3-quasi-Sasakian Manifolds

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- An almost contact metric manifold is an odd-dimensional manifold M which carries a $(1,1)$-tensor field ϕ, a vector field ξ, a 1-form η, satisfying

$$\phi^2 = -I + \eta \otimes \xi \quad \textit{and} \quad \eta(\xi) = 1,$$

and a *compatible metric* g such that

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y)$$

for all $X, Y \in \Gamma(TM)$.

Basic definitions

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for all $X, Y \in \Gamma(TM)$.

- An almost contact metric manifold (M, ϕ, ξ, η, g) is said to be normal if

$$N^{(1)} := [\phi, \phi] + 2d\eta \otimes \xi = 0.$$

Quasi-Sasakian manifolds

- Quasi-Sasakian manifolds were introduced by D. E. Blair in 1967 in the attempt to unify Sasakian and cosymplectic geometry.
- A quasi-Sasakian structure on a $(2n+1)$-dimensional manifold M is a normal almost contact metric structure (ϕ, ξ, η, g) whose fundamental 2-form Φ defined by $\Phi(X, Y) = g(X, \phi Y)$ is closed.

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- A quasi-Sasakian manifold is said to be of rank $2p+1$ if $\eta \wedge (d\eta)^p \neq 0$ and $(d\eta)^{p+1} = 0$ for some $p \leq n$.
- For instance, in a cosymplectic manifold η has rank 1 and in a Sasakian manifold it has maximal rank $2n+1$.

3-structures

- An almost 3-contact metric manifold is a $(4n+3)$-dim. smooth manifold M endowed with three almost contact structures $(\phi_\alpha, \xi_\alpha, \eta_\alpha)$ $(\alpha \in \{1,2,3\})$ satisfying, for any even permutation, the relations

$$\phi_\gamma = \phi_\alpha\phi_\beta - \eta_\beta \otimes \xi_\alpha = -\phi_\beta\phi_\alpha + \eta_\alpha \otimes \xi_\beta,$$
$$\xi_\gamma = \phi_\alpha\xi_\beta = -\phi_\beta\xi_\alpha, \quad \eta_\gamma = \eta_\alpha \circ \phi_\beta = -\eta_\beta \circ \phi_\alpha,$$

  and a Riemannian metric g compatible with each of them.
- M is said to be *hyper-normal* if each almost contact structure is normal.

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  and a Riemannian metric g compatible with each of them.
- M is said to be *hyper-normal* if each almost contact structure is normal.

- By putting $\mathcal{H} = \bigcap_{\alpha=1}^{3} \ker(\eta_\alpha)$ one obtains a $4n$-dim. distribution on M and the tangent bundle splits as the orthogonal sum

$$TM = \mathcal{H} \oplus \mathcal{V},$$

  where $\mathcal{V} = \langle \xi_1, \xi_2, \xi_3 \rangle$.

Definition

A 3-quasi-Sasakian manifold is a hyper-normal almost 3-contact metric manifold $(M, \phi_\alpha, \xi_\alpha, \eta_\alpha, g)$ such that each fundamental 2-form Φ_α is closed.

The class of 3-quasi-Sasakian manifolds includes as special cases the 3-cosymplectic ($d\eta_\alpha = 0$) and the 3-Sasakian manifolds ($d\eta_\alpha = \Phi_\alpha$) which have been widely studied in recent years.

The canonical foliation of a 3-quasi-Sasakian manifold

Starting from a generalization to the setting of 3-structures of the characterization of quasi-Sasakian manifolds obtained by Kanemaki we proved the following theorem.

Theorem

Let $(M, \phi_\alpha, \xi_\alpha, \eta_\alpha, g)$ *be a 3-quasi-Sasakian manifold. Then the 3-dimensional distribution* $\mathcal{V}$ *generated by* ξ_1, ξ_2, ξ_3 *is integrable. Moreover,* $\mathcal{V}$ *defines a totally geodesic and Riemannian foliation.*

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Corollary

Let $(M, \phi_\alpha, \xi_\alpha, \eta_\alpha, g)$ be a 3-quasi-Sasakian manifold. Then, for all $X \in \Gamma(\mathcal{H})$ and for all $\alpha, \beta \in \{1, 2, 3\}$, $d\eta_\alpha(X, \xi_\beta) = 0$. Moreover, each Reeb vector field ξ_α is an infinitesimal automorphism with respect to the distribution $\mathcal{H}$.

Structure of the leaves of $\mathcal{V}$

Theorem

Let $(M, \phi_\alpha, \xi_\alpha, \eta_\alpha, g)$ be a 3-quasi-Sasakian manifold. Then, for any even permutation (α, β, γ) of $\{1, 2, 3\}$ and for some $c \in \mathbb{R}$

$$[\xi_\alpha, \xi_\beta] = c\xi_\gamma.$$

So we can divide 3-quasi-Sasakian manifolds in two main classes according to the behaviour of the leaves of $\mathcal{V}$: those 3-quasi-Sasakian manifolds for which each leaf of $\mathcal{V}$ is locally $SO(3)$ (or $SU(2)$) (which corresponds to take in the above theorem the constant $c \neq 0$), and those for which each leaf of $\mathcal{V}$ is locally an abelian group (the case $c = 0$).

An example of 3-quasi-Sasakian manifold

Note that 3-Sasakian manifolds and 3-cosymplectic manifolds are representatives of each of the above classes but they do not exhaust the two classes as it is shown in the following example.

Example

Let us denote the canonical global coordinates on $\mathbb{R}^{4n+3}$ by $x_1, \dots, x_n$, $y_1, \dots, y_n, u_1, \dots, u_n, v_1, \dots, v_n, z_1, z_2, z_3$. We consider the open submanifold M of $\mathbb{R}^{4n+3}$ obtained by removing the points where $\sin(z_2) = 0$ and define three vector fields on M by

$$
\begin{aligned}
\xi_1 &= c\frac{\partial}{\partial z_1}, \\
\xi_2 &= c\left(\cos(z_1)\cot(z_2)\frac{\partial}{\partial z_1} + \sin(z_1)\frac{\partial}{\partial z_2} - \frac{\cos(z_1)}{\sin(z_2)}\frac{\partial}{\partial z_3}\right), \\
\xi_3 &= c\left(-\sin(z_1)\cot(z_2)\frac{\partial}{\partial z_1} + \cos(z_1)\frac{\partial}{\partial z_2} + \frac{\sin(z_1)}{\sin(z_2)}\frac{\partial}{\partial z_3}\right)
\end{aligned}
$$

for some non-zero real number c, and and three 1-forms by

An example of 3-quasi-Sasakian manifold

Example (continued)

$$\eta_1 = \frac{1}{c}\left(dz_1 + \cos\left(z_2\right) dz_3\right),$$
$$\eta_2 = \frac{1}{c}\left(\sin\left(z_1\right) dz_2 - \cos\left(z_1\right)\sin\left(z_2\right) dz_3\right),$$
$$\eta_3 = \frac{1}{c}\left(\cos\left(z_1\right) dz_2 + \sin\left(z_1\right)\sin\left(z_2\right) dz_3\right),$$

so that $[\xi_\alpha, \xi_\beta] = c\xi_\gamma$ for any cyclic permutation, and $\eta_\alpha\left(\xi_\beta\right) = \delta_{\alpha\beta}$.
We define a Riemannian metric g on M by declaring that the set $\{X_i = \frac{\partial}{\partial x_i},\ Y_i = \frac{\partial}{\partial y_i},\ U_i = \frac{\partial}{\partial u_i},\ V_i = \frac{\partial}{\partial v_i}, \xi_1, \xi_2, \xi_3\}$ $(i = 1, \ldots, n)$ is a global orthonormal frame. Finally, the tensor fields ϕ_1, ϕ_2, ϕ_3 are defined by putting

$$\phi_\alpha \xi_\beta = \epsilon_{\alpha\beta\gamma}\xi_\gamma$$

and

$$\phi_1 X_i = Y_i,\ \phi_1 Y_i = -X_i,\ \phi_1 U_i = V_i,\ \phi_1 V_i = -U_i,$$
$$\phi_2 X_i = U_i,\ \phi_2 Y_i = -V_i,\ \phi_2 U_i = -X_i,\ \phi_2 V_i = Y_i,$$
$$\phi_3 X_i = V_i,\ \phi_3 Y_i = U_i,\ \phi_3 U_i = -Y_i,\ \phi_3 V_i = -X_i.$$

An example of 3-quasi-Sasakian manifold

Example (Concluding Remarks)

- $(M, \phi_\alpha, \xi_\alpha, \eta_\alpha, g)$ is a 3-quasi-Sasakian manifold which is neither 3-cosymplectic, since the Reeb vector fields do not commute, nor 3-Sasakian, since it admits a Darboux-like coordinate system.
- Furthermore, M is η-Einstein, its Ricci tensor being given by $\mathrm{Ric} = \frac{c^2}{2}(\eta_1 \otimes \eta_1 + \eta_2 \otimes \eta_2 + \eta_3 \otimes \eta_3)$. Thus, differently from 3-Sasakian and 3-cosymplectic geometry, there are 3-quasi-Sasakian manifolds which are not Einstein.

The rank of a 3-quasi-Sasakian manifold

For a 3-quasi-Sasakian manifold one can consider the ranks of the three 1-forms η_1, η_2, η_3. We prove that these ranks coincide allowing us to classify 3-quasi-Sasakian manifolds.

The rank of a 3-quasi-Sasakian manifold

Theorem

Let $(M, \phi_\alpha, \xi_\alpha, \eta_\alpha, g)$ be a 3-quasi-Sasakian manifold of dimension $4n+3$. Then the 1-forms η_1, η_2 and η_3 have the same rank $4l+3$ or $4l+1$, for some $l \leq n$, according to $[\xi_\alpha, \xi_\beta] = c\xi_\gamma$ with $c \neq 0$, or $[\xi_\alpha, \xi_\beta] = 0$, respectively.

- The above theorem allows to define the rank of a 3-quasi-Sasakian manifold $(M, \phi_\alpha, \xi_\alpha, \eta_\alpha, g)$ as the rank shared by the 1-forms η_1, η_2 and η_3.
- We may thus classify 3-quasi-Sasakian manifolds of dimension $4n+3$, according to their rank.
- The total number of classes amounts to $2n+2$.

The rank of a 3-quasi-Sasakian manifold

- $\mathcal{E}^{4m} := \{X \in \Gamma(\mathcal{H}) \mid i_X d\eta_\alpha = 0\}$,
- $\mathcal{E}^{4l}$ the orthogonal complement of $\mathcal{E}^{4m}$ in $\Gamma(\mathcal{H})$,
- $\mathcal{E}^{4m+3} := \mathcal{E}^{4m} \oplus \Gamma(\mathcal{V})$,
- $\mathcal{E}^{4l+3} := \mathcal{E}^{4l} \oplus \Gamma(\mathcal{V})$.

We now consider the class of 3-quasi-Sasakian manifolds such that $[\xi_\alpha, \xi_\beta] = c\xi_\gamma$ with $c \neq 0$ and let $4l+3$ be the rank. In this case, we define for each α two $(1,1)$-tensor fields ψ_α and θ_α by putting

$$\psi_\alpha X = \begin{cases} \phi_\alpha X, & \text{if } X \in \mathcal{E}^{4l+3}; \\ 0, & \text{if } X \in \mathcal{E}^{4m}; \end{cases} \qquad \theta_\alpha X = \begin{cases} 0, & \text{if } X \in \mathcal{E}^{4l+3}; \\ \phi_\alpha X, & \text{if } X \in \mathcal{E}^{4m}. \end{cases}$$

Note that $\phi_\alpha = \psi_\alpha + \theta_\alpha$. Next, we define a new (pseudo-Riemannian, in general) metric $\bar{g}$ on M setting

$$\bar{g}(X,Y) = \begin{cases} -d\eta_\alpha(X, \phi_\alpha Y), & \text{for } X, Y \in \mathcal{E}^{4l}; \\ g(X,Y), & \text{elsewhere.} \end{cases}$$

The rank of a 3-quasi-Sasakian manifold

The following decomposition theorem holds.

Theorem

Let $(M^{4n+3}, \phi_\alpha, \xi_\alpha, \eta_\alpha, g)$ *be a 3-quasi-Sasakian manifold of rank* $4l+3$ *with* $[\xi_\alpha, \xi_\beta] = 2\xi_\gamma$. *Assume* $[\theta_\alpha, \theta_\alpha] = 0$ *for some* $\alpha \in \{1,2,3\}$ *and* $\bar{g}$ *positive definite on* $\mathcal{E}^{4l}$. *Then* M^{4n+3} *is locally the product of a 3-Sasakian manifold* M^{4l+3} *and a hyper-Kählerian manifold* M^{4m} *with* $m = n - l$.

The rank of a 3-quasi-Sasakian manifold

We now consider the class of 3-quasi-Sasakian manifolds such that $[\xi_\alpha, \xi_\beta] = 0$ and let $4l + 1$ be the rank. In this case we define for each α two $(1,1)$-tensor fields ψ_α and θ_α by putting

$$\psi_\alpha X = \begin{cases} \phi_\alpha X, & \text{if } X \in \mathcal{E}^{4l}, \\ 0, & \text{if } X \in \mathcal{E}^{4m+3}, \end{cases} \qquad \theta_\alpha X = \begin{cases} 0, & \text{if } X \in \mathcal{E}^{4l}, \\ \phi_\alpha X, & \text{if } X \in \mathcal{E}^{4m+3}. \end{cases}$$

Note that for each α the maps $-\psi_\alpha^2$ and $-\theta_\alpha^2 + \eta_\alpha \otimes \xi_\alpha$ define an almost product structure which is integrable if and only if $[-\psi_\alpha^2, -\psi_\alpha^2] = 0$ or, equivalently, $[\psi_\alpha, \psi_\alpha] = 0$.

Theorem

Let $(M, \phi_\alpha, \xi_\alpha, \eta_\alpha, g)$ be a 3-quasi-Sasakian manifold such that $[\xi_\alpha, \xi_\beta] = 0$ for any $\alpha, \beta \in \{1,2,3\}$ and $[\psi_\alpha, \psi_\alpha] = 0$ for some $\alpha \in \{1,2,3\}$. Then M is a 3-cosymplectic manifold.

Recall that the corrected energy $\mathcal{D}(\mathcal{V})$ of a p-dim. distribution $\mathcal{V}$ on a compact oriented Riemannian manifold (M^m, g) is

$$\mathcal{D}(\mathcal{V}) = \int_M \left(\sum_{a=1}^{m} \|\nabla_{e_a}\xi\|^2 + q(q-2)\|\overrightarrow{H}_{\mathcal{H}}\|^2 + p^2\|\overrightarrow{H}_{\mathcal{V}}\|^2 \right) d\mathrm{vol},$$

where $\{e_1, \dots, e_m\}$ is a local adapted frame with $e_1, \dots, e_p \in \mathcal{V}_x$, $e_{p+1}, \dots,\ e_{m=p+q} \in \mathcal{H}_x$, and $\xi = e_1 \wedge \cdots \wedge e_p$ is a p-vector which determines the distribution $\mathcal{V}$ regarded as a section of the bundle $G\,(p, M^m)$ of oriented p-planes in the tangent spaces of M^m. Finally $\overrightarrow{H}_{\mathcal{H}}$ and $\overrightarrow{H}_{\mathcal{V}}$ are resp. the mean curvatures of $\mathcal{H}$ and $\mathcal{V}$.

Theorem

The canonical foliation $\mathcal{V}$ defined by the Reeb vector fields ξ_1, ξ_2, ξ_3 of a compact oriented 3-quasi-Sasakian manifold $(M, \phi_\alpha, \xi_\alpha, \eta_\alpha, g)$ represents a minimum of the corrected energy $\mathcal{D}(\mathcal{V})$.

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