4-manifolds

(symplectic or not)

notes for a course at the IFWGP 2007

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Introduction

These notes review some state of the art and open questions on (smooth) 4-manifolds from the point of view of symplectic geometry. They developed from the first two lectures of a short course presented at at the *International Fall Workshop on Geometry and Physics 2007* which took place in Lisbon, 5-8/September 2007. The original course included a third lecture explaining the existence on 4-manifolds of structures related to symplectic forms:

1st lecture – 4-Manifolds

Whereas (closed simply connected) topological 4-manifolds are completely classified, the panorama for smooth 4-manifolds is quite wild: the existence of a smooth structure imposes strong topological constraints, yet for the same topology there can be infinite different smooth structures. We discuss constructions of (smooth) 4-manifolds and invariants to distinguish them.

2nd lecture – Symplectic 4-Manifolds

Existence and uniqueness of symplectic forms on a given 4-manifold are questions particularly relevant to 4-dimensional topology and to mathematical physics, where symplectic manifolds occur as building blocks or as key examples. We describe some constructions of symplectic 4-manifolds and invariants to distinguish them.

3rd lecture - Folded Symplectic 4-Manifolds

Any orientable 4-manifold admits a folded symplectic form, that is, a closed 2-form which is symplectic except on a separating hypersurface where the form singularities are like the pullback of a symplectic form by a folding map. We explain how, for orientable even-dimensional manifolds, the existence of a stable almost complex structure is necessary and sufficient to warrant the existence of a folded symplectic form.

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1 4-Manifolds

1.1 intersection form

Very little was known about 4-dimensional manifolds[1] until 1981, when Freedman [24] provided a complete classification of closed[2] simply connected *topological* 4-manifolds, and soon thereafter Donaldson [12] showed that the panorama for *smooth* 4-manifolds was much wilder. Key to this understanding was the *intersection form*.

The **intersection form** of an oriented topological closed 4-manifold M is the symmetric bilinear pairing

$$Q_M : H^2(M;\mathbb{Z}) \times H^2(M;\mathbb{Z}) \to \mathbb{Z} \ , \qquad Q_M(\alpha,\beta) := \langle \alpha \cup \beta, [M] \rangle \ ,$$

where $\alpha \cup \beta$ is the *cup product* and $[M]$ is the *fundamental class*. For smooth simply connected manifolds and smooth differential forms representing (non-torsion) cohomology classes, this pairing is simply $Q_M([f],[g]) = \int_M f \wedge g$ on 2-forms $[f]$ and $[g]$.[3]

Since the intersection form Q_M always vanishes on torsion elements, it descends to the quotient group $H^2(M;\mathbb{Z})/\text{torsion}$ where it is represented by a

[1] An n-dimensional **topological manifold** is a second countable Hausdorff space in which every point has a neighborhood homeomorphic to an open euclidean n-dimensional ball, $B^n = \{(x_1, x_2, \ldots, x_n) | x_1^2 + x_2^2 + ... + x_n^2 < 1\}$.

An n-dimensional **smooth manifold** is an an n-dimensional topological manifold admitting homeomorphisms on overlapping neighborhoods which are diffeomorphisms, so that we may define a set of differentiable functions on the whole manifold as functions which are differentiable in each neighborhood.

Second countability amounts to the existence of a countable collection of open sets forming a basis for the topology. The *Hausdorff condition* amounts to every pair of distinct points admitting non-intersecting open neighborhoods. These two conditions for smooth manifolds ensure, for instance, the existence of partitions of unity.

Other kinds of manifolds may be considered with additional structure, the structure on each map being consistent with the overlapping maps.

[2] A **closed** manifold is just a compact manifold (without boundary).

[3] For smooth closed oriented 4-manifolds, every element of $H_2(M;\mathbb{Z})$ can be represented by an embedded surface: elements of $H^2(M;\mathbb{Z})$ are in one-to-one correspondence with complex line bundles over M via the Euler class; the zero set of a generic section of a bundle with Euler class α is a smooth surface representing the Poincaré dual of α. If Σ_α and Σ_β are generic surface representatives of the Poincaré duals of $\alpha, \beta \in H^2(M;\mathbb{Z})$, so that their intersections are transverse, then $Q_M(\alpha,\beta)$ is the number of intersection points in $\Sigma_\alpha \cap \Sigma_\beta$ counted with signs depending on the matching of orientations – called the **intersection number**, $\Sigma_\alpha \cdot \Sigma_\beta$.

matrix after choosing a basis. The quotient Q_M is a unimodular pairing: the determinant of a matrix representing Q_M is ± 1 by Poincaré duality. Hence, the corresponding (symmetric) matrix is diagonalizable over $\mathbb{R}$ with eigenvalues ± 1. We denote by b_2^+ (respectively, b_2^-) the number of positive (resp. negative) eigenvalues of Q_M counted with multiplicities, i.e., the dimension of a maximal subspace where Q_M is positive-definite (resp. negative-definite).

The **signature** of M is the difference $\sigma := b_2^+ - b_2^-$, whereas the second Betti number is the sum $b_2 = b_2^+ + b_2^-$, i.e., the **rank** of Q_M (the dimension of the domain). The **type** of an intersection form is **definite** if it is positive or negative definite (i.e., $|\sigma| = b_2$) and **indefinite** otherwise. We say that the **parity** of an intersection form Q_M is **even** when $Q_M(\alpha, \alpha)$ is always even, and is **odd** otherwise.

1.2 up to the 80's

We restrict to closed simply connected[4] topological 4-manifolds.

Before the work of Freedman, it had been proved by Rokhlin [62] in 1952 that if such a smooth manifold M has even intersection form Q_M (this amounts to the second Stiefel-Whitney class $w_2(M)$ vanishing), then the signature of Q_M must be a multiple of 16. Milnor [52] showed that, as a consequence of a theorem of Whitehead, two such topological manifolds are homotopy equivalent if and only if they have the same intersection form.

Example. Consider the even positive-definite form given by the matrix

$$E_8 := \begin{bmatrix} 2 & 1 & & & & & & \\ 1 & 2 & 1 & & & & & \\ & 1 & 2 & 1 & & & & \\ & & 1 & 2 & 1 & & & \\ & & & 1 & 2 & 1 & & 1 \\ & & & & 1 & 2 & 1 & \\ & & & & & 1 & 2 & \\ & & & & 1 & & & 2 \end{bmatrix}$$

related to the lattice of the Lie algebra with the same name. Since E_8 has signature 8, by Rokhlin's result it cannot occur as the intersection form of a smooth

[4] A manifold is **simply connected** if it is path-connected and every path between two points can be continuously transformed into every other.

4-manifold. Indeed there is a topological manifold, called the E_8 *manifold*, with this intersection form, built by plumbing based on the E_8 Dynkin diagram. $\diamondsuit$

Freedman [24] showed that, modulo homeomorphism, such topological manifolds are essentially classified by their intersection forms:

- for an *even* intersection form there is exactly one class, whereas
- for an *odd* intersection form there are exactly two classes distinguished by the *Kirby-Siebenmann invariant* in $\mathbb{Z}/2$, at most one of which admits smooth representatives (smoothness requires vanishing invariant).

Example. For instance, whereas the standard complex projective plane $\mathbb{CP}^2$ has odd intersection form

$$Q_{\mathbb{CP}^2} = [1] \ ,$$

there is a topological manifold, called the *fake projective plane*, with the same intersection form (hence the same homotopy type) yet not homeomorphic to $\mathbb{CP}^2$ and admitting no smooth structure. $\diamondsuit$

The 4-dimensional topological Poincaré conjecture is a corollary of Freedman's theorem: when $H_2(M) = 0$, the manifold M must be homeomorphic to the sphere S^4.

By the way, Freedman's work can extend to a few other simple enough fundamental groups. Very little is known when the fundamental group is large. Yet, any finitely presented group occurs as the fundamental group of a closed (smooth) 4-manifold, and such groups are not classifiable.

Freedman's work reduced the classification of closed simply connected topological 4-manifolds to the algebraic problem of classifying unimodular symmetric bilinear forms. Milnor and Husemoller [54] showed that indefinite forms are classified by rank, signature and parity. Up to isomorphism, intersection forms can be:

- if odd indefinite, then $n[1] \oplus m[-1]$;
- if even indefinite, then $\pm 2nE_8 \oplus kH$;
- if definite, there are too many possibilities. For each rank three is a finite number which grows very fast. For instance, there are more than 10^{50} different definite intersection forms with rank 40, so this classification is hopeless in practice.

On the other hand, Donaldson [12] showed that for a smooth manifold an intersection form which is definite must be a diagonal either of 1s or of -1s which we represent by $n[1]$ and $m[-1]$. In particular, it cannot be an even form (unless it is empty, i.e., $H_2(M) = \{0\}$).

Consequently, *the homeomorphism class of a connected simply connected closed oriented smooth 4-manifold is determined by the two integers* (b_2, σ) *– the second Betti number and the signature – and the parity of the intersection form.*

Whereas the existence of a smooth structure imposes strong constraints on the topological type of a manifold, Donaldson also showed that for the same topological manifold there can be infinite different smooth structures. In other words, by far not all intersection forms can occur for smooth 4-manifolds and the same intersection form may correspond to nondiffeomorphic manifolds.

Donaldson's tool was a set of gauge-theoretic invariants, defined by counting with signs the equivalence classes (modulo gauge equivalence) of connections on $\mathrm{SU}(2)$- (or $\mathrm{SO}(3)$-) bundles over M whose curvature has vanishing self-dual part. For a dozen years there was hard work on the invariants discovered by Donaldson but limited advancement on the understanding of smooth 4-manifolds.

1.3 topological coordinates

As a consequence of the work of Freedman and Donaldson in the 80's, the numbers (b_2, σ) – the second Betti number and the signature – can be treated as **topological coordinates** determining, together with the parity, the homeomorphism class of a connected simply connected closed oriented *smooth* 4-manifold. Yet, for each pair (b_2, σ) there could well be infinite different (i.e., nondiffeomorphic) smooth manifolds.

Traditionally, the numbers used are $(c_1^2, c_2) := (3\sigma + 2\chi, \chi) = (3\sigma + 4 + 2b_2, 2 + b_2)$, and frequently just the **slope** c_1^2/c_2 is considered. If M admits an almost complex structure J, then (TM, J) is a complex vector bundle, hence has Chern classes [11] $c_1 = c_1(M, J)$ and $c_2 = c_2(M, J)$. Both $c_1^2 := c_1 \cup c_1$ and c_2 may be regarded as numbers since $H^4(M; \mathbb{Z}) \simeq \mathbb{Z}$. They satisfy

- $c_1^2 = 3\sigma + 2\chi$ (by Hirzebruch's signature formula) [81] and
- $c_2 = \chi$ (because the top Chern class is always the Euler class),

justifying the notation for the topological coordinates in this case.

Examples.

- The manifold $\mathbb{CP}^2$ has $(b_2, \sigma) = (1, 1)$, i.e., $(c_1^2, c_2) = (9, 3)$. We have that $H_2(\mathbb{CP}^2) \simeq \mathbb{Z}$ is generated by the class of a complex projective line inside $\mathbb{CP}^2$. The corresponding intersection form is represented by the matrix $[1]$, translating the fact that two lines meet at one point. Reversing the orientation, $\overline{\mathbb{CP}^2}$ has $(b_2, \sigma) = (1, -1)$, i.e., $(c_1^2, c_2) = (3, 3)$. The intersection form is now represented by $[-1]$.

- The connected sum[5] $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$ has $(b_2, \sigma) = (2, 0)$, i.e., $(c_1^2, c_2) = (8, 0)$. The corresponding intersection form is represented by
$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} .$$

- The product $S^2 \times S^2$ also has $(b_2, \sigma) = (2, 0)$ i.e., $(c_1^2, c_2) = (8, 4)$. But $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$ has an *odd* intersection form whereas $S^2 \times S^2$ has an *even* intersection form represented by
$$H := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} .$$
The standard generators of $H_2(S^2 \times S^2)$ are the classes of each factor times a point in the other factor.

- The quartic hypersurface in $\mathbb{CP}^3$
$$K3 = \{[z_0, z_1, z_2, z_3] \in \mathbb{CP}^3 \mid z_0^4 + z_1^4 + z_2^4 + z_3^4 = 0\}$$
(apparently named in honor of Kummer, Kähler and Kodaira, or/and after the famous K2 mountain in the Himalayas) has intersection form represented by
$$-2E_8 \oplus 3H \ .$$
This can be seen from studying $K3$ as a singular fibration $E(2)$.

◊

[5]A **connected sum** $M \# N$ of two 4-manifolds M and N is a manifold formed by cutting out a 4-ball inside each of M and N and identifying the resulting boundary 3-spheres. The intersection form of a connected sum $M \# N$ is (isomorphic to) the direct sum of the intersection forms of the manifold summands: $Q_{M\#N} \simeq Q_M \oplus Q_N$. *Topologically*, the converse is also true as a consequence of Freedman's theorem: if for a simply connected manifold the intersection form splits as a direct sum of two forms, then that manifold is the connected sum of two topological manifolds with those forms.

Geography problems are problems on the existence of simply connected closed oriented 4-dimensional manifolds with some additional structure (such as, a symplectic form or a complex structure) for each pair of topological coordinates; see Section 2.4.

1.4 smooth representatives

Donaldson's work together with work of Furuta [26] in the 90's showed that, if Q_M is the intersection form of a smooth manifold M, then

- Q_M odd $\quad\Longrightarrow\quad$ $Q_M \simeq n[1] \oplus m[-1]$,
- Q_M even $\quad\Longrightarrow\quad$ $Q_M \simeq \pm 2nE_8 \oplus kH$ with $k > 2n$ or Q_M is trivial (i.e., $H_2(M) = \{0\}$).

The first set is realized by connected sums

$$M = \left(\#_n \mathbb{CP}^2\right) \# \left(\#_m \overline{\mathbb{CP}^2}\right) .$$

For the second set, notice that with $k > 2n$ and $n \neq 0$ we have

$$\frac{b_2}{|\sigma|} = \frac{16n + 2k}{16n} > \frac{16n + 4n}{16n} = \frac{5}{4} .$$

When $k \geq 3n$, the forms $\pm 2nE_8 \oplus kH$ are represented by

$$\left(\#_n \overline{K3}\right) \# \left(\#_{k-3n} S^2 \times S^2\right) \qquad \text{and} \qquad \left(\#_n K3\right) \# \left(\#_{k-3n} S^2 \times S^2\right) .$$

Indeed, recall that

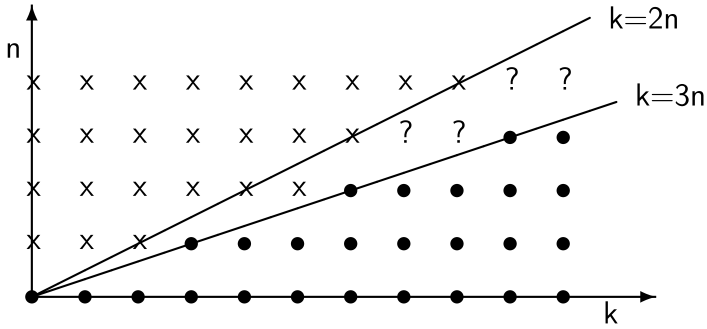
$$Q_{K3} = -2E_8 \oplus 3H \qquad \text{and} \qquad Q_{S^2 \times S^2} = H$$

and notice that $H \simeq -H$ by flipping the sign of one of the generators, and $E_8 \oplus (-E_8) \simeq 8H$. In the case $k \geq 3n$, and with $n \neq 0$, we have that

$$\frac{b_2}{|\sigma|} \geq \frac{16n + 6n}{16n} = \frac{11}{8} .$$

The question of whether the forms $\pm 2nE_8 \oplus kH$ are realized as the intersection forms Q_M and $Q_{\overline{M}}$ for a smooth manifold M has thus been answered affirmatively for $k \geq 3n$ (represented by dots in the following diagram) and negatively for $k \leq 2n$ (represented by crosses).

The $\frac{11}{8}$ *conjecture* [43] claims that the answer is also no for all points between the two lines. The case corresponding to $n = 2$ and $k = 5$ has been checked, yet all others (starting with $n = 3$ and $k = 7$ for which the rank is 62), represented by question marks in the following diagram, are still open.



1.5 exotic manifolds

In dimensions up to 3, each topological manifold has exactly one smooth structure, and in dimensions 5 and higher each topological manifold has at most finitely many smooth structures. Yet there are no known finiteness results for the smooth types of a given topological 4-manifold. Using riemannian geometry, Cheeger [10] showed that there are at most *countably many* different smooth types for closed 4-manifolds.

For open manifolds, the contrast of behavior for dimensions 4 and other is at least as striking. Whereas each topological $\mathbb{R}^n$, $n \neq 4$, admits a unique smooth structure, Taubes showed that the topological $\mathbb{R}^4$ admits uncountably many smooth structures.

A manifold homeomorphic but not diffeomorphic to a smooth manifold M is called an **exotic** M. Finding exotic smooth structures on closed simply connected manifolds with small b_2, dubbed *small 4-manifolds*, has long been an interesting problem, especially in view of the smooth Poincaré conjecture for 4-manifolds: if M is a closed smooth 4-manifold homotopy equivalent to the sphere S^4, is M necessarily diffeomorphic to S^4?

Examples.

- The first exotic smooth structures on a rational surface $\mathbb{CP}^2 \#_n \overline{\mathbb{CP}^2}$ were found in the late 80's when Donaldson [13] proved that the Dolgachev surface $E(1)_{2,3}$ is homeomorphic but not diffeomorphic to $\mathbb{CP}^2 \#_9 \overline{\mathbb{CP}^2}$ by using his invariant based on $\mathrm{SU}(2)$ gauge theory. Shortly thereafter, Friedman and Morgan [25] and Okonek and Van de Ven [57] produced an infinite family of manifolds homeomorphic but not diffeomorphic to $\mathbb{CP}^2\#_9\overline{\mathbb{CP}^2}$. Later Kotschick [40] and Okonek and Van de Ven [58] applied $\mathrm{SO}(3)$ gauge theory to prove that the Barlow surface is homeomorphic but not diffeomorphic to $\mathbb{CP}^2\#_8\overline{\mathbb{CP}^2}$.

- There was no progress until work of Jongil Park [60] in 2004 constructing a symplectic exotic $\mathbb{CP}^2\#_7\overline{\mathbb{CP}^2}$ and using this to exhibit a third distinct smooth structure on $\mathbb{CP}^2\#_8\overline{\mathbb{CP}^2}$, thus illustrating how the existence of symplectic forms links to the existence of different smooth structures. This stimulated research by Fintushel, J. Park, Stern, Stipsicz and Szabó [69, 21, 61], which shows that there are infinitely many exotic smooth structures on $\mathbb{CP}^2\#_n\overline{\mathbb{CP}^2}$ for $n = 5, 6, 7, 8$.
- Last year Fintushel, Doug Park and Stern [20] announced an infinite family of distinct smooth structures on $\mathbb{CP}^2\#_3\overline{\mathbb{CP}^2}$, following work by Akhmedov and D. Park [1] and Balbridge and Kirk [8] providing one such exotic structure.

$\diamondsuit$

Still, up to date there is no classification of smooth structures on any given *smoothable* topological 4-manifold, and there are not even standing conjectures, except for a vague belief that any such manifold has infinite smooth structures. It was speculated that perhaps any simply connected closed smooth 4-manifold other than S^4 is diffeomorphic to a connected sum of symplectic manifolds, where any orientation is allowed on each summand – the so-called *minimal conjecture* for smooth 4-manifolds. Szabó [70, 71] provided counterexamples in a family of irreducible[6] simply connected closed non-symplectic smooth 4-manifolds.

[6] A (smooth) manifold is **irreducible** when it is not a connected sum of other (smooth) manifolds except if one of the summands is a *homotopy sphere*. A **homotopy sphere** is a closed n-manifold which is homotopy equivalent to the n-sphere.

2 Symplectic 4-Manifolds

2.1 kähler structures and co.

A **symplectic 4-manifold** (M, ω) is a smooth oriented 4-manifold M equipped with a closed 2-form ω such that $\omega \wedge \omega$ is a volume form. In other dimensions, necessarily even, a symplectic manifold is a smooth manifold equipped with a closed nondegenerate 2-form. The form ω is then called a **symplectic form**. Hence, both an algebraic condition – nondegeneracy – and an analytic condition – closedness – come into *symplectiness*. Just as any n-dimensional manifold is locally diffeomorphic to $\mathbb{R}^n$, the *Darboux theorem* states that any symplectic manifold (M^{2n}, ω) is locally *symplectomorphic* to $(\mathbb{R}^{2n}, \omega_0)$ where $\omega_0 = \sum_{i=1}^n dx_i \wedge dy_i$ in terms of linear coordinates $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ in $\mathbb{R}^{2n}$. A **symplectomorphism** is a diffeomorphism from one symplectic manifold to another taking one symplectic form to the other.

A **complex manifold** (M, ω) is a smooth manifold M equipped with an atlas of complex coordinate charts for which the transition maps are biholomorphic. On such a manifold, multiplication by i induces a field of linear maps on the tangent spaces $J_p : T_pM \to T_pM$ with $J_p^2 = -\mathrm{Id}$ for each $p \in M$, called an **almost complex structure** J. More concretely, if z_1, z_2 are local complex coordinates on a complex surface (real 4-manifold) with $z_k = x_k + iy_k$, then $\frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial y_2}$ span the tangent space at each point and we have that

$$J_p\left(\left.\frac{\partial}{\partial x_k}\right|_p\right) = \left.\frac{\partial}{\partial y_k}\right|_p \qquad \text{and} \qquad J_p\left(\left.\frac{\partial}{\partial y_k}\right|_p\right) = -\left.\frac{\partial}{\partial x_k}\right|_p .$$

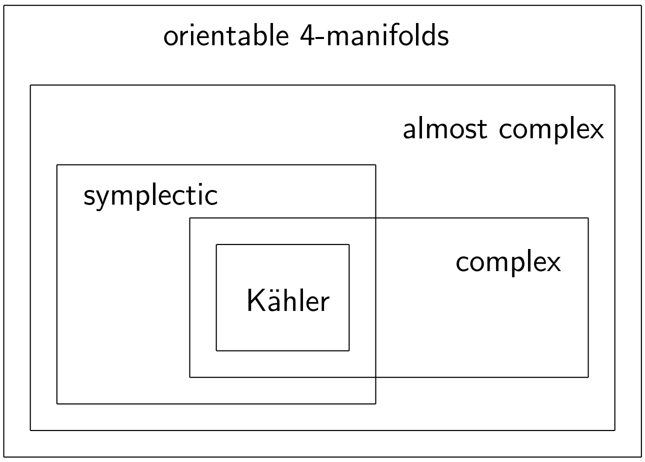
This is globally well-defined thanks to the Cauchy-Riemann equations.

A **Kähler manifold** is a symplectic manifold (M, ω) which is also a complex manifold and where the map that assigns to each point $p \in M$ the bilinear pairing $g_p : T_pM \times T_pM \to \mathbb{R}$, $g_p(u, v) := \omega_p(u, J_pv)$ is a riemannian metric, the map J being the almost complex structure induced by the complex coordinates. This **compatibility condition** comprises the *positivity* $\omega_p(v, J_pv) > 0$ for all $v \neq 0$ and the *symmetry* which translates into $\omega_p(J_pu, J_pv) = \omega_p(u, v)$ for all u, v. The symplectic form ω is then called a **Kähler form**.

A linear algebra argument known as the *polar decomposition* shows that any symplectic vector space, i.e., a vector space equipped with a nondegenerate skew-symmetric bilinear pairing Ω, admits a *compatible* linear complex structure, that is, a linear complex structure J such that $\Omega(\cdot, J\cdot)$ is an inner product. It is enough to start with a choice of an arbitrary inner product G, take the matrix A that satisfies the relation $\Omega(u,v) = G(Au, v)$ for all vectors u, v and consider $J := (\sqrt{AA^t})^{-1}A$.

Being canonical after the choice of G, the above argument may be performed *smoothly* on a symplectic manifold with some riemannian metric. This shows that any symplectic manifold admits compatible almost complex structures.

The following diagram faithfully represents the relations among these structures for *closed* 4-manifolds, where each region admits representatives.



Not all 4-dimensional manifolds are almost complex. A result of Wu [81] gives a necessary and sufficient condition in terms of the signature σ and the Euler characteristic χ of a 4-dimensional closed manifold M for the existence of an almost complex structure: $3\sigma + 2\chi = h^2$ *for some* $h \in H^2(M;\mathbb{Z})$ *congruent with the second Stiefel-Whitney class* $w_2(M)$ *modulo 2*.

Example. The sphere S^4 and $(S^2 \times S^2)\#(S^2 \times S^2)$ are not almost complex. $\diamondsuit$

When an almost complex structure exists, the first Chern class of the tangent bundle (regarded as a complex vector bundle) satisfies the condition for h. The sufficiency of Wu's condition is the remarkable part.[7]

The Newlander-Nirenberg theorem [56] gives a necessary and sufficient condition for an almost complex manifold (M, J) to actually be complex, i.e., for a J to be actually induced by an underlying complex atlas. That condition can

[7] Moreover, such solutions h are in one-to-one correspondence with *isomorphism* classes of almost complex structures.

be phrased in terms of a Dolbeault operator or in terms of the vanishing of the *Nijenhuis tensor*:

$$\mathcal{N}(X,Y) := [JX,JY] - J[JX,Y] - J[X,JY] - [X,Y] \ ,$$

for vector fields X and Y on M, $[\cdot,\cdot]$ being the usual bracket.[8]

According to Kodaira's classification [39], a closed complex surface admits a Kähler structure if and only if its first Betti number b_1 is even. The necessity of this condition is a Hodge relation on the Betti numbers: for a compact Kähler manifold, the Hodge theorems [37] imply that the Betti numbers must be the sum of *Hodge numbers* $b^k = \sum_{\ell+m=k} h^{\ell,m}$ where $h^{\ell,m} = h^{m,\ell}$ are integers, hence the odd Betti numbers must be even.

Examples.

- The complex projective plane $\mathbb{CP}^2$ with the Fubini-Study form[9] might be called the simplest example of a closed Kähler 4-manifold.

- The **Kodaira-Thurston example** [76] first demonstrated that a manifold that admits both a symplectic and a complex structure does not have to admit any Kähler structure. Take $\mathbb{R}^4$ with $dx_1 \wedge dy_1 + dx_2 \wedge dy_2$, and Γ the discrete group generated by the four symplectomorphisms:

$$\begin{array}{ccc} (x_1,x_2,y_1,y_2) & \longmapsto & (x_1+1,x_2,y_1,y_2) \\ (x_1,x_2,y_1,y_2) & \longmapsto & (x_1,x_2+1,y_1,y_2) \\ (x_1,x_2,y_1,y_2) & \longmapsto & (x_1,x_2+y_2,y_1+1,y_2) \\ (x_1,x_2,y_1,y_2) & \longmapsto & (x_1,x_2,y_1,y_2+1) \end{array}$$

  Then $M = \mathbb{R}^4/\Gamma$ is a symplectic manifold that is a 2-torus bundle over a 2-torus. Kodaira's classification [39] shows that M has a complex structure. However, $\pi_1(M) = \Gamma$, hence $H_1(\mathbb{R}^4/\Gamma;\mathbb{Z}) = \Gamma/[\Gamma,\Gamma]$ has rank 3, so $b_1 = 3$ is *odd*.

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[8] The **bracket** of vector fields X and Y is the vector field $[X,Y]$ characterized by the property that $\mathcal{L}_{[X,Y]}f := \mathcal{L}_X(\mathcal{L}_Y f) - \mathcal{L}_Y(\mathcal{L}_X f)$, for $f \in C^\infty(M)$, where $\mathcal{L}_X f = df(X)$.

[9] The 2-form

$$\omega_{\rm FS} = \tfrac{i}{2}\partial\bar{\partial}\log(|z|^2+1)$$

is a Kähler form on $\mathbb{C}^n$, called the **Fubini-Study form** on $\mathbb{C}^n$. Since $\omega_{\rm FS}$ is preserved by the transition maps of the usual complex atlas on $\mathbb{CP}^n$, it induces forms on each chart which glue well together to form the **Fubini-Study form** on $\mathbb{CP}^n$.

- Fernández-Gotay-Gray [19] first exhibited symplectic manifolds that do not admit any complex structure at all. Their examples are circle bundles over circle bundles (i.e., a *tower* of circle bundles) over a 2-torus.

- The **Hopf surface** is the complex surface diffeomorphic to $S^1 \times S^3$ obtained as the quotient $\mathbb{C}^2 \backslash \{0\}/\Gamma$ where $\Gamma = \{2^n \mathrm{Id} \mid n \in \mathbb{Z}\}$ is a group of *complex* transformations, i.e., we factor $\mathbb{C}^2 \backslash \{0\}$ by the equivalence relation $(z_1, z_2) \sim (2z_1, 2z_2)$. The Hopf surface is not symplectic because $H^2(S^1 \times S^3) = 0$.

- The manifold $\mathbb{CP}^2 \# \mathbb{CP}^2 \# \mathbb{CP}^2$ is almost complex but is neither complex (since it does not fit Kodaira's classification [39]), nor symplectic as shown by Taubes [72] using Seiberg-Witten invariants (Section 2.3).

$\diamondsuit$

We could go through the previous discussion restricting to closed 4-dimensional examples *with a specific fundamental group*. Let us consider only simply connected examples. It is a consequence of Wu's result [81] that a simply connected manifold admits an almost complex structure if and only if b_2^+ is odd. By Kodaira's classification [39], a simply connected complex surface always admits a compatible symplectic form (since $b^1 = 0$ is even), i.e., it is always Kähler. Hence, the previous picture collapses in this class where

$$\text{complex} \qquad \Longrightarrow \qquad \text{symplectic (Kähler)}.$$

Examples.

- The connected sum $\#_m \mathbb{CP}^2 \#_n \overline{\mathbb{CP}^2}$ (of m copies of $\mathbb{CP}^2$ with n copies of $\overline{\mathbb{CP}^2}$) has an almost complex structure if and only if m is odd.

- The simply connected manifolds S^4, $\mathbb{CP}^2 \# \mathbb{CP}^2 \# \mathbb{CP}^2$ and $\mathbb{CP}^2$ provide examples of not almost complex, almost complex but not symplectic and complex (Kähler) 4-manifolds, respectively. All of $\mathbb{CP}^2 \#_m \overline{\mathbb{CP}^2}$ are also simply connected Kähler manifolds because they are *pointwise blow-ups*[10] of $\mathbb{CP}^2$ and the *blow-down map* is holomorphic.

[10] *Symplectic blow-up* is the extension to the symplectic category of the blow-up operation in algebraic geometry. It is due to Gromov according to the first printed exposition of this operation in [44]. Let $L = \{([p], z) \mid p \in \mathbb{C}^n \setminus \{0\},\ z = \lambda p \text{ for some } \lambda \in \mathbb{C}\}$ be the **tautological line bundle** over $\mathbb{CP}^{n-1}$ with projection to $\mathbb{CP}^{n-1}$ given by $\pi : ([p], z) \mapsto [p]$. The fiber of L over the point $[p] \in \mathbb{CP}^{n-1}$ is the complex line in $\mathbb{C}^n$ represented by that

- There is a family of manifolds obtained from $\mathbb{CP}^2\#_9\overline{\mathbb{CP}^2} =: E(1)$ by a *knot surgery* that were shown by Fintushel and Stern [22] to be symplectic and confirmed by Jongil Park [59] not to admit a complex structure.[11]

$\diamondsuit$

point. The **blow-up of $\mathbb{C}^n$ at the origin** is the total space of the bundle L, sometimes denoted $\widetilde{\mathbb{C}}^n$. The corresponding **blow-down map** is the map $\beta : L \to \mathbb{C}^n$, $\beta([p], z) = z$. The zero section of L is called the **exceptional divisor** E, is diffeomorphic to $\mathbb{CP}^{n-1}$ and gets mapped to the origin by β. On the other hand, the restriction of β to the complementary set $L \setminus E$ is a diffeomorphism onto $\mathbb{C}^n \setminus \{0\}$. Hence, we may regard L as being obtained from $\mathbb{C}^n$ by smoothly replacing the origin by a copy of $\mathbb{CP}^{n-1}$. Every biholomorphic map $f : \mathbb{C}^n \to \mathbb{C}^n$ with $f(0) = 0$ lifts uniquely to a biholomorphic map $\widetilde{f} : L \to L$ with $\widetilde{f}(E) = E$. The lift is given by the formula

$$\widetilde{f}([p], z) = \begin{cases} ([f(z)], f(z)) & \text{if } z \neq 0 \\ ([p], 0) & \text{if } z = 0 \ . \end{cases}$$

The map β is $\mathrm{U}(n)$-equivariant for the action of the unitary group $\mathrm{U}(n)$ on L induced by the standard linear action on $\mathbb{C}^n$. For instance, $\beta^*\omega_0 + \pi^*\omega_{FS}$ is a $\mathrm{U}(n)$-invariant Kähler form on L. A **blow-up symplectic form** on the tautological line bundle L is a $\mathrm{U}(n)$-invariant symplectic form ω such that the difference $\omega - \beta^*\omega_0$ is compactly supported, where $\omega_0 = \frac{i}{2}\sum_{k=1}^n dz_k \wedge d\bar{z}_k$ is the standard symplectic form on $\mathbb{C}^n$. Two blow-up symplectic forms are **equivalent** if one is the pullback of the other by a $\mathrm{U}(n)$-equivariant diffeomorphism of L. Guillemin and Sternberg [36] showed that two blow-up symplectic forms are equivalent if and only if they have equal restrictions to the exceptional divisor $E \subset L$. Let Ω^ε $(\varepsilon > 0)$ be the set of all blow-up symplectic forms on L whose restriction to the exceptional divisor $E \simeq \mathbb{CP}^{n-1}$ is $\varepsilon\omega_{\mathrm{FS}}$, where ω_{FS} is the Fubini-Study form. An **ε-blow-up** of $\mathbb{C}^n$ at the origin is a pair (L, ω) with $\omega \in \Omega^\varepsilon$. Let (M, ω) be a $2n$-dimensional symplectic manifold. It is shown in [36] that, for ε small enough, we can perform an ε-blow-up of M at p modeled on $\mathbb{C}^n$ at the origin, without changing the symplectic structure outside of a small neighborhood of p. The resulting manifold is called an **ε-blow-up of M at p**. As a manifold, the blow-up of M at a point is diffeomorphic to the *connected sum* $M\#\overline{\mathbb{CP}^n}$, where $\overline{\mathbb{CP}^n}$ is the manifold $\mathbb{CP}^n$ equipped with the orientation opposite to the natural complex one. When $(\mathbb{CP}^{n-1}, \omega_{FS})$ is symplectically embedded in a symplectic manifold (M, ω) with image X and normal bundle isomorphic to the tautological bundle L, it can be subject to a *blow-down* operation [50, §7.1]. Moreover, we can define a *blow-up of the symplectic manifold* (M, ω) *along the symplectic submanifold.*

[11] The first example of a closed simply connected symplectic manifold that cannot be Kähler, was a 10-dimensional manifold obtained by McDuff [44] by *blowing-up* $(\mathbb{CP}^5, \omega_{FS})$ along the image of a symplectically embedded [35, 77] Kodaira-Thurston example $\mathbb{R}^4/\Gamma$.

2.2 pseudoholomorphic curves

Since the mid 80's, symplectic geometry has been revolutionized by powerful analogues of complex-geometric tools, such as pseudoholomorphic curves (leading to Gromov-Witten invariants and Floer homology) and Lefschetz pencils (leading to a topological description of symplectic 4-manifolds; see Section 2.5).

Whereas an almost complex manifold (M, J) tends to have no J-holomorphic functions $M \to \mathbb{C}$ at all,[12] it has plenty of *J-holomorphic curves* $\mathbb{C} \to M$. Gromov first realized that *pseudoholomorphic curves* provide a powerful tool in symplectic topology in an extremely influential paper [34].

Fix a closed Riemann surface (Σ, j), that is, a closed complex 1-dimensional manifold Σ equipped with the canonical almost complex structure j. A (parametrized) **pseudoholomorphic curve** (or J-**holomorphic curve**) in (M, J) is a (smooth) map $u : \Sigma \to M$ whose differential intertwines j and J, that is, $du_p \circ j_p = J_p \circ du_p$, $\forall p \in \Sigma$. The last condition, requiring that du_p be complex-linear, amounts to the **Cauchy-Riemann equation**: $du + J \circ du \circ j = 0$, a well-behaved (elliptic) system of first order partial differential equations.

When J is a compatible almost complex structure on a symplectic manifold (M, ω), pseudoholomorphic curves are related to parametrized 2-dimensional symplectic submanifolds.[13] If a pseudoholomorphic curve $u : \Sigma \to M$ is an embedding, then its image $S := u(\Sigma)$ is a 2-dimensional almost complex submanifold, hence a symplectic submanifold. Conversely, the inclusion $i : S \hookrightarrow M$ of a 2-dimensional symplectic submanifold can be seen as a pseudoholomorphic curve. An appropriate compatible almost complex structure J on (M, ω) can be constructed starting from S, such that TS is J-invariant. The restriction j of J to TS is necessarily integrable because S is 2-dimensional.

The group G of complex diffeomorphisms of (Σ, j) acts on (parametrized) pseudoholomorphic curves by reparametrization: $u \mapsto u \circ \gamma$, for $\gamma \in G$. This normally means that each curve u has a noncompact orbit under G. The orbit space $\mathcal{M}_g(A, J)$ is the set of unparametrized pseudoholomorphic curves in (M, J) whose domain Σ has genus g and whose image $u(\Sigma)$ has homology class $A \in H_2(M; \mathbb{Z})$. The space $\mathcal{M}_g(A, J)$ is called the **moduli space of unparametrized pseudoholomorphic curves** of genus g representing the class A. For generic J, Fredholm theory shows that pseudoholomorphic curves occur

[12] However, the study of *asymptotically J-holomorphic functions* has been recently developed to obtain important results [14, 16, 7]; see Section 2.5.

[13] A **symplectic submanifold** of a symplectic manifold (M, ω) is a submanifold X of M where, at each $p \in X$, the restriction of ω_p to the subspace T_pX is nondegenerate.

in finite-dimensional smooth families, so that the moduli spaces $\mathcal{M}_g(A, J)$ can be manifolds, after avoiding singularities given by *multiple coverings*.[14]

Example. Often, Σ is the Riemann sphere $\mathbb{CP}^1$, whose complex diffeomorphisms are those given by *fractional linear transformations* (or *Möbius transformations*). So the 6-dimensional noncompact group of projective linear transformations $\mathrm{PSL}(2;\mathbb{C})$ acts on **pseudoholomorphic spheres** by reparametrization $u \mapsto u \circ \gamma_A$, where $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{PSL}(2;\mathbb{C})$ acts by $\gamma_A : \mathbb{CP}^1 \to \mathbb{CP}^1$, $\gamma_A[z,1] = [\frac{az+b}{cz+d}, 1]$. $\diamondsuit$

When J is an almost complex structure *compatible* with a symplectic form ω, the area of the image of a pseudoholomorphic curve u (with respect to the metric $g_J(\cdot,\cdot) = \omega(\cdot, J\cdot)$) is determined by the class A that it represents. The number

$$E(u) := \omega(A) = \int_\Sigma u^*\omega = \text{ area of the image of } u \text{ with respect to } g_J$$

is called the **energy** of the curve u and is a topological invariant: it only depends on $[\omega]$ and on the homotopy class of u. Gromov proved that the constant energy of all the pseudoholomorphic curves representing a homology class A ensured that the space $\mathcal{M}_g(A, J)$, though not necessarily compact, had natural **compactifications** $\overline{\mathcal{M}}_g(A, J)$ by including what he called *cusp-curves*.

Theorem 2.1 (Gromov's compactness theorem) *If (M, ω) is a compact manifold equipped with a generic compatible almost complex structure J, and if u_j is a sequence of pseudoholomorphic curves in $\mathcal{M}_g(A, J)$, then there is a subsequence that weakly converges to a cusp-curve in $\overline{\mathcal{M}}_g(A, J)$.*

Hence the cobordism class of the compactified moduli space $\overline{\mathcal{M}}_g(A, J)$ might be a nice symplectic invariant of (M, ω), as long as it is not empty or null-cobordant. Actually, a nontrivial regularity criterion for J ensures the existence of pseudoholomorphic curves. And even when $\overline{\mathcal{M}}_g(A, J)$ is null-cobordant, we can define an invariant to be the (signed) number of pseudoholomorphic curves of genus g in class A that intersect a specified set of representatives of homology classes in M [64, 73, 80]. For more on pseudoholomorphic curves, see for instance [49] (for a comprehensive discussion of the genus 0 case) or [5]

[14] A curve $u : \Sigma \to M$ is a **multiple covering** if u factors as $u = u' \circ \sigma$ where $\sigma : \Sigma \to \Sigma'$ is a holomorphic map of degree greater than 1.

(for higher genus). Here is a selection of applications of (developments from) pseudoholomorphic curves:

- Proof of the **nonsqueezing theorem** [34]: for $R > r$ there is no symplectic embedding of a ball B_R^{2n} of radius R into a cylinder $B_r^2 \times \mathbb{R}^{2n-2}$ of radius r, both in $(\mathbb{R}^{2n}, \omega_0)$.

- Proof that there are *no lagrangian spheres*[15] in $(\mathbb{C}^n, \omega_0)$, except for the circle in $\mathbb{C}^2$, and more generally *no compact exact lagrangian submanifolds*, in the sense that the tautological 1-form α restricts to an exact form [34].

- Proof that if (M, ω) is a connected symplectic 4-manifold symplectomorphic to $(\mathbb{R}^4, \omega_0)$ outside a compact set and containing no symplectic S^2's, then (M, ω) symplectomorphic to $(\mathbb{R}^4, \omega_0)$ [34].

- Study questions of **symplectic packing** [9, 48, 78] such as: for a given $2n$-dimensional symplectic manifold (M, ω), what is the maximal radius R for which there is a symplectic embedding of N disjoint balls B_R^{2n} into (M, ω)?

- Study **groups of symplectomorphisms** of 4-manifolds (for a review see [47]). Gromov [34] showed that the groups of symplectomorphisms of $(\mathbb{CP}^2, \omega_{_{\mathrm{FS}}})$ and of $(S^2 \times S^2, \mathrm{pr}_1^*\sigma \oplus \mathrm{pr}_2^*\sigma)$ deformation retract onto the corresponding groups of standard isometries.

- Development of **Gromov-Witten invariants**[16] allowing to prove, for instance, the nonexistence of symplectic forms on $\mathbb{CP}^2 \# \mathbb{CP}^2 \# \mathbb{CP}^2$ or the classification of symplectic structures on *ruled surfaces*.[17]

[15] A submanifold X of a symplectic manifold (M, ω) is **lagrangian** if, at each $p \in X$, the restriction of ω_p to the subspace T_pX is trivial and $\dim X = \frac{1}{2} \dim M$.

[16] For defining Gromov-Witten invariants, n marked points are introduced on the surface Σ of genus g so that the Euler characteristic of the n-punctured surface, $2-2g-n$, becomes negative, the group of reparametrizations preserving the marked points becomes finite, and an actual count of curves is possible. In fact, a *virtual* version of this count is most commonly used yielding rational numbers.

[17] A (rational) **ruled surface** is a complex (Kähler) surface that is the total space of a holomorphic fibration over a Riemann surface with fiber $\mathbb{CP}^1$. When the base is also a sphere, these are the **Hirzebruch surfaces** $\mathbb{P}(L \oplus \mathbb{C})$ where L is a holomorphic line bundle over $\mathbb{CP}^1$. A **symplectic ruled surface** is a symplectic 4-manifold (M, ω) that is the total space of an S^2-fibration where ω is nondegenerate on the fibers.

- Development of **Floer homology** to prove the Arnold conjecture [2, Appendix 9] on the fixed points of symplectomorphisms of compact symplectic manifolds, or on the intersection of lagrangian submanifolds (see, for instance, [17, 65]).
- Development of **symplectic field theory** introduced by Eliashberg, Givental and Hofer [18] extending Gromov-Witten theory, exhibiting a rich algebraic structure and also with applications to *contact geometry*.

2.3 invariants for 4-manifolds

In 1994 Witten brought about a revolution in Donaldson theory by introducing a new set of invariants – the **Seiberg-Witten invariants** – which are much simpler to calculate and to apply. This new viewpoint was inspired by developments due to Seiberg and Witten in the understanding of $N = 2$ *supersymmetric Yang-Mills*.

Let M be a smooth oriented closed 4-dimensional manifold with $b_2^+(M) > 1$ (there is a version for $b_2^+(M) = 1$). All such 4-manifolds M (with any $b_2^+(M)$) admit a spin-c structure, i.e., a $\mathrm{Spin}^c(4)$-bundle over M with an isomorphism of the associated $\mathrm{SO}(4)$-bundle to the bundle of oriented frames on the tangent bundle for some chosen riemannian metric. Let $\mathcal{C}_M = \{a \in H^2(M;\mathbb{Z}) \mid a \equiv w_2(TM)(2)\}$ be the set of characteristic elements, and let $\mathrm{Spin}^c(M)$ be the set of spin-c structures on M. For simplicity, assume that M is simply connected (or at least that $H_1(M;\mathbb{Z})$ has no 2-torsion), so that $\mathrm{Spin}^c(M)$ is isomorphic to $\mathcal{C}_M$ with isomorphism given by the first Chern class of the *determinant line bundle* (the **determinant line bundle** is the line bundle associated by a natural group homomorphism $\mathrm{Spin}^c(4) \to \mathrm{U}(1)$). Fix an orientation of a maximal-dimensional positive-definite subspace $H^2_+(M;\mathbb{R}) \subset H^2(M;\mathbb{R})$. The Seiberg-Witten invariant is the function

$$\mathrm{SW}_M : \mathcal{C}_M \longrightarrow \mathbb{Z}$$

defined as follows. Given a spin-c structure $\alpha \in \mathrm{Spin}^c(M) \simeq \mathcal{C}_M$, the image $\mathrm{SW}_M(\alpha) = [\mathcal{M}] \in H_d(\mathcal{B}^*;\mathbb{Z})$ is the homology class of the moduli space $\mathcal{M}$ of solutions (called **monopoles**) of the Seiberg-Witten (SW) equations modulo gauge equivalence. The SW equations are non-linear differential equations on a pair of a connection A on the determinant line bundle of α and of a section φ of an associated $\mathrm{U}(2)$-bundle, called the positive (half) spinor bundle:

$$F_A^+ = iq(\varphi) \qquad \text{and} \qquad D_A\varphi = 0 \ ,$$

where F_A^+ is the self-dual part of the (imaginary) curvature of A, q is a squaring operation taking sections of the positive spinor bundle to self-dual 2-forms, and D_A is the corresponding Dirac operator. For a generic perturbation of the equations (replacing the first equation by $F_A^+ = iq(\varphi) + i\nu$, where ν is a self-dual 2-form) and of the riemannian metric, a transversality argument shows that the moduli space $\mathcal{M}$ is well-behaved and actually inside the space $\mathcal{B}^*$ of gauge-equivalence classes of irreducible pairs (those (A, φ) for which $\varphi \neq 0$), which is homotopy-equivalent to $\mathbb{CP}^\infty$ and hence has even-degree homology groups $H_d(\mathcal{B}^*; \mathbb{Z}) \simeq \mathbb{Z}$. When the dimension d of $\mathcal{M}$ is odd or when $\mathcal{M}$ is empty, the invariant $\mathrm{SW}_M(\alpha)$ is set to be zero. The **basic classes** are the classes $\alpha \in \mathcal{C}_M$ for which $\mathrm{SW}_M(\alpha) \neq 0$. The set of basic classes is always finite, and if α is a basic class then so is $-\alpha$. The main results are that the Seiberg-Witten invariants are invariants of the diffeomorphism type of the 4-manifold M and satisfy vanishing and nonvanishing theorems, which allowed to answer an array of questions about specific manifolds.

Taubes [73] discovered an equivalence between Seiberg-Witten and Gromov invariants (using pseudoholomorphic curves) for symplectic 4-manifolds, by proving the existence of pseudoholomorphic curves from solutions of the Seiberg-Witten equations and vice-versa. As a consequence, he proved:

Theorem 2.2 (Taubes) *Let (M, ω) be a compact symplectic 4-manifold.*

If $b_2^+ > 1$, then $c_1(M, \omega)$ admits a smooth pseudoholomorphic representative.

If $M = M_1 \# M_2$, then one of the M_i's has negative definite intersection form.

There are results also for $b_2^+ = 1$, and follow-ups describe the set of basic classes of a connected sum $M \# N$ in terms of the set of basic classes of M when N is a manifold with negative definite intersection form (starting with $\overline{\mathbb{CP}^2}$).

In an attempt to understand other 4-manifolds via Seiberg-Witten and Gromov invariants, some analysis of pseudoholomorphic curves has been extended to nonsymplectic 4-manifolds by equipping these with a *nearly nondegenerate closed 2-form*. In particular, Taubes [75] has related Seiberg-Witten invariants to pseudoholomorphic curves for compact oriented 4-manifolds with $b_2^+ > 0$. Any compact oriented 4-manifold M with $b_2^+ > 0$ admits a closed 2-form that vanishes along a union of circles and is symplectic elsewhere [27, 38]. In fact, for a generic metric on M, there is a self-dual harmonic form ω which is transverse to zero as a section of $\Lambda^2 T^* M$. The vanishing locus of ω is the union of a finite number of embedded circles, and ω is symplectic elsewhere.

The generic behavior of closed 2-forms on orientable 4-manifolds is partially understood [3, pp.23-24]. Here is a summary. Let ω be a generic closed 2-form on a 4-manifold M. At the points of some hypersurface Z, the form ω has rank 2. At a generic point of M, ω is nondegenerate; in particular, has the Darboux normal form $dx_1 \wedge dy_1 + dx_2 \wedge dy_2$. There is a codimension-1 submanifold Z where ω has rank 2, and there are no points where ω vanishes. At a generic point of Z, the kernel of $\widetilde{\omega}$ is transverse to Z; the normal form near such a point is $x_1 dx_1 \wedge dy_1 + dx_2 \wedge dy_2$. There is a curve C where the kernel of $\widetilde{\omega}$ is not transverse to Z, hence sits in TZ. At a generic point of C, the kernel of $\widetilde{\omega}$ is transverse to C; there are two possible normal forms near such points, called *elliptic* and *hyperbolic*, $d(x - \frac{z^2}{2}) \wedge dy + d(xz \pm ty - \frac{z^3}{3}) \wedge dt$. The hyperbolic and elliptic sections of C are separated by *parabolic* points, where the kernel is tangent to C. It is known that there exists at least one continuous family of inequivalent degeneracies in a parabolic neighborhood [28].

2.4 geography and botany

Symplectic geography [32, 68] addresses the following existence question (cf. Section 1.3):

– What is the set of pairs of integers $(m, n) \in \mathbb{Z} \times \mathbb{Z}$ for which there exists a connected simply connected closed *symplectic* 4-manifold M having second Betti number $b_2(M) = m$ and signature $\sigma(M) = n$?

This problem includes the usual geography of simply connected complex surfaces, since all such surfaces are Kähler according to Kodaira's classification [39]. Often, instead of the numbers (b_2, σ), the question is equivalently phrased in terms of the Chern numbers $(c_1^2, c_2) = (3\sigma + 2\chi, \chi)$ for a compatible almost complex structure, where $\chi = b_2 + 2$ is the *Euler number*, cf. Section 1.3. Usually only *minimal*[18] or *irreducible* (Section 1.5) manifolds are considered to avoid trivial examples. These questions could be posed for other fundamental groups.

A first attempt to produce new symplectic manifolds from old is to use con-

[18] Following algebraic geometry, we call **minimal** a $2n$-dimensional symplectic manifold (M, ω) without any symplectically embedded $(\mathbb{CP}^{n-1}, \omega_{FS})$, so that (M, ω) is not the blow-up at a point of another symplectic manifold. In dimension 4, a manifold is minimal if it does not contain any embedded sphere S^2 with self-intersection -1. Indeed, by the work of Taubes [72, 74], if such a sphere S exists, then either the homology class $[S]$ or its symmetric $-[S]$ can be represented by a *symplectically* embedded sphere with self-intersection -1.

nected sums. Yet, in dimensions other than 2 and 6, a connected sum $M_0 \# M_1$ of closed symplectic manifolds (M_0, ω_0) and (M_1, ω_1) does not admit a symplectic form isotopic to ω_i on each M_i minus a ball, $i = 0, 1$. The reason [4] is that such a symplectic form on $M_0 \# M_1$ would allow to construct an almost complex structure on the sphere formed by the union of the two removed balls, which is known not to exist except on S^2 and S^6.

For connected sums to work in the symplectic category, in particular for 4-manifolds, they should be done along codimension 2 symplectic submanifolds. The following construction, already mentioned in [35], was dramatically explored by Gompf [29]. Let (M_0, ω_0) and (M_1, ω_1) be two $2n$-dimensional symplectic manifolds. Suppose that a compact symplectic manifold (X, α) of dimension $2n - 2$ admits symplectic embeddings to both $i_0 : X \hookrightarrow M_0$, $i_1 : X \hookrightarrow M_1$. For simplicity, assume that the corresponding normal bundles are trivial (in general, they need to have symmetric Euler classes). By the symplectic neighborhood theorem, there exist symplectic embeddings $j_0 : X \times B_\varepsilon \to M_0$ and $j_1 : X \times B_\varepsilon \to M_1$ (called **framings**) where B_ε is a ball of radius ε and centered at the origin in $\mathbb{R}^2$ such that $j_k^* \omega_k = \alpha + dx \wedge dy$ and $j_k(p, 0) = i_k(p)$ $\forall p \in X$, $k = 0, 1$. Chose an area- and orientation-preserving diffeomorphism ϕ of the annulus $B_\varepsilon \setminus B_\delta$ for $0 < \delta < \varepsilon$ that interchanges the two boundary components. Let $\mathcal{U}_k = j_k(X \times B_\delta) \subset M_k$, $k = 0, 1$. A **symplectic sum** of M_0 and M_1 along X is defined to be

$$M_0 \#_X M_1 := (M_0 \setminus \mathcal{U}_0) \cup_\phi (M_1 \setminus \mathcal{U}_1)$$

where the symbol $\cup_\phi$ means that we identify $j_1(p, q)$ with $j_0(p, \phi(q))$ for all $p \in X$ and $\delta < |q| < \varepsilon$. As ω_0 and ω_1 agree on the regions under identification, they induce a symplectic form on $M_0 \#_X M_1$. The result depends on j_0, j_1, δ and ϕ.

Gompf [29] used *symplectic sums* to prove the following theorem, and also showed that his surgery construction can be adapted to produce *non*-Kähler examples. Since finitely-presented groups are not classifiable, this shows that compact symplectic 4-manifold are not classifiable.

Theorem 2.3 (Gompf) *Every finitely-presented group occurs as the fundamental group $\pi_1(M)$ of a compact symplectic 4-manifold (M, ω).*

Symplectic botany [23] addresses the following uniqueness question (cf. Section 1.3):

– Given a pair of integers $(m, n) \in \mathbb{Z} \times \mathbb{Z}$, what are all the connected simply connected closed *symplectic* 4-manifolds M having second Betti number $b_2(M) = m$ and signature $\sigma(M) = n$ (up to diffeomorphism)?

The answer here is still less clear. There has been significant research on classes of surgery operations that can be used to produce examples, such as fiber sums, surgery on tori, blow-up and rational blow-downs, some of which discussed in the next couple of sections. In particular, if a symplectic 4-manifold has a nontrivial Seiberg-Witten invariants and contains a symplectically embedded minimal genus torus with self-intersection zero and with simply connected complement, then by *knot surgery* one can show that it also admits infinitely many distinct smooth symplectic structures (as well as infinitely many distinct smooth nonsymplectic structures) [23].

Instead of smoothly, the uniqueness question can be addressed symplectically, where different identifications compete. Let (M, ω_0) and (M, ω_1) be two symplectic manifolds (with the same underlying manifold M).

- (M, ω_0) and (M, ω_1) are **symplectomorphic** if there is a diffeomorphism $\varphi : M \to M$ such that $\varphi^* \omega_1 = \omega_0$.
- (M, ω_0) and (M, ω_1) are **strongly isotopic** if there is an isotopy $\rho_t : M \to M$ such that $\rho_1^* \omega_1 = \omega_0$.
- (M, ω_0) and (M, ω_1) are **deformation-equivalent** if there is a smooth family ω_t of symplectic forms joining ω_0 to ω_1.
- (M, ω_0) and (M, ω_1) are **isotopic** if they are deformation-equivalent and the de Rham cohomology class $[\omega_t]$ is independent of t.
- (M, ω_0) and (M, ω_1) are **equivalent** if they are related by a combination of deformation-equivalences and symplectomorphisms.

Hence, *equivalence* is the relation generated by deformations and diffeomorphisms. The corresponding equivalence classes can be viewed as the connected components of the moduli space of symplectic forms up to diffeomorphism. *Equivalence* deserves this simple designation because this notion allows the cleanest statements about uniqueness when focusing on topological properties.

Examples.

1. The complex projective plane $\mathbb{CP}^2$ has a unique symplectic structure up to symplectomorphism and scaling. This was shown by Taubes [73] relating Seiberg-Witten invariants (Section 2.3) to pseudoholomorphic curves

to prove the existence of a pseudoholomorphic sphere. Previous work of Gromov [34] and McDuff [46] showed that the existence of a pseudoholomorphic sphere implies that the symplectic form is standard.

2. Lalonde and McDuff [41] concluded similar classifications for symplectic ruled surfaces and for symplectic rational surfaces. [19] The symplectic form on a symplectic ruled surface is unique up to symplectomorphism in its cohomology class, and is isotopic to a standard Kähler form. In particular, any symplectic form on $S^2 \times S^2$ is symplectomorphic to $a\pi_1^*\sigma + b\pi_2^*\sigma$ for some $a, b > 0$ where σ is the standard area form on S^2.

3. Li and Liu [42] showed that the symplectic structure on $\mathbb{CP}^2 \#_n \overline{\mathbb{CP}^2}$ for $2 \leq n \leq 9$ is unique up to equivalence.

4. McMullen and Taubes [51] first exhibited simply connected closed 4-manifolds admitting inequivalent symplectic structures. Their examples were constructed using 3-dimensional topology, and distinguished by analyzing the structure of Seiberg-Witten invariants to show that the first Chern classes of the two symplectic structures lie in disjoint orbits of the diffeomorphism group. In higher dimensions there were previously examples of manifolds with inequivalent symplectic forms; see for instance [63].

5. With symplectic techniques and avoiding gauge theory, Smith [66] showed that, for each $n \geq 2$, there is a simply connected closed 4-manifold that admits at least n inequivalent symplectic forms, also distinguished via the first Chern classes. It is not yet known whether there exist inequivalent symplectic forms on a 4-manifold with the same first Chern class.

$\diamondsuit$

2.5 lefschetz pencils

Lefschetz pencils in symplectic geometry imitate linear systems in complex geometry. Whereas holomorphic functions on a projective surface must be constant, there are interesting functions on the complement of a finite set, and generic such functions have only quadratic singularities. A Lefschetz pencil can be viewed as a complex Morse function [53] or as a very singular fibration, in the sense that,

[19] A **symplectic rational surface** is a symplectic 4-manifold (M, ω) that can be obtained from the standard $(\mathbb{CP}^2, \omega_{FS})$ by blowing up and blowing down.

not only some fibers are singular (have ordinary double points) but all *fibers* go through some points.

A **Lefschetz pencil** on an oriented 4-manifold M is a map $f : M\backslash\{b_1, \ldots, b_n\} \to \mathbb{CP}^1$ defined on the complement of a finite set in M, called the **base locus**, that is a submersion away from a finite set $\{p_1, \ldots, p_{n+1}\}$, and obeying local models $(z_1, z_2) \mapsto z_1/z_2$ near the b_j's and $(z_1, z_2) \mapsto z_1 z_2$ near the p_j's, where (z_1, z_2) are oriented local complex coordinates.

Usually it is also required that each fiber contains at most one singular point. By blowing-up M at the b_j's, we obtain a map to $\mathbb{CP}^1$ on the whole manifold, called a **Lefschetz fibration**.[20] Lefschetz pencils and Lefschetz fibrations can be defined on higher dimensional manifolds where the b_j's are replaced by codimension 4 submanifolds. By working on the Lefschetz fibration, Gompf [30, 31] proved that a structure of Lefschetz pencil (with a nontrivial base locus) gives rise to a symplectic form, canonical up to isotopy, such that the fibers are symplectic.

Using asymptotically holomorphic techniques [6, 14], Donaldson [16] proved that symplectic 4-manifolds admit Lefschetz pencils. More precisely:

Theorem 2.4 (Donaldson) *Let J be a compatible almost complex structure on a compact symplectic 4-manifold (M, ω) where the class $[\omega]/2\pi$ is integral. Then J can be deformed through almost complex structures to an almost complex structure J' such that M admits a Lefschetz pencil with J'-holomorphic fibers.*

The closure of a smooth fiber of the Lefschetz pencil is a symplectic submanifold Poincaré dual to $k[\omega]/2\pi$. The starting point was actually a theorem of Donaldson's on the existence of such manifolds:

Theorem 2.5 (Donaldson) *Let (M, ω) be a compact symplectic manifold. Assume that the cohomology class $[\omega]$ is integral, i.e., lies in $H^2(M; \mathbb{Z})$. Then, for every sufficiently large integer k, there exists a connected codimension-2 symplectic submanifold X representing the Poincaré dual of the integral cohomology class $k[\omega]$.*

[20] A **fibration** (or *fiber bundle*) is a manifold M (called the **total space**) with a submersion $\pi : M \to X$ to a manifold X (the **base**) that is locally trivial in the sense that there is an open covering of X, such that, to each set $\mathcal{U}$ in that covering corresponds a diffeomorphism of the form $\varphi_{\mathcal{U}} = (\pi, s_{\mathcal{U}}) : \pi^{-1}(\mathcal{U}) \to \mathcal{U} \times F$ (a **local trivialization**) where F is a fixed manifold (the **model fiber**). A collection of local trivializations such that the sets $\mathcal{U}$ cover X is called a **trivializing cover** for π. Given two local trivializations, the second entry of the composition $\varphi_{\mathcal{V}} \circ \varphi_{\mathcal{U}}^{-1} = (\mathrm{id}, \psi_{\mathcal{UV}})$ on $(\mathcal{U} \cap \mathcal{V}) \times F$ gives the corresponding **transition function** $\psi_{\mathcal{UV}}(x) : F \to F$ at each $x \in \mathcal{U} \cap \mathcal{V}$.

Other perspectives on Lefschetz pencils have been explored, including in terms of representations of the free group $\pi_1(\mathbb{CP}^1 \setminus \{p_1, \ldots, p_{n+1}\})$ in the mapping class group Γ_g of the generic fiber surface [67].

Similar techniques were used by Auroux [7] to realize symplectic 4-manifolds as *branched covers* of $\mathbb{CP}^2$, and thus reduce the classification of symplectic 4-manifolds to a (hard) algebraic question about factorization in the braid group. Let M and N be compact oriented 4-manifolds, and let ν be a symplectic form on N.

A map $f : M \to N$ is a **symplectic branched cover** if for any $p \in M$ there are complex charts centered at p and $f(p)$ such that ν is positive on each complex line and where f is given by: a local diffeomorphism $(x, y) \to (x, y)$, or a simple branching $(x, y) \to (x^2, y)$, or an ordinary cusp $(x, y) \to (x^3 - xy, y)$.

Theorem 2.6 (Auroux) *Let (M, ω) be a compact symplectic 4-manifold where the class $[\omega]$ is integral, and let k be a sufficiently large integer. Then there is a symplectic branched cover $f_k : (M, k\omega) \to \mathbb{CP}^2$, that is canonical up to isotopy for k large enough. Conversely, given a symplectic branched cover $f : M \to N$, the domain M inherits a symplectic form canonical up to isotopy in the class $f^*[\nu]$.*

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