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Journal of Computational and Applied Mathematics 189 (2006) 412–423
JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS
www.elsevier.com/locate/cam

Numerical methods for a Volterra integral equation with non-smooth solutions

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Received 28 September 2004; received in revised form 28 April 2005

Abstract

We consider the numerical treatment of a singular Volterra integral equation with an infinite set of solutions, one of which is smooth and all others have infinite gradient at the origin. This equation has been the subject of previous works, where we have dealt with the approximation of the smooth solution. Here we present numerical methods which enable us to obtain approximations to any of the infinite class of solutions. Some numerical examples are given which illustrate the performance of the methods employed.
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Keywords: Volterra integral equation; Singular kernel; Product integration methods; Multiple solutions

1. Introduction

We consider a class of singular Volterra integral equations of the form

$$u(t) = \int_0^t \frac{s^{\mu-1}}{t^{\mu}} u(s)\, \mathrm{d}s + g(t), \quad t \in (0, T], \tag{1.1}$$

where $\mu > 0$ and g is a given function. This equation with noncompact kernel has been the subject of several works (see, for example, [8,3,5,6]). We note that for values of $\mu > 1$, the kernel is singular only at $t = 0$. In this case a smooth forcing function g leads to a smooth solution u. However, if $0 < \mu < 1$, there is a singularity at $t = 0$ and at $s = 0$ for any positive values of t. It turns out that in this case the equation has an infinite set of solutions.

The following explicit representation for the solutions of (1.1) is given in [4].

Lemma 1.1. (a) *If* $0 < \mu \leqslant 1$ *and* $g \in C^1[0, T]$ (*with* $g(0) = 0$ *if* $\mu = 1$) *then* (1.1) *has a family of solutions* $u \in C[0, T]$ *given by the formula*

$$u(t) = c_0 t^{1-\mu} + g(t) + \gamma + t^{1-\mu} \int_0^t s^{\mu-2}(g(s) - g(0))\, \mathrm{d}s, \tag{1.2}$$

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doi:10.1016/j.cam.2005.10.019

where

$$\gamma := \begin{cases} \dfrac{1}{\mu - 1} g(0) & \text{if } \mu < 1, \\ 0 & \text{if } \mu = 1, \end{cases} \tag{1.3}$$

and c_0 *is an arbitrary constant. Out of the family of solutions there is one particular solution* $u \in C^1[0, T]$. *Such a solution is unique and can be obtained from* (1.2) *by taking* $c_0 = 0$.

(b) *If* $\mu > 1$ *and* $g \in C^m[0, T]$, $m \geqslant 0$, *then the unique solution* $u \in C^m[0, T]$ *of* (1.1) *is*

$$u(t) = g(t) + t^{1-\mu} \int_0^t s^{\mu-2} g(s)\, ds. \tag{1.4}$$

We note that (1.4) *can be obtained from* (1.2) *with* $c_0 = 0$. *Indeed, it follows from* (1.2) *that*

$$c_0 = \lim_{t \to 0^+} t^{\mu-1}\, u(t), \tag{1.5}$$

and this limit is zero when $\mu > 1$.

In principle, if we know the value of c_0 we may use (1.2) to obtain numerical approximations of the solution. However, due to the singularity in the integrand at $s = 0$, the use of a standard quadrature rule to approximate the integral (like the repeated midpoint rule) will yield a method of order μ, assuming $0 < \mu < 1$. Efficient methods to compute the solution which take into account its singular behaviour will be presented in the next sections.

2. Numerical methods

Let us reformulate Eq. (1.1) in the following way. We choose some fixed real number $\varepsilon > 0$. Substituting t by $t + \varepsilon$ in (1.1) gives

$$u(t + \varepsilon) = \frac{1}{(t + \varepsilon)^\mu} \int_0^\varepsilon s^{\mu-1} u(s)\, ds + \int_\varepsilon^{t+\varepsilon} \frac{s^{\mu-1}}{(t + \varepsilon)^\mu} u(s)\, ds + g(t + \varepsilon),$$

or, equivalently,

$$u(t + \varepsilon) = \frac{I_\varepsilon}{(t + \varepsilon)^\mu} + \int_0^t \frac{(s + \varepsilon)^{\mu-1}}{(t + \varepsilon)^\mu} u(s + \varepsilon)\, ds + g(t + \varepsilon), \tag{2.1}$$

where

$$I_\varepsilon := \int_0^\varepsilon s^{\mu-1} u(s)\, ds. \tag{2.2}$$

Here we suppose that I_ε is known exactly for a chosen exact solution and then apply a numerical method to (2.1). We note that the kernel of this new equation is regular in $\{(t, s) : 0 \leqslant s \leqslant t \leqslant T - \varepsilon\}$, so that any standard numerical scheme for regular second kind integral equations can be applied.

Let $X_h := \{t_i = ih + \varepsilon,\ 0 \leqslant i \leqslant N\}$ be a uniform grid of the interval $[\varepsilon, T]$, with stepsize h. Setting $t = nh$ in (2.1) we get

$$u(t_n) = \frac{I_\varepsilon}{t_n^\mu} + \int_0^{nh} \frac{(s + \varepsilon)^{\mu-1}}{t_n^\mu} u(s + \varepsilon)\, ds + g(t_n). \tag{2.3}$$

In Euler's method we approximate $u(s + \varepsilon)$ by $u(t_j)$ on each subinterval $[jh, (j + 1)h]$. Defining

$$D_j := \int_{jh}^{(j+1)h} (s + \varepsilon)^{\mu-1}\, ds = (t_{j+1}^\mu - t_j^\mu)/\mu, \quad j = 0, 1, \ldots, n - 1, \tag{2.4}$$

we obtain the algorithm

$$u_n^h = \frac{I_\varepsilon}{t_n^\mu} + \frac{1}{t_n^\mu}\sum_{j=0}^{n-1} D_j u_j^h + g(t_n), \quad n = 1, 2, \ldots, N. \tag{2.5}$$

In the Trapezoidal method we use a piecewise linear approximation for u, that is

$$u(s+\varepsilon) \simeq [u(t_{j+1})(s-jh) + u(t_j)((j+1)h - s)]/h, \quad s \in [jh, (j+1)h]. \tag{2.6}$$

In this case we are led to the algorithm

$$u_n^h = \frac{I_\varepsilon}{t_n^\mu} + \frac{1}{t_n^\mu}\sum_{j=0}^{n-1}(D_j^1 u_{j+1}^h - D_j^2 u_j^h)/h + g(t_n), \quad n = 1, 2, \ldots, N, \tag{2.7}$$

where

$$D_j^1 = \int_{jh}^{(j+1)h} (s+\varepsilon)^{\mu-1}(s-jh)\,\mathrm{d}s, \quad D_j^2 = \int_{jh}^{(j+1)h} (s+\varepsilon)^{\mu-1}(s-(j+1)h)\,\mathrm{d}s$$

can be evaluated analytically. Starting with an initial value $u_0^h = u(\varepsilon)$, the above algorithms give approximate values u_n^h of $u(t_n)$.

3. Convergence analysis

In the case $\varepsilon = 0$ the methods considered in the previous section reduce to the traditional product integration methods. In particular, for the product Euler's method we have proved the following results in [6].

Theorem 3.1. *Consider Eq.* (1.1) *with* $0 < \mu \leqslant 1$ *and* $g \in C^1[0, T]$. *Then the approximate solution obtained by the product Euler's method* (2.5), *with* $\varepsilon = 0$, *converges to the particular solution of* (1.1) *which is in* $C^1[0, T]$. *Moreover, if* $g \in C^2[0, T]$ *and* $g'(0) = 0$ *we have first order convergence. However, if* $g'(0) \neq 0$ *we have convergence of order* μ.

Remark 3.1. In the traditional product trapezoidal method (that is, (2.7) with $\varepsilon = 0$) a convergence proof has not been available yet. However, the numerical experiments support the conjecture that if $g \in C^3[0, T]$ and $g'(0) = g''(0) = 0$ then the trapezoidal method converges with order 2. However, if $g''(0) \neq 0$ the order seems to be only $1 + \mu$ (cf. Table 5).

We now state two convergence results for the methods of the previous section, in the case when $\varepsilon \neq 0$.

Theorem 3.2. *Consider Eq.* (1.1) *with* $0 < \mu \leqslant 1$ *and* $g \in C^1[0, T]$. *Let* $\varepsilon \neq 0$ *be fixed in the equivalent equation* (2.1) *and assume the integral* I_ε *is known* (*exactly*) *for a chosen particular solution* (*corresponding to a certain value of the parameter* c_0). *Then the approximate solution obtained by the product Euler's method* (2.5) *converges with order* 1 *to that particular exact solution.*

Proof. The exact solution u satisfies

$$u(t_n) = \frac{I_\varepsilon}{t_n^\mu} + \frac{1}{t_n^\mu}\sum_{j=0}^{n-1} D_j u(t_j) + g(t_n) + \delta(h, t_n), \tag{3.1}$$

where $\delta(h, t_n)$ is the consistency error given by

$$\delta(h, t_n) = \int_\varepsilon^{t_n} \frac{s^{\mu-1}u(s)\,\mathrm{d}s}{t_n^\mu} - \frac{1}{t_n^\mu}\sum_{j=0}^{n-1} D_j u(t_j). \tag{3.2}$$

Setting $e_i = u(t_i) - u_i^h$ and subtracting Eqs. (3.1) and (2.5), we obtain

$$e_n = \frac{1}{t_n^\mu} \sum_{j=0}^{n-1} e_j \int_{t_j}^{t_{j+1}} s^{\mu-1}\,\mathrm{d}s + \delta(h, t_n), \quad n \geqslant 1. \tag{3.3}$$

Now

$$\frac{1}{t_n^\mu} \int_{t_j}^{t_{j+1}} s^{\mu-1}\,\mathrm{d}s \leqslant \frac{h t_j^{\mu-1}}{t_n^\mu} \leqslant h \left(\frac{t_j}{t_n}\right)^\mu \frac{1}{t_j} \leqslant \frac{h}{\varepsilon}. \tag{3.4}$$

Taking modulus in (3.3) and using (3.4) yields

$$|e_n| \leqslant \frac{h}{\varepsilon} \sum_{j=0}^{n-1} |e_j| + |\delta(h, t_n)|, \quad n \geqslant 1. \tag{3.5}$$

On the other hand, from (3.2) and (2.4), we have

$$|\delta(h, t_n)| \leqslant \frac{1}{t_n^\mu} \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} s^{\mu-1} |u(s) - u(t_j)|\,\mathrm{d}s \leqslant \frac{h}{t_n^\mu} \max_{s \in [\varepsilon, T]} |u'(s)| \int_\varepsilon^{t_n} s^{\mu-1}\,\mathrm{d}s. \tag{3.6}$$

Defining $M(\varepsilon) = \max_{s \in [\varepsilon, T]} |u'(s)|$, we get the following bound:

$$|\delta(h, t_n)| \leqslant \left(1 - \frac{\varepsilon^\mu}{t_n^\mu}\right) \frac{M(\varepsilon) h}{\mu}. \tag{3.7}$$

Using (3.7) and applying a known discrete Gronwall Lemma to (3.5), we obtain the estimate

$$|e_n| \leqslant \left(1 - \frac{\varepsilon^\mu}{t_n^\mu}\right) \frac{M(\varepsilon) h}{\mu} \mathrm{e}^{(T-\varepsilon)/\varepsilon} \tag{3.8}$$

and first order convergence follows. □

Remark 3.2. In the case of the choice $c_0 = 0$ (that is, the C^1 solution), the result of the last theorem is an improvement over the convergence order $k = \mu$ obtained with the traditional method ($\varepsilon = 0$), when $g'(0) \neq 0$ (cf. Theorem 3.1).

An analogous result to Theorem 3.2 can be proved for the Trapezoidal method.

Theorem 3.3. *Consider Eq.* (1.1) *with* $0 < \mu \leqslant 1$ *and* $g \in C^2[0, T]$. *Let* $\varepsilon \neq 0$ *be fixed in the equivalent equation* (2.1) *and assume the integral* I_ε *is known* (*exactly*) *for a chosen particular solution* (*corresponding to a certain value of the parameter* c_0). *Then the approximate solution obtained by the product Trapezoidal method* (2.7) *converges with order* 2 *to that particular exact solution.*

Remark 3.3. In the case of the choice $c_0 = 0$ (that is, the C^1 solution), the result of Theorem 3.3 is an improvement over the convergence order $k = 1 + \mu$ obtained with the traditional method ($\varepsilon = 0$), when $g''(0) \neq 0$ (cf. Remark 3.1).

4. Allowing ε to vary

So far we have assumed that ε remains fixed when $h \to 0$. In practice, we may need to approximate I_ε and it is natural to expect that the error in the approximation will increase with ε. Therefore, if ε is too large in comparison with h this may lead to significant initial errors. As a way to deal with this we may consider ε as a function of h such that $\varepsilon(h) \to 0$ as $h \to 0$.

Theorem 4.1. *Consider Eq.* (1.1) *with* $0 < \mu \leqslant 1$ *and* $g \in C^1[0, T]$ *satisfying* $g(0) = 0$. *Assume that the integral* I_ε (*with* $\varepsilon \neq 0$) *in* (2.1) *is known* (*exactly*) *for a chosen particular solution* (*corresponding to a certain value of the parameter* c_0). *Then the error in the approximate solution obtained by the product Euler's method* (2.5) *satisfies*

$$\max_{t_n \in X_h} |e(t_n)| := \max |u(t_n) - u_n^h| \leqslant C \frac{h}{\varepsilon}, \tag{4.1}$$

where C *is a constant independent of* h *and* ε.

Proof. Let

$$L_\mu u(t) := u(t) - \int_0^t \frac{s^{\mu-1}}{t^\mu} u(s)\,\mathrm{d}s. \tag{4.2}$$

Then Eq. (2.1) can be written as

$$L_\mu u(t + \varepsilon) = g(t + \varepsilon). \tag{4.3}$$

Let us also write Euler's method in operator form. Consider in $\mathbb{R}^{N+1}$ the maximum norm

$$\|v^h\| := \max_{0 \leqslant i \leqslant N} |v_i| \tag{4.4}$$

and associate with the uniform grid $X_h = \{t_i = ih + \varepsilon,\ 0 \leqslant i \leqslant N\}$ the linear restriction operator $r^h : C[0, T] \to \mathbb{R}^{N+1}$ defined by

$$(r^h f(t))_i := f(t_i), \quad 0 \leqslant i \leqslant N. \tag{4.5}$$

Let us associate with the operator L_μ defined by (4.2) the linear discrete operator $L_\mu^h : \mathbb{R}^{N+1} \to \mathbb{R}^{N+1}$ such that

$$(L_\mu^h v^h)_k := \begin{cases} -\dfrac{I_\varepsilon}{\varepsilon^\mu} + v_0^h, & k = 0, \\ v_k^h - \dfrac{I_\varepsilon}{t_k^\mu} - t_k^{-\mu} \displaystyle\sum_{i=0}^{k-1} D_i v_i^h, & 1 \leqslant k \leqslant N, \end{cases}$$

where the D_i coefficients are defined by (2.4).

Using the above operator Euler's method (2.5) is given by

$$L_\mu^h u^h = r^h g. \tag{4.6}$$

From (3.2), we note that

$$(L_\mu^h e^h)_n = (r^h(L_\mu u(t)) - L_\mu^h r^h u(t))_n = -\delta(h, t_n). \tag{4.7}$$

Using similar arguments to the ones used in [5, Theorem 3.3] we may prove that there exists a function $C_1(t) \in C[\varepsilon, T]$ such that

$$\delta(h, t_n) = C_1(t_n)h + \psi(h), \quad \psi(h) = \mathrm{o}(h), \tag{4.8}$$

where

$$C_1(t_n) = t_n^{-\mu} \alpha_0 \int_0^{nh} (t + \varepsilon)^{\mu-1} u'(t + \varepsilon)\,\mathrm{d}t = \alpha_0 u'(\xi) \int_0^{nh} \frac{(t + \varepsilon)^{\mu-1}}{t_n^\mu}\,\mathrm{d}t, \quad \xi \in (0, nh), \tag{4.9}$$

with $\alpha_0 = \frac{1}{2}$. Then we can associate with $C_1(t_n)$ a continuous function $C_1(t)$ such that

$$\max_{t \in [\varepsilon, T]} |C_1(t)| \leqslant \frac{M}{2\mu}, \tag{4.10}$$

with $M = \max_{t \in [\varepsilon, T]} |u'(t)|$, and where we have used the fact that

$$\left[1 - \left(\frac{\varepsilon}{t}\right)^\mu\right] < 1.$$

Hence, using the results of Marchuk and Shaidurov [7], we may conclude that there exists a function $e_1(t) \in C([\varepsilon, T])$ such that

$$L_\mu e_1(t) = C_1(t) \tag{4.11}$$

and

$$e(t) = e_1(t)h + \psi_2(h), \tag{4.12}$$

with $\psi_2(h) = \mathrm{o}(h)$.

Then, from (4.11) and (1.2), it follows that

$$e_1(t) = c_0 t^{1-\mu} + w(t), \tag{4.13}$$

where $w(t)$ is the differentiable solution of (4.11). We note that, since $\mu < 1$, from (4.11) it is not possible to get a bound on $w(t)$. However, by dividing both sides of the equation by t, it can be transformed into the equivalent equation

$$L_{\mu+1}\frac{e_1(t)}{t} = \frac{C_1(t)}{t}. \tag{4.14}$$

Since $\mu + 1 > 1$, we can use the solution formula (1.4) to obtain

$$\frac{e_1(t)}{t} = \frac{C_1(t)}{t} + t^{-\mu}\int_0^t s^{\mu-1}\frac{C_1(s)}{s}\,\mathrm{d}s. \tag{4.15}$$

Applying modulus and using (4.10) gives

$$\max_{t\in[\varepsilon,T]}\left|\frac{e_1(t)}{t}\right| \leqslant \max_{t\in[\varepsilon,T]}\left|\frac{C_1(t)}{t}\right| + \max_{r\in[\varepsilon,T]}\left|\frac{C_1(r)}{r}\right|\left|\int_0^t \frac{s^{\mu-1}}{t^\mu}\,\mathrm{d}s\right| \leqslant \frac{M}{2\mu\varepsilon}\left(1+\frac{1}{\mu}\right).$$

Therefore,

$$\max_{t\in[\varepsilon,T]}|e_1(t)| \leqslant \frac{TM}{2\mu\varepsilon}\left(\frac{\mu+1}{\mu}\right), \tag{4.16}$$

which, substituted into (4.12), yields

$$|e(t_n)| \leqslant \frac{kh}{\varepsilon} + \mathrm{O}(h^2), \tag{4.17}$$

where $k = TM(\mu+1)/(2\mu^2)$. From here (4.1) follows. □

Let us now suppose for example that ε is of the form $\varepsilon = Ch^\theta$.

Corollary 4.1. *Let us assume that the conditions of Theorem* 4.1 *are satisfied. If* $\varepsilon = h^\theta,\ 0 < \theta < 1$, *then*

$$\max_{t_n\in X_h}|e(t_n)| \leqslant Ch^{1-\theta}. \tag{4.18}$$

In Section 6, a numerical example is given where the obtained results are in good agreement with the statement of this corollary (cf. Table 6).

5. Approximation of I_ε and the solution in [0, ε]

So far, we have considered I_ε, used to compute the numerical solution, as a known exact value. This enables us to specify which particular solution is being approximated out of the infinite set. However, in applications it often happens that I_ε is not known. In this case other kind of initial data are needed in order to compute the required solution. A possible way of specifying a particular trajectory is by giving its value at ε (for details see [1]). Then it follows from (2.1) that

$$I_\varepsilon = \varepsilon^\mu [u(\varepsilon) - g(\varepsilon)]. \tag{5.1}$$

On the other hand, supposing we know $u(\varepsilon)$ (or I_ε), it is possible to compute approximations for the values $u(t)$, $0 < t < \varepsilon$. From (1.2) we can write

$$u(t) = c_0 t^{1-\mu} + u_r(t), \tag{5.2}$$

where the smooth solution $u_r(t)$ satisfies the original equation (1.1). Setting $t = \varepsilon$ in (5.2) gives

$$c_0 = (u(\varepsilon) - u_r(\varepsilon))/\varepsilon^{1-\mu}. \tag{5.3}$$

We note that $u_r(t)$ is differentiable at $t = 0$ and therefore may be approximated by standard methods applied to (1.1) (cf. Theorem 3.1). Once $u_r(\varepsilon)$ is known, we can obtain c_0 from (5.3) which, substituted into (5.2), yields the value $u(t)$. Some numerical results for $u_r(\varepsilon)$ and the corresponding values for I_ε are presented in Section 6 (Table 7).

We conclude this section by establishing how an error in the value of I_ε will affect the numerical solution of (1.1). With this purpose, let us recall from [1] that if two solutions of Eq. (1.1), with $\mu < 1$, have different values at $t = \varepsilon$, say, $u(\varepsilon)$ and $\tilde{u}(\varepsilon)$, then the separation between these solutions will grow as $t^{1-\mu}$, that is, for $t = T$, we get

$$|u(T) - \tilde{u}(T)| = \left(\frac{T}{\varepsilon}\right)^{1-\mu} |u(\varepsilon) - \tilde{u}(\varepsilon)|. \tag{5.4}$$

On the other hand, from (2.1), we can conclude that, for any solution u,

$$u(\varepsilon) = g(\varepsilon) + \frac{I_\varepsilon}{\varepsilon^\mu}. \tag{5.5}$$

Therefore, if two different values I_ε and $\tilde{I}_\varepsilon$ are given, the values of the corresponding solutions at $t = \varepsilon$ are such that

$$|u(\varepsilon) - \tilde{u}(\varepsilon)| = \frac{1}{\varepsilon^\mu} |I_\varepsilon - \tilde{I}_\varepsilon|. \tag{5.6}$$

Finally, from (5.6) and (5.4) it follows that, if $|I_\varepsilon - \tilde{I}_\varepsilon| < \delta$, then the difference between the two solutions will satisfy

$$|u(T) - \tilde{u}(T)| < \frac{T^{1-\mu}}{\varepsilon} \delta. \tag{5.7}$$

Moreover, as follows from a result of [2], the numerical methods for solving the Eq. (1.1) exhibit the correct exponential order of growth over the long term. Therefore, if we denote by $\tilde{u}_N^h$ the numerical solution at $t = T$, computed from a non-exact value of I_ε, the distance between this solution and the one computed by the same numerical method, using the exact value of I_ε, must satisfy an inequality analogous to (5.7):

$$|u_N^h - \tilde{u}_N^h| \leqslant \frac{T^{1-\mu}}{\varepsilon} \delta. \tag{5.8}$$

From (5.8) it follows that the total error of $\tilde{u}_N^h$ satisfies

$$|u(T) - \tilde{u}_N^h| \leqslant |\tilde{u}_N^h - u_N^h| + |u(T) - u_N^h| \leqslant \frac{T^{1-\mu}}{\varepsilon} \delta + |e(T)|. \tag{5.9}$$

Hence, if we require that the error of $\tilde{I}_\varepsilon$ should not affect strongly the total error at $t = T$, we must have

$$\frac{T^{1-\mu}}{\varepsilon} \delta \ll |e(T)|. \tag{5.10}$$

6. Numerical examples

In this section we consider two examples to which we have applied the methods described in the previous section.

Example 1. If we set $g(t) = 1 + t + t^2$ and $0 < \mu < 1$ in Eq. (1.1), then, using (1.2), we obtain the general form of its family of solutions

$$u(t) = c_0 t^{1-\mu} + \frac{\mu}{\mu - 1} + \frac{\mu+1}{\mu} t + \frac{\mu+2}{\mu+1} t^2, \quad c_0 \text{ arbitrary constant.} \tag{6.1}$$

Example 2. We set $\mu = 0.75$ and choose g such that the general solution of (1.1) is given by

$$u(t) = c_0 t^{0.25} + t^{1/2} \sin(2\pi t), \quad c_0 \text{ arbitrary real constant.} \tag{6.2}$$

In Tables 1–5 we have considered $t \in [0, 1]$. The following quantity has been used as an estimate of the convergence order

$$k := -\log_2 \left(\frac{\|r^h u - u^h\|}{\|r^{2h} u - u^{2h}\|} \right). \tag{6.3}$$

Table 1 shows the errors obtained with Euler's method (2.5) applied to Example 1, with $\mu = 0.5$ and $c_0 = 2$. That is, for each fixed value of ε, we have used the exact value of I_ε corresponding to this particular solution. We see that the errors decrease as ε increases and this could be expected since we are fixing the exact solution over a larger interval. The results indicate convergence to the chosen particular solution ($c_0 = 2$ in (6.1)) and this is clearly illustrated in Fig. 1, which also shows the smooth solution ($c_0 = 0$). Moreover, different choices of ε (Table 1) and μ (Table 3) do not affect the expected first order of convergence. Similar results are obtained if instead of the particular solution corresponding to $c_0 = 2$ we choose any other member of the family. As an example, we consider the smooth solution, that is, we take $c_0 = 0$ in (6.1) and put in (2.5) the exact value of I_ε for this particular solution. In Table 2 the

Table 1
Error norms and convergence orders for Euler's method in Example 1

N	$\varepsilon = 0.01$		$\varepsilon = 0.02$		$\varepsilon = 0.05$	
	$\|e(t_i)\|_\infty$	k	$\|e(t_i)\|_\infty$	k	$\|e(t_i)\|_\infty$	k
100	5.93E − 1		3.84E − 1		2.01E − 1	
200	3.37E − 1	0.81	2.06E − 1	0.90	1.04E − 1	0.96
400	1.82E − 1	0.89	1.07E − 1	0.94	5.26E − 2	0.98
800	9.44E − 2	0.94	5.47E − 2	0.97	2.65E − 2	0.99
1600	4.82E − 2	0.97	2.76E − 2	0.98	1.33E − 2	0.99

The effect of varying ε ($c_0 = 2$, $\mu = 0.5$).

Table 2
Error norms and convergence orders for Euler's method in Example 1

N	$\varepsilon = 0$		$\varepsilon = 0.02$		$\varepsilon = 0.05$	
	$\|e(t_i)\|_\infty$	k	$\|e(t_i)\|_\infty$	k	$\|e(t_i)\|_\infty$	k
100	4.03E − 1		1.82E − 1		1.17E − 1	
200	2.85E − 1	0.496	9.65E − 2	0.912	6.00E − 2	0.959
400	2.02E − 1	0.499	4.99E − 2	0.952	3.04E − 2	0.979
800	1.43E − 1	0.500	2.54E − 2	0.975	1.53E − 2	0.989
1600	1.01E − 1	0.500	1.28E − 2	0.987	7.69E − 3	0.995

Approximation of the smooth solution ($c_0 = 0$), with $\mu = 0.5$.

Table 3
Error norms and convergence orders for Euler's method in Example 1

N	$\mu=0.3$		$\mu=0.5$		$\mu=0.8$	
	$\lVert e(t_i)\rVert_\infty$	k	$\lVert e(t_i)\rVert_\infty$	k	$\lVert e(t_i)\rVert_\infty$	k
100	1.05E + 0		5.93E − 1		2.47E − 1	
200	6.09E − 1	0.78	3.37E − 1	0.81	1.35E − 1	0.87
400	3.32E − 1	0.87	1.81E − 1	0.89	7.09E − 2	0.93
800	1.74E − 1	0.93	9.44E − 2	0.94	3.64E − 2	0.96
1600	8.94E − 2	0.96	4.82E − 2	0.97	1.84E − 2	0.98

The effect of varying μ ($c_0=2$, $\varepsilon=0.01$).

Table 4
Error norms and convergence orders for the Trapezoidal method in Example 1

N	$\varepsilon=0.01$		$\varepsilon=0.02$		$\varepsilon=0.05$	
	$\lVert e(t_i)\rVert_\infty$	k	$\lVert e(t_i)\rVert_\infty$	k	$\lVert e(t_i)\rVert_\infty$	k
100	1.77E − 2		4.51E − 3		5.72E − 4	
200	4.81E − 3	1.88	1.16E − 3	1.96	1.44E − 4	1.99
400	1.23E − 3	1.96	2.92E − 4	1.99	3.60E − 5	2.00
800	3.11E − 4	1.99	7.30E − 5	2.00	9.00E − 6	2.00
1600	7.78E − 5	2.00	1.83E − 5	2.00	2.25E − 6	2.00

The effect of varying ε ($c_0=2$, $\mu=0.5$).

Table 5
Error norms and convergence orders for the Trapezoidal method in Example 1

N	$\varepsilon=0$		$\varepsilon=0.02$		$\varepsilon=0.05$	
	$\lVert e(t_i)\rVert_\infty$	k	$\lVert e(t_i)\rVert_\infty$	k	$\lVert e(t_i)\rVert_\infty$	k
100	1.81E − 3		3.23E − 4		1.74E − 4	
200	6.47E − 4	1.487	8.09E − 5	1.996	4.35E − 5	1.999
400	2.30E − 4	1.491	2.02E − 5	1.999	1.09E − 5	2.000
800	8.18E − 5	1.494	5.06E − 6	2.000	2.72E − 6	2.000
1600	2.90E − 5	1.496	1.26E − 6	2.000	6.80E − 7	1.999

Approximation of the smooth solution ($c_0=0$), with $\mu=0.5$.

errors and convergence rates produced with $\varepsilon=0.02$ and $\varepsilon=0.05$ confirm the predicted first order of convergence. For the sake of comparison, we have also applied the conventional product Euler's method (which corresponds to setting $\varepsilon=0$) to the same example. The results displayed in the first column of Table 2 are in agreement with the theoretical order $k=\mu$ given in Theorem 3.1 (see also Remark 3.1). An increase in order is thus obtained with $\varepsilon\neq 0$. Figs. 2 and 3 illustrate the performance of Euler's method for Example 2.

Table 4 contains the errors obtained with the Trapezoidal method (2.7) for Example 1, with $\mu=0.5$. First, we have taken $c_0=2$ and convergence of order 2 to the chosen particular solution is confirmed, independently of the choice of each fixed value of ε. Similarly to Euler's method (Table 2) we have also considered the smooth solution ($c_0=0$) and the errors in the approximation are displayed in Table 5. Again the algorithm (2.7) with $\varepsilon\neq 0$ gives a higher convergence order than the conventional product Trapezoidal method (that is, when $\varepsilon=0$). Table 6 shows the error norms and convergence orders of Euler's method when $\varepsilon=h^\theta$, for several values of θ. The results indicate that the convergence order is $1-\theta$, thus confirming the statement of Corollary 4.1.

In Table 7 are displayed approximate values $\tilde{u}_r(\varepsilon)$ for $u_r(\varepsilon)$, in the case of Example 1; these values were obtained by the Trapezoidal method applied to (1.1) in interval $[0,\varepsilon]$, with $\varepsilon=0.01$. The corresponding approximations to $\tilde{I}_\varepsilon$ were

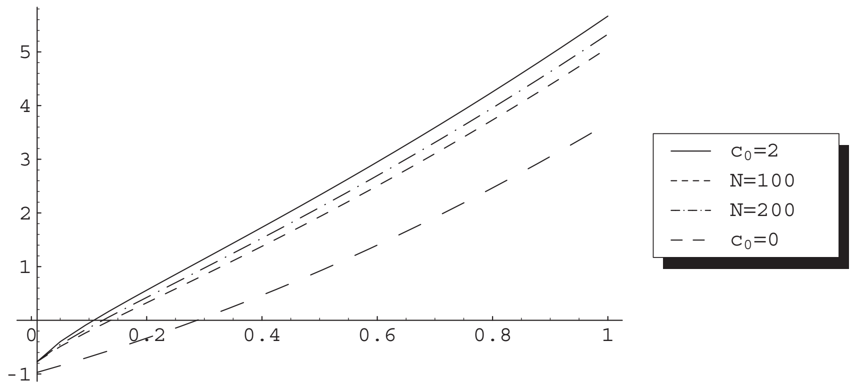

Fig. 1. Euler's method for Example 1: approximate solutions ($N = 100$, 200) and exact solutions with $c_0 = 0$, $c_0 = 2$ ($\mu = 0.5$).

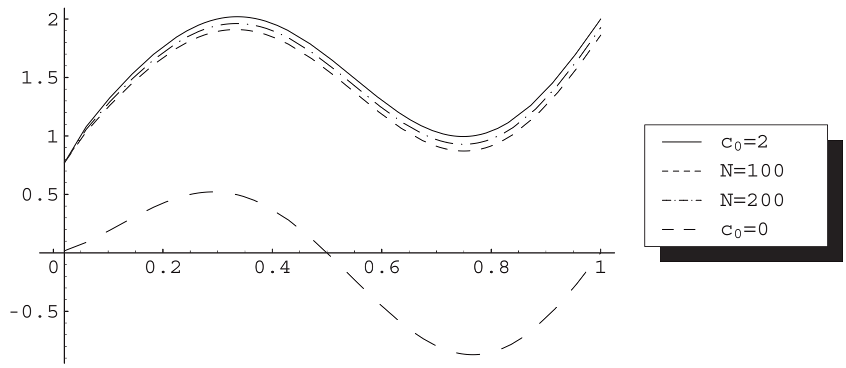

Fig. 2. Euler's method for Example 2: approximate solutions ($N = 100$, 200) and exact solutions with $c_0 = 0$, $c_0 = 2$.

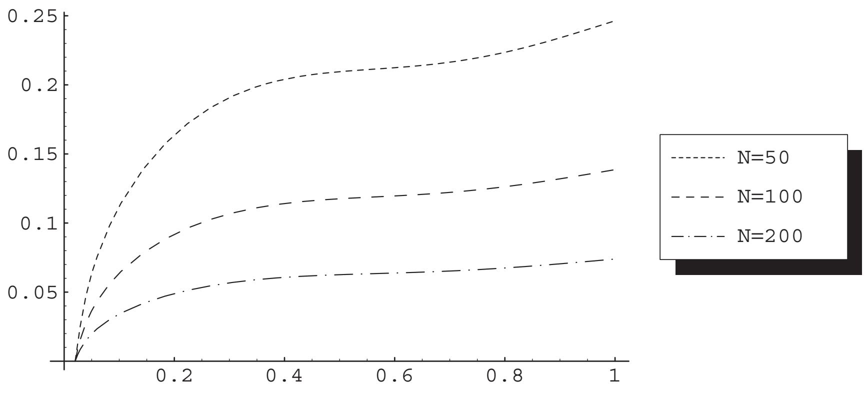

Fig. 3. Absolute errors $|e(t_i)|$ of Euler's method in Example 2, with $c_0 = 2$.

calculated by using (5.1). According to (5.8), the effect of the error $I_\varepsilon - \tilde{I}_\varepsilon$ on the computed value $u(T)$ is given by $T^{1-\mu}|I_\varepsilon - \tilde{I}_\varepsilon|/\varepsilon < 5.36\text{E}-5$, with $T = 1$, which is negligible when compared with the errors obtained by Euler's method on $[\varepsilon, T]$ (compare Tables 2, 3). The last column illustrates the application of formula (5.3) to obtain the constant c_0, using the given value $u(\varepsilon)$ and the approximations $u_r(\varepsilon)$. Finally, Table 8 shows the influence on the final result $u(T)$ of the errors in initial data. We replace I_ε (here considered as initial data) by $\tilde{I}_\varepsilon$, such that $|I_\varepsilon - \tilde{I}_\varepsilon| = \delta = 10^{-2}$ and we

Table 6
Error norms and convergence orders for Euler's method in Example 1, with $\mu = 0.5,\ c_0 = 10,\ \varepsilon = h^\theta,\ \theta \in \{0.2, 0.6, 0.8\},\ T = 1 + \varepsilon$

h	$\theta = 0.2$		$\theta = 0.6$		$\theta = 0.8$	
	$\Vert e(t_i)\Vert_\infty$	k	$\Vert e(t_i)\Vert_\infty$	k	$\Vert e(t_i)\Vert_\infty$	k
0.01	0.099989		0.476465		1.025023	
0.005	0.056365	0.827	0.352668	0.434	0.875238	0.228
0.0025	0.031752	0.828	0.261367	0.432	0.750982	0.221
0.00125	0.017888	0.828	0.194073	0.429	0.647112	0.215
0.000625	0.010081	0.827	0.144423	0.426	0.559551	0.210
0.000313	0.005684	0.827	0.107719	0.423	0.485150	0.206

Table 7
Approximate values of $u_r(\varepsilon)$, by the Trapezoidal method, and the corresponding values for I_ε; Example 1, with $\varepsilon = 0.01,\ \mu = 0.5,\ c_0 = 5$

N	$\tilde{u}_r(\varepsilon)$	$\lvert u_r(\varepsilon) - \tilde{u}_r(\varepsilon)\rvert$	Rate	$\tilde{I}_\varepsilon$	$\lvert I_\varepsilon - \tilde{I}_\varepsilon\rvert$	$\tilde{c}_0$	$\lvert c_0 - \tilde{c}_0\rvert$
10	−0.969827974	5.359E − 6		−0.147992797	5.359E − 7	4.99994641	5.36E − 5
20	−0.969831381	1.952E − 6	1.46	−0.147993138	0.952E − 7	4.99998048	1.95E − 5
40	−0.969832629	7.045E − 7	1.47	−0.147993263	7.046E − 8	4.999992957	7.043E − 6
80	−0.969833081	2.527E − 7	1.48	−0.147993308	2.527E − 8	4.999997477	2.523E − 6
160	−0.969833243	9.024E − 8	1.49	−0.147993324	9.025E − 9	4.999999097	9.033E − 7
320	−0.969833301	3.213E − 8	1.49	−0.14799333	3.214E − 9	4.999999677	3.233E − 7

Table 8
Influence of the error in the evaluation of I_ε.

(a) varying T with constant $\varepsilon = 0.05$

μ	$T = 1$		$T = 2$		$T = 3$	
	$\lvert e(T)\rvert$	Δ	$\lvert e(T)\rvert$	Δ	$\lvert e(T)\rvert$	Δ
0.25	0.749	0.187	1.325	0.314	1.837	0.425
0.50	0.431	0.191	0.653	0.270	0.830	0.331
0.75	0.225	0.196	0.299	0.232	0.354	0.257

(b) varying ε with constant $T = 1$

	$\varepsilon = 0.01$		$\varepsilon = 0.02$		$\varepsilon = 0.03$		$\varepsilon = 0.04$	
	$\lvert e(T)\rvert$	Δ	$\lvert e(T)\rvert$	Δ	$\lvert e(T)\rvert$	Δ	$\lvert e(T)\rvert$	Δ
0.25	2.526	0.748	1.566	0.426	1.144	0.299	0.905	0.230
0.50	1.498	0.820	0.906	0.448	0.658	0.309	0.520	0.236
0.75	0.783	0.903	0.466	0.473	0.339	0.321	0.269	0.243

The difference $\Delta = \lvert u_N^h - \tilde{u}_N^h\rvert$ and the errors of Euler's method are displayed for Example 1 with different values of μ; $\delta = 10^{-2}$ (perturbation of I_ε); $h = 0.01;\ c_0 = 7$.

display the difference between the two resulting values of $u(T)$, obtained by Euler's method. Firstly, we keep ε constant ($\varepsilon = 0.05$) and vary T; in this case, the difference $\lvert u_N^h - \tilde{u}_N^h\rvert$ grows as $T^{1-\mu}$, according to formula (5.8). Secondly, we keep T constant and vary ε; in this case, the considered difference decreases approximately as $1/\varepsilon$, which is also in agreement with (5.8). The absolute error at $t = T$ of Euler's method (without perturbation of I_ε) is also given for the sake of comparison. As we can see, in this case the error resulting from the non-exact evaluation of I_ε is nearly as large as the error of Euler's method.

7. Conclusion

This work was concerned with the solution of the singular Volterra integral equation (1.1), with an infinite family of solutions in the case $0 < \mu < 1$, of which only one has finite gradient at the origin. We have introduced a technique based on splitting up the integral $\int_0^t = \int_0^\varepsilon + \int_\varepsilon^t$, where $\varepsilon > 0$ is a fixed real number. Supposing that the value of the solution $u(\varepsilon)$ is given, this specifies a particular trajectory $u(t)$ for which we can easily evaluate $I_\varepsilon = \int_0^\varepsilon u(t)\,\mathrm{d}t$ (cf. (5.1)). Product integration quadrature methods can then be applied and convergence to the specified solution is obtained. While in previous works [1,6] we have only dealt with the approximation of the smooth solution, the present methods enable us to approximate any member of the family of solutions. Moreover, in the case of approximating the smooth solution we get higher convergence orders than with the conventional product integration methods used in the referred works.

We have also considered the case in which the integral I_ε (which was considered as initial data for the computation of any particular solution) is known only approximately. We have estimated how this approximation will affect the error in the solution and performed some numerical experiments, whose results agreed with the theory.

It will be natural to think that if ε tends to zero as a certain power of h then the error in evaluating I_ε will also tend to zero. Having this in mind, we have determined the convergence order of Euler's method in the case when $\varepsilon = h^\theta$.

It is worth noting that, however, the error in the evaluation of I_ε should tend to 0 as $\varepsilon \to 0$, the total error of the numerical method on $[\varepsilon, T]$ grows as $\varepsilon \to 0$ (see Theorem 4.1 for the case of Euler's method). Therefore, there should be, for a given numerical algorithm and a given stepsize, an optimal value of ε which provides the best approximation. This will be the subject of further investigation.

Acknowledgements

T. Diogo, P. Lima and S. Valtchev are pleased to acknowledge support from FCT, project POCTI/MAT/45700/2002.

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