\xi^{-1} \in \Omega$, called the *inverse of* ξ, such that
   (a) $\alpha(\xi^{-1}) = \beta(\xi)$ and $\beta(\xi^{-1}) = \alpha(\xi)$;
   (b) $\xi^{-1}\xi = 1_{\alpha(\xi)}$ and $\xi\xi^{-1} = 1_{\beta(\xi)}$.

The groupoid Ω will be said *transitive* if, for every pair $x, y \in M$, the set of elements that have x as source and y as target, *i.e.* $\Omega_{x,y} := \alpha^{-1}(x) \cap \beta^{-1}(y)$, is not empty.
A subset $\Omega' \subset \Omega$ is said to be a *subgroupoid of* Ω *over* M if itself is a groupoid over M with the restriction of the structural maps of Ω.

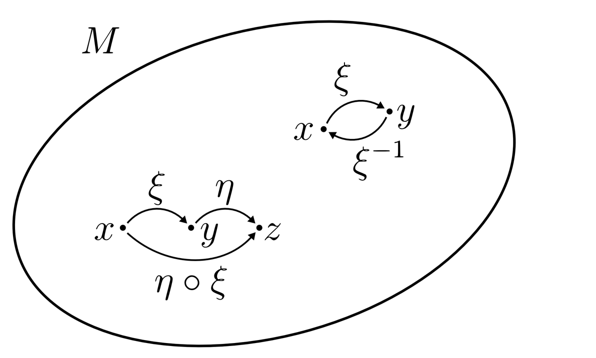

Figure 1: *The arrow picture.*

Definition 1.2. We say that a groupoid Ω over M is a *differential groupoid* if the groupoid Ω and the base M are endowed by respective differential structures such that:

1. the source and the target projections $\alpha, \beta : \Omega \to M$ are smooth surjective submersions;
2. the *unity* or *inclusion map* $i : x \in M \mapsto 1_x \in \Omega$ is smooth;
3. and the composition law, defined on Ω_Δ, is smooth.

Additionally if Ω is transitive, then we call it a *Lie groupoid*.
A subgroupoid Ω' of a differential groupoid Ω which is in turn a differential groupoid with the restricted differential structure is called a *differential subgroupoid*.

Example 1.3. • The *frame groupoid* of a smooth manifold,

$$\Pi(M) = \bigcup_{x,y \in M} \mathrm{Iso}(T_xM, T_yM). \tag{1.1}$$

• The *unimodular groupoid* of a smooth manifold with respect to a volume form (or density) ρ,

$$\mathcal{U}(M) = \det{}_\rho^{-1}(\pm 1), \tag{1.2}$$

• The *orthogonal groupoid* of a Riemannian manifold,

$$\mathcal{O}(M) = \left\{ A \in \Pi(M) \ : \ A^{-1} = A^T \right\}. \tag{1.3}$$

These groupoids are Lie and, whenever they are well defined, we have the relations:

$$\mathcal{O}(M) \leq \mathcal{U}(M) \leq \Pi(M). \tag{1.4}$$

2. G-structures

Definition 2.1. Given a smooth manifold M, a G*-structure* $G(M)$ is a G-reduction of the frame bundle $\mathcal{F}M$.

Note that there may be different G-structures with the same structure group. Once it is fixed, the linear frames that lie in the G-structure are called *adapted* or *distinguished* references. As an example of G-structure, consider the set of orthonormal references of a Riemannian manifold (M^n, g). This set is a $O(n)$-reduction of $\mathcal{F}M$ and, in fact, any $O(n)$-structure on M is equivalent to a Riemannian structure (see [4]).

Proposition 2.2 (*cf.* [6])**.** *Let* Ω *be a Lie groupoid over a smooth manifold* M *with source and target projections* α *and* β*, respectively. Given any point* $x \in B$*, we have that:*

1. $\Omega_{x,x}$ *is a Lie group and*
2. $\Omega_x = \alpha^{-1}(x)$ *is a principal* $\Omega_{x,x}$*-bundle over* B *whose canonical projection is the restriction of* β*.*

Any reference z at some point x of a smooth n-manifold M may be seen as the linear mapping $e_i \in \mathbb{R}^n \mapsto v_i \in T_xM$, where $(e_1, \dots, e_n)$ is the canonical basis of $\mathbb{R}^n$ and $(v_1, \dots, v_n)$ the basis defined by z.

Theorem 2.3 (*cf.* [6])**.** *Given a smooth* n*-dimensional manifold* M*, let* Ω *be a Lie subgroupoid of the frame groupoid* $\Pi(M)$ *and denote by* α *and* β *the respective source and target projections of* Ω*. We have that for any point* $x \in M$ *and any frame reference* $z \in \mathcal{F}M$ *at* x*:*

1. $G_z := z^{-1} \cdot \Omega_{x,x} \cdot z$ *is a Lie subgroup of* $\mathrm{Gl}(n)$ *and*
2. *the set* Ω_z *of all the linear frames obtained by translating* z *by* Ω_x*, that is*

$$\Omega_z = \left\{ g_{x,y} \cdot z \ : \ g_{x,y} \in \Omega_x \right\}, \tag{2.1}$$

is a G_z*-structure on* M*.*

Theorem 2.4 (*cf.* [1])**.** *Let* ω *be a* G*-structure over an* n*-dimensional smooth manifold* M*, then the set*

$$\Omega = \left\{ A \in \Pi(M) \ : \ Az \in \omega \ \forall z \in \omega_{\alpha(A)} \right\}, \tag{2.2}$$

where $\Pi(M)$ *is the frame groupoid of* M *and* α *the source projection, is a Lie soubgroupoid of* $\Pi(M)$*. Furthermore, for any reference frame* $z \in \omega$*, the* G*-structure associated to* Ω *and given by theorem 2.3 coincides with* ω*,* i.e.

$$\Omega_z = \omega \quad \text{and} \quad G_z = G. \tag{2.3}$$

Proposition 2.5. *Let* M *be a manifold. If* Ω *and* $\tilde{\Omega}$ *are two subgroupoids of the frame groupoid* $\Pi(M)$*, then their intersection* $\hat{\Omega} := \Omega \cap \tilde{\Omega}$ *is again a subgroupoid of* $\Pi(M)$ *(and of the original ones). Furthermore, if they are Lie groupoids, then we have the following relations:*

$$\hat{\Omega}_z = \Omega_z \cap \tilde{\Omega}_z \quad \text{and} \quad \hat{G}_z = G_z \cap \tilde{G}_z, \tag{2.4}$$

where $z \in \mathcal{F}M$ *is a fixed frame and* Ω_z*,* $\tilde{\Omega}_z$*,* $\hat{\Omega}_z$*,* G_z*,* $\tilde{G}_z$ *and* $\hat{G}_z$ *are the respective* G*-structures and structural groups.*

Definition 2.6. A G-structure $G(M)$ over a manifold M is said to be *integrable* if there exists an atlas $\{(U_\alpha, \phi_\alpha)\}_{\alpha \in A}$ of the base manifold, such that the induced cross-sections $\sigma_\alpha(x) := (T_x\phi_\alpha)^{-1}$ take values in $G(M)$.

Part II
Continuum Mechanics

3. The Constitutive Equation

In the more general sense (see [7], for instance), a *body* is a smooth manifold B^n that can be embedded in a Riemannian manifold (S^n, g), the *ambient space*. Usually, the body B is a simply connected open set of $\mathbb{R}^n$ and the ambient space is $\mathbb{R}^n$ itself with the standard metric. Each embedding $K : B \to S$ is called a *configuration* and its tangent map $TK : TB \to TS$ is called an *infinitesimal configuration*. If we fix a configuration K_r (the *reference configuration*) and we pick an arbitrary configuration K, then the embedding compositon $\phi = K \circ K_r^{-1} : K_r(B) \subset S \to S$ is considered as a body *deformation* and we call the *infinitesimal deformation* $T\phi$ the *deformation gradient*, usually denoted by F. Since (S, g) is a Riemannian manifold, we can induce a riemmanian metric on B by the pull-back of g by the reference configuration K_r. Thus, the metric on B is not canonical, since it depends on the chosen reference configuration. In the case of solid materials, we are able to define canonically a Riemannian metric on them (section §6).

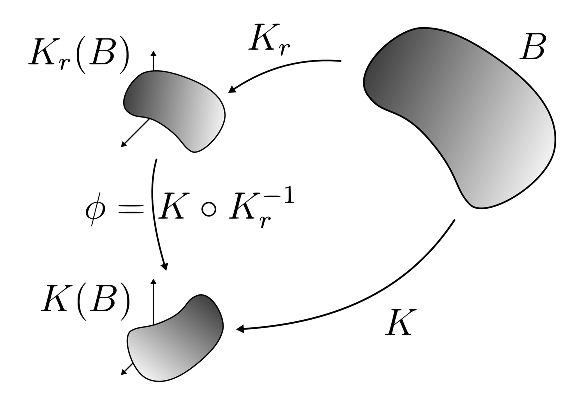

Figure 2: *Deformation in a reference configuration.*

As stated by *the principle of determinism*, the mechanical and thermal behaviors of a material are determined by a relation called *the constitutive equation*. In our case of interest, *elastic materials*, the constitutive equation establishes that, in a given reference configuration, *the Cauchy stress tensor* depends only on the material points and on the infinitesimal deformations applied on them, that is

$$\sigma = \sigma(F_{K_r}, K_r(X)). \tag{3.1}$$

This relation is simplified in the particular case of *hyperelastic materials*, for which equation (3.1) becomes

$$W = W(F_{K_r}, K_r(X)). \tag{3.2}$$

where W is a scalar valued funtion which measures the stored energy per unit volume.
Among other postulates (*principle of determinism*, *principle of local action*, *principle of frame-indifference*, etc.), it is claimed that the constitutive equation must not depend on the reference configuration. It turns out that equation (3.1) (and (3.2)) now can be written in the form

$$\sigma = \sigma(F, X) \quad (W = W(F, X), \text{ respectively}), \tag{3.3}$$

where F stands for the tangent map at X of a local configuration (deformation).

Definition 3.1. We say that two points $X, Y \in B$ are *materially isomorphic* (made of same material) if there exists a linear isomorphism $P_{XY} : T_XB \to T_YB$ such that

$$\sigma(F \cdot P_{XY}, X) = \sigma(F, Y), \tag{3.4}$$

for any deformation F at Y.
The linear isomorphism P_{XY} is called a *material isomorphism* when $X \neq Y$ and a *material symmetry* when $X = Y$. The set of material symmetries at a material point $X \in B$ is denoted by $\mathcal{G}(X)$ and it is called the *symmetry group* of B at X.

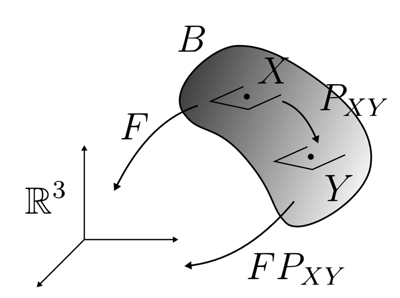

Figure 3: *Material isomorphism.*

Even if the definition of material isomorphism and material symmetries are mathematically the same, there is an important conceptual difference. While the symmetry group of a point characterizes the material behaviors of that point, a material isomorphism establishes a relation between two points.

4. Uniformity and Homogeneity

Definition 4.1. The set of material isomorphisms and symmetries of a body B is called the *material groupoid*, that is the set

$$\mathcal{G}(B) = \{ P_{XY} \in \Pi(B) \text{ satisfying (3.4)} \}. \tag{4.1}$$

It is easy to check that the material groupoid $\mathcal{G}(B)$ is actually a groupoid. Furthermore, it is a subgroupoid of the groupoid of isomorphism $\Pi(B)$. But note that it is not necessarily a Lie groupoid or a transitive one as the frame groupoid.

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Definition 4.2. Given a material body B, we say that it is *uniform* if the material groupoid $\mathcal{G}(B)$ is transitive, and *smoothly uniform* when the material groupoid is a transitive differential groupoid (and hence a Lie subgroupoid of $\Pi(B)$).

Remark 4.3. If two points $X, Y \in B$ are materially isomorphic, the corresponding symmetry groups are conjugate, namely

$$\mathcal{G}(Y) = P \cdot \mathcal{G}(X) \cdot P^{-1} \tag{4.2}$$

for any material isomorphism $P \in \mathcal{G}_{XY}(B)$. Thus, in a uniform body, the symmetry groups are pairwise conjugate.

Definition 4.4. A *material G-structure* of a smoothly uniform body B is any of the G_z-structures induced by the material groupoid as shown in theorem 2.3. The chosen reference frame $z \in \mathcal{F}B$ is called the *reference crystal*.

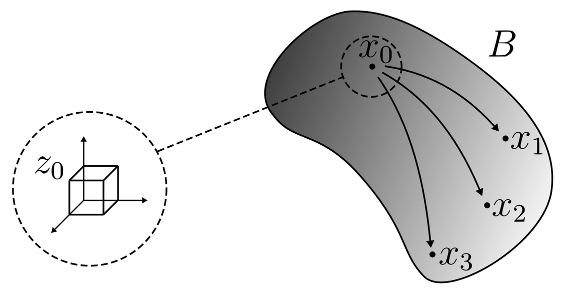

Figure 4: *The reference crystal.*

Definition 4.5. Given a smoothly uniform body B, a configuration K that induces a cross-section of a material G-structure will be called *uniform*. If there exists an atlas $\{(U_\alpha, K_\alpha)\}_{\alpha \in A}$ of B of local uniform configurations for a fixed material G-structure, the body B will be said *locally homogeneous*, and *(globally) homogeneous* if the body B may be covered by just one uniform configuration.

The material concept of homogeneity corresponds to the mathematical concept of integrability. Integrability conditions of smoothly uniform bodies are developed in [6] through a clasification of Lie subgroups of $\mathrm{Gl}(3)$.

5. Unisymmetry and Homosymmetry

As we have seen, the concept of homogeneity must be understood within the framework of uniformity. But, there are materials that are not uniform, the so called *functionally graded materials*, or FGM for short. This type of material is made by some techniques that do it gradually different from point to point: for instance, ceramic metal composite used in aeronautics, which consists in a plate made of ceramic at one side that continously change to some metal at the opposite face. The material properties are given through constitutive equations like (3.3). Therefore, we will have a notion of symmetry group as in the case of uniform materials. For a FGM material, the symmetry groups at two different points are still conjugate, accordingly to the following definition.

Definition 5.1. Given a body B, let be $X, Y \in B$; we say that a linear map $A : T_X B \longrightarrow T_Y B$ is a *unisymmetric isomorphism* if it conjugates the symmetry groups of X and Y, namely

$$\mathcal{G}(Y) = A \cdot \mathcal{G}(X) \cdot A^{-1}. \tag{5.1}$$

Definition 5.2. Given a body B, the set of unisymmetric isomorphism, that is the set

$$\mathcal{N}(B) = \left\{ A \in \Pi(B) \ : \ \mathcal{G}(Y) = A \cdot \mathcal{G}(X) \cdot A^{-1} \right\}, \tag{5.2}$$

will be called the *extended material groupoid*.

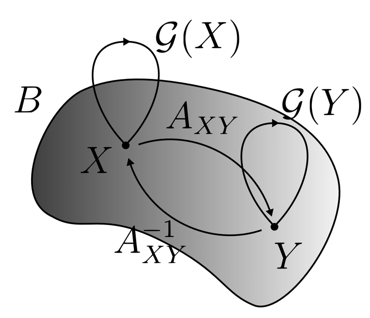

Figure 5: *The extended material groupoid.*

Note that the group over a base point in the extended groupoid is the normalizer of the symmetry group over this point, *i.e.*

$$\mathcal{N}(X) = N(\mathcal{G}(X)). \tag{5.3}$$

The difference between a subgroup and its normalizer can be huge, as in the trivial group, for which the normalizer is the whole ambient group. So, in a triclinic graded material, the extended groupoid is the whole frame groupoid.

Definition 5.3. A body B will be said *unisymmetric* if the extended material groupoid $\mathcal{N}(B)$ is transitive, and *smoothly unisymmetric* if it is a Lie groupoid.

Definition 5.4. Let B be a smoothly unisymmetric body. Any of the asociated G-strutures $\mathcal{N}_z(B)$, with $z \in \mathcal{F}B$, will be called a *material N-structure*. A cross-section of a material N-structure will be a *unisymmetric cross-section* and a configuration inducing such a cross-section will be a *unisymmetric configuration*. If for any of the material N-structures there exists a covering by unisymmetric configurations, the body B will be said *locally homosymmetric*, and *(globally) homosymmetric* if the covering consists of only one unisymmetric configuration.

6. Elastic Solids

In some materials, points are know to exhibit prefered states, which is the case of elastic solid materials. In such class of materials, each point has an undistorted state in which the isotropy group may be seen as a subgroup of the rotation group. If furthermore it is posible to choose locally a configuration of undistorted states then the material will be in a natural or relaxed configuration.

Definition 6.1. A smoothly uniform body B is said to be an *elastic solid* if there is a material G-structure such that G is a subgroup of the orthogonal group $O(3)$. Such a material structure is said to be *undistorted*.

Theorem 6.2. *Let B be an elastic solid. Each undistorted material G-structure defines a Riemannian metric g on B invariant under material isomorphisms.*

The metric g is given in the following way, let (U, σ) be a local cross-section of a fixed undistorted material G-structure,

$$g_X(v, w) = \left\langle \sigma(X)^{-1} \cdot v, \sigma(X)^{-1} \cdot w \right\rangle, \tag{6.1}$$

where $X \in U$ and $v, w \in T_X B$.

Definition 6.3. A body B is an *elastic solid* if there exists a covering Σ of B by local cross-sections of $\mathcal{F}B$ verifying:

1. $\sigma(X)^{-1} \cdot \mathcal{G}(X) \cdot \sigma(X) \leq O(3) \ \forall X \in U \ \forall (U, \sigma) \in \Sigma$;
2. $\sigma(X)^{-1} \cdot \tau(X) \in O(3) \ \forall X \in U \cap V \ \forall (U, \sigma), (V, \tau) \in \Sigma$.

Such a cover Σ will be called an *undistorted solid atlas* and it will be supposed maximal.

Remark 6.4. Theorem 6.2 remains valid in the sense that each undistorted material atlas Σ defines a Riemannian metric g_Σ invariant under material symmetries.

Definition 6.5. A solid material B will be said to be *relaxable* if for some undistorted atlas Σ there exists a subcovering $\Sigma_0 \subset \Sigma$ of B induced by local configurations; that is, if the $O(3)$-structure given by the metric g_Σ is integrable or, equivalently, if the Riemannian curvature vanish identically. The elements of Σ_0 are called *relaxed configurations*.

Definition 6.6. We say that a body B is *homosymmetrically relaxable* if B is an unisymmetric solid material for which there exists a cover Σ_0 of local configuration that are both, unisymmetric and relaxed configurations.

Theorem 6.7 (see [1])**.** *If B is relaxable elastic solid that is also homosymmetric, we have*

$$\bar{\mathcal{N}}(B) = \mathcal{N}(B) \cap \mathcal{O}(B), \tag{6.2}$$

where $\bar{\mathcal{N}}(B)$ consits in the orthogonal part of the isomorphisms of $\mathcal{N}(B)$. Therefore, B will be homosymmetrically relaxable if and only if the reduced material groupoid *$\bar{\mathcal{N}}_z(B)$ is integrable (where $z \in \mathcal{F}B$).*

Example 6.8. Let B a relaxable and homosymmetric elastic solid.

- If B is *fully isotropic*, which means there is a material atlas Σ in which the symmetry group of each point $\mathcal{G}(X)$, $X \in B$, is equal to the orthogonal group $O(T_X B, g_\Sigma)$, where g_Σ is the Riemannian metric related to Σ. Thus, the reduced material groupoid $\bar{\mathcal{N}}(B)$ would coincide with the orthogonal grupoid $\mathcal{O}(B)$.
- If B is *triclinic* (the only element of the symmetry group is the identity map), the extended material groupoid $\mathcal{N}(B)$ is the full frame groupoid $\Pi(B)$, and thus $\bar{\mathcal{N}}(B) = \mathcal{O}(B)$ as before.
- If B is *transversally isotropic*, at each point $X \in B$ there exists a basis of $T_X B$ in which the material symmetries $g \in \mathcal{G}(X)$ may be represented by matrices of the form:

  $$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$$

  Thus, for this basis, the normalizer of $\mathcal{G}(X)$ is

  $$\mathcal{N}(X) = \left\langle \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}, \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \beta \end{pmatrix} \right\rangle$$

  where the brackets denote the group generated by the elements within it, where θ, α, β are real numbers and where α, β are in addition positive. Therefore, the group at any base point of the reduced normalizoid coincides with the respective symmetry group, that is

  $$\bar{\mathcal{N}}(X) = \mathcal{G}(X) \quad \forall x \in B.$$

  This means that, even if the material groupoid $\mathcal{G}(B)$ is not transitive (*i.e.* B is not uniform), so it is the reduced material groupoid $\bar{\mathcal{N}}(B)$ and it coincides with $\mathcal{G}(B)$ on the symmetry groups. Thus, there is some kind of uniformity that generalizes the classical one. Finally, note that any G-structure related to $\bar{\mathcal{N}}(B)$ will have a transversally istropic structural group as mentioned before.

7. Elastic Fluids

The standard definition of an elastic fluid is a uniform elastic material which posses a unimodular material structure, that is a $U(3)$-structure (see [9] for instance), even though there are smaller fluid structures as the ones of fluid crystals (*cf.* [6]).

Definition 7.1. A body B is *(an elastic) fluid* if there exists a covering Σ of B by smooth local cross-sections of $\mathcal{F}B$ verifying:

1. $\sigma(X)^{-1} \cdot \mathcal{G}_X \cdot \sigma(X) \leq U(3) \ \forall X \in U \ \forall (U, \sigma) \in \Sigma$;
2. $\sigma(X)^{-1} \cdot \tau(X) \in U(3) \ \forall X \in U \cap V \ \forall (U, \sigma), (V, \tau) \in \Sigma$.

Such a cover Σ will be called an *undistorted fluid atlas* and supposed maximal.

Proposition 7.2. *Let B be a fluid material, then each undistorted fluid atlas Σ defines a volume form ρ_Σ invariant under material symmetries.*

From the volume form ρ_Σ is straightforward to define a determinant on $\Pi(B)$, for which the material symmetries will be unitary.

Proposition 7.3 (see [1])**.** *If B is a unisymmetric elastic fluid, then*

$$\mathcal{N}^1(B) = \mathcal{N}(B) \cap \mathcal{U}(B), \tag{7.1}$$

where $\mathcal{N}^1(B)$ is the reduced material groupoid *which consits in the unimodular part of the isomorphisms of $\mathcal{N}(B)$, that is*

$$\mathcal{N}^1(B) = \mathcal{N}^1(B)/\det_\rho. \tag{7.2}$$

Example 7.4. Let B a fluid crystal of first kind (see [6, 10]), that is, an elastic fluid as in 7.1 such that, for each material point $x \in B$, the symmetry group G_x may be represented for some reference z at X by matrices of the form

$$A = \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ e & f & g \end{pmatrix}$$

with $\det(A) = \pm 1$. The normalizer in $\mathrm{Gl}(3)$ of this group of matrices is the set of matrices of the same form but with the restriction $\det(A) \neq 0$. Therefore, when we intersect the normalizer with $U(3)$ we obtain the original group of matrices. This means that $\mathcal{N}^1(X) = \mathcal{G}(X)$ for every material point $X \in B$.

The latter example shows us how a fluid material, which is not necessarilly uniform, preserves uniformly the symmetry group structure across the body.

References

[1] Cédric M. Campos, Marcelo Epstein and Manuel de León, *Uniformity in functionally graded materials*, preprint.

[2] Marek Elżanowski, Marcelo Epstein, and Jędrzej Śniatycki, *G-structures and material homogeneity*, J. Elasticity **23** (1990), no. 2-3, 167–180. MR MR1074674 (91j:73008)

[3] Marcelo Epstein and Manuel de León, *Homogeneity without uniformity: towards a mathematical theory of functionally graded materials*, Internat. J. Solids Structures **37** (2000), no. 51, 7577–7591. MR MR1804274 (2001k:74020)

[4] Atsuo Fujimoto, *Theory of G-structures*, Study Group of Geometry, Department of Applied Mathematics, College of Liberal Arts and Science, Okayama University, Okayama, 1972, English edition, translated from the original Japanese, Publications of the Study Group of Geometry, Vol. 1. MR MR0348664 (50 #1161)

[5] K. Mackenzie, *Lie groupoids and Lie algebroids in differential geometry*, London Mathematical Society Lecture Note Series, vol. 124, Cambridge University Press, Cambridge, 1987. MR MR896907 (89g:58225)

[6] David Marín and Manuel de León, *Classification of material G-structures*, Mediterr. J. Math. **1** (2004), no. 4, 375–416. MR MR2112746 (2005i:53024)

[7] Jerrold E. Marsden and Thomas J. R. Hughes, *Mathematical foundations of elasticity*, Dover Publications Inc., New York, 1994, Corrected reprint of the 1983 original. MR MR1262126 (95h:73022)

[8] Clifford A. Truesdell, *A first course in rational continuum mechanics. Vol. 1*, Academic Press [Harcourt Brace Jovanovich Publishers], New York, 1977, General concepts, Pure and Applied Mathematics. MR MR559731 (81a:73001)

[9] Clifford A. Truesdell and Walter Noll, *The nonlinear field theories of mechanics*, second ed., Springer-Verlag, Berlin, 1992. MR MR1215940 (94c:73002)

[10] C. C. Wang, *On the geometric structures of simple bodies. A mathematical foundation for the theory of continuous distributions of dislocations*, Arch. Rational Mech. Anal. **27** (1967/1968), 33–94. MR MR0229420 (37 #4994)

[11] C. C. Wang and C. Truesdell, *Introduction to rational elasticity*, Noordhoff International Publishing, Leyden, 1973, Monographs and Textbooks on Mechanics of Solids and Fluids: Mechanics of Continua. MR MR0468442 (57 #8276)

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