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Lectures on Open/Closed Duality

Modern Trends in String Theory II

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Routes to gauge theory/geometry correspondence

- Spacetime picture motivated by D-branes

Either backreaction on spacetime *or* worldvolume degrees of freedom

- Large N expansion and open $\longrightarrow$ closed worldsheets ‘t Hooft
- Non-critical strings and the Liouville extra dimension Polyakov
- Gauge theory loop equations and all that

$\ldots\ldots$

Time to go back seriously to a “microscopic” worldsheet picture: how to really “close the holes”?

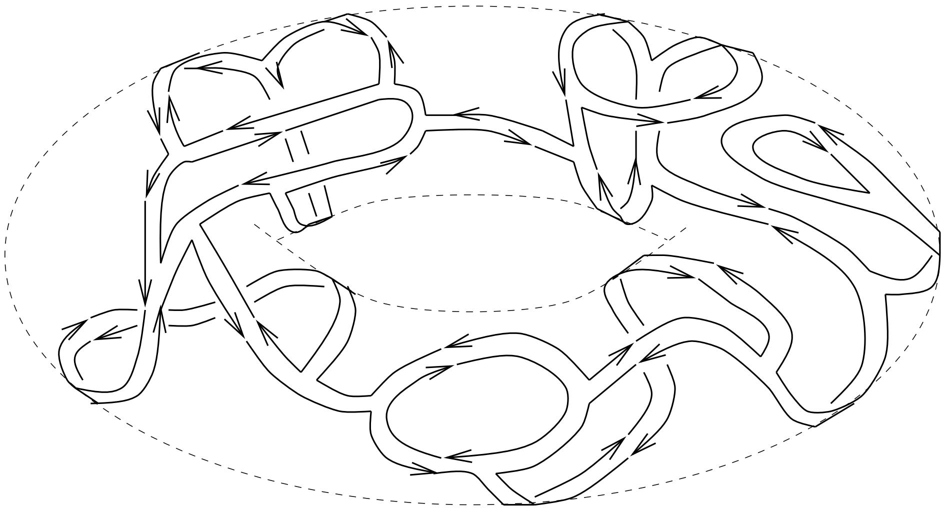
In these lecture we will do this in simple examples.

The mechanism, we claim, is quite general.

Large N

Gauge theory partition function: 't Hooft

$$\log \mathcal{Z}^{open}(g_{YM}, N) = \sum_{g=0}^{\infty} \sum_{h=1}^{\infty} N^{2-2g} (g_{YM}^2 N)^h F_{g,h}$$



$$t = g_{YM}^2 N$$

Closed string partition function:

$$\log \mathcal{Z}^{closed}(N, t) = \sum_{g=0}^{\infty} N^{2-2g} F_g(t) \,, \quad F_g(t) = \sum_{h=2}^{\infty} F_{g,h} t^h \,.$$

One possibility

"Summing over h, we fill in the surface."
This is the intuition of "old matrix models".

$\exists$ critical value t_c such that as $t \to t_c$,
$F_g(t)$ dominated by surfaces with $h \to \infty$. Critical behaviour:

$$F_g(t) \sim f_g \, (t_c - t)^{\alpha(2-2g)}$$

Theory near this critical point is effectively a closed string theory.

Taking double-scaling limit

$$N \to \infty \,, \quad t \to t_c \,, \quad \frac{1}{\kappa} \equiv (t_c - t)^{\alpha} N \;\; \text{fixed} \,,$$

we find a sum over closed surfaces,

$$\log \mathcal{Z} = \sum_g \kappa^{2g-2} f_g \,.$$

This allows to exactly solve closed string theories with $c \leq 1$.

However...

In AdS/CFT and related examples, N can be kept finite.

Moreover, t is a free geometric parameter of the closed string background.

Riemann surfaces with h holes $\leftrightarrow$

Closed Riemann surfaces with h extra closed string insertions

D-branes in imaginary time offer a precise example Gaiotto Itzhaki L.R.

$$t\, b_0 \int d\rho\, \rho^{L_0}\, |\mathcal{B}\rangle_P \leftrightarrow t\, \mathcal{W}(P)$$

Summing over holes $\sim \exp(t \int d^2z\, \mathcal{W}(z))$

Topological strings seem to work similarly Ooguri Vafa

Basic Setup

Open string side

In these lectures, the poor man's version of a gauge theory, a matrix model.

Interpret it as the open string field theory (OSFT) on N branes of some appropriate string theory:
fatgraphs $\equiv$ open string field Feynman diagrams.
(We really mean the full OSFT, not some effective low-energy limit).

Coupling constants in the matrix model $\sim$ open string moduli $\{z_i\}$, $i = 1, \ldots N$ (choices of open string boundary conditions).

Natural class of observables encoded in the vacuum amplitude

$$\log \mathcal{Z}^{open}(g_o, N) = \sum_{g=0}^{\infty} \sum_{h=1}^{\infty} {g_o}^{-2+2g} ({g_o}^2 N)^h F_{g,h}^{open}(\{z_i\}) \, .$$

Closed string side

Natural observables are correlators of closed strings physical states $\{\mathcal{O}_k\}$, encoded in

$$\log \mathcal{Z}^{closed}(g_s, \{t_k\}) = \sum_{g=0}^{\infty} {g_s}^{2g-2} \langle \exp(\sum_k t_k \mathcal{O}_k) \rangle_g \, .$$

Open/closed duality:

open diagram with h holes $\leftrightarrow$ closed diagram with h punctures.

$$b_0 \int d\rho \rho^{L_0} |\mathcal{B}_z\rangle \leftrightarrow \sum_k c_k(z) \mathcal{O}_k$$

$$\mathcal{Z}^{open}(g_o, \{z_i\}) = \mathcal{Z}^{closed}\left(g_s = g_o^2, t_k = \sum_i c_k(z_i) \right)$$

Moduli spaces

For this to make sense, appropriate moduli spaces of open and closed Riemann surfaces must be closely related

$\mathcal{M}_{g,h}^{open} \equiv$ moduli space of open Riemann surfaces of genus g, with h holes

$\dim(\mathcal{M}_{g,h}^{open}) = 6g - 6 + 3h$

$\mathcal{M}_{g,p}^{closed} \equiv$ moduli space of closed Riemann surfaces of genus g, with p punctures

$\dim(\mathcal{M}_{g,p}^{closed}) = 6g - 6 + 2p$

Natural isomorphism Penner, Kontsevich

$$\mathcal{M}_{g,h}^{open} \cong \mathbf{R}_{+}^{h} \times \mathcal{M}_{g,p=h}^{closed}$$

Brief review of c < 1 closed string models

p	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
4	○	●	●	●	●	$\cdots$
3	○	●	●	●	●	$\cdots$
2	○	●	●	●	●	$\cdots$
	1	2	3	4	5	q

Noncritical bosonic string theories from (p, q) minimal models + Liouville.

Exact solution from double-scaling of $p-1$ matrix model.

Douglas Shenker, Brezin Kasakov, Gross Migdal

Models in the same row related by turning on deformations, $S = S_0 + t_n \mathcal{O}_n$.

$(p, 1)$ column: topological models

- Focus first on $(2, 1)$: $c = -2$ matter $\oplus$ $c = 28$ Liouville.

Alternative powerful formulation as topological 2d gravity, intersection theory on moduli space of Riemann surfaces. Witten

$(2,1)$ **model**

Closed string observables $\mathcal{O}_{2k+1}$, $k = 0, 1, \ldots, \infty$

($\mathcal{O}_{2k+1} \leftrightarrow c_1(\mathcal{L})^k$ in intersection theory)

$$\log \mathcal{Z}^{closed}(g_s, t_k) = \sum_{g=0}^{\infty} {g_s}^{2g-2} \langle \exp(\sum_k t_k \mathcal{O}_{2k+1}) \rangle_g \, .$$

$\mathcal{Z}(t) = \tau(t)$ is a τ-function of the KP(KdV) hierarchy. Douglas

Uniquely determined by Virasoro algebra of constraints: Dijkgraaf Verlinde Verlinde

$$\frac{\partial}{\partial t_k} \mathcal{Z}(t_k) = \mathcal{L}_{2k-2} \mathcal{Z}(t_k)$$

Kontsevich matrix integral

Beautiful representation of $\mathcal{Z}(t)$ from an integral over $N \times N$ hermitian matrices

$$\mathcal{Z}^{closed}(t) = \rho(Z)^{-1} \int [dX] \exp\left(-\frac{1}{g_o^2}\mathrm{Tr}\left[\frac{1}{2}ZX^2 + \frac{1}{6}X^3\right]\right)$$

$$\rho(Z) = \int [dX] \exp\left(-\frac{1}{2g_o^2}\mathrm{Tr} Z X^2\right) .$$

Closed string sources t_k encoded in the N eigenvalues of matrix Z:

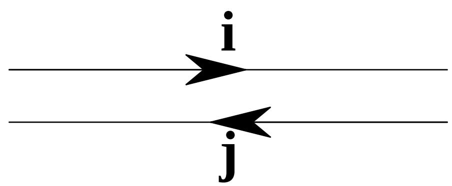
$$t_k = \frac{g_o^2}{2k+1}\,\mathrm{Tr}\, Z^{-2k-1} = \frac{g_o^2}{2k+1}\sum_{n=1}^{N}\frac{1}{{z_n}^{2k+1}}$$

Different from double-scaled matrix model.

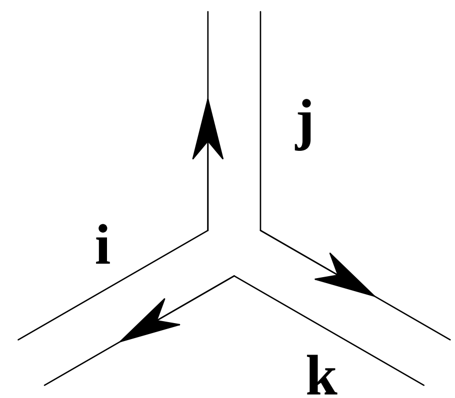
Here we can keep g_s and N finite, or do ordinary 't Hooft expansion with t_k fixed.

• Konsevitch integral resembles cubic open SFT...

Feynman rules



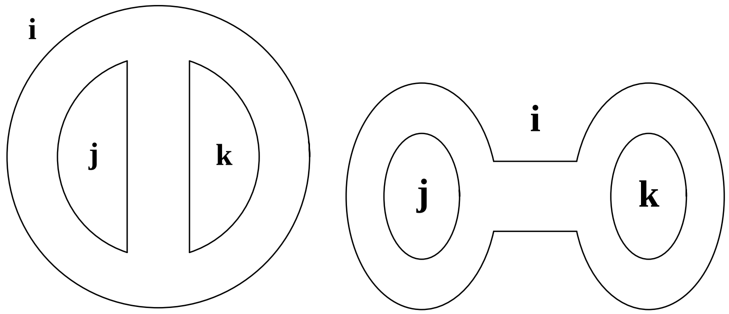
$$\frac{2\,g_s}{z_i + z_j}$$



$$\frac{1}{g_s}$$

Example: $\langle \mathcal{O}_1 \mathcal{O}_1 \mathcal{O}_1 \rangle_{S_2}$

Sphere with three holes: Assign Chan-Paton indexes to each hole i,j,k



Four contributions:

$$\frac{2g_o^2}{(z_i+z_j)(z_j+z_k)(z_k+z_i)} + \left[\frac{g_o^2}{z_i(z_i+z_j)(z_i+z_k)} + (i \to j) + (i \to k)\right] = \frac{g_o^2}{z_i z_j z_k}$$

Sum over Chan-Paton:

$$\frac{g_o^2}{6}\left(\sum_{i=1}^{N}\frac{1}{z_i}\right)^3 \equiv \frac{1}{6g_s^2}{t_0}^3 \longrightarrow \langle \mathcal{O}_1 \mathcal{O}_1 \mathcal{O}_1 \rangle_{S_2} = 1\,.$$

Hole in Feymann diagram $\leftrightarrow$ closed string puncture $\sim g_s \sum \frac{\mathcal{O}_{2k+1}}{z^{2k+1}}$

- Summing over number of holes exponentiates the puncture to a closed background

Our Proposal

Kontsevich integral = open SFT on N stable D-branes

- $|\mathcal{B}\rangle_z = ($FZZT brane for Liouville with $\mu = 0$, $\mu_B = z) \otimes |\mathcal{B}^{matter}\rangle$

Introducing a brane is equivalent to a shift of the closed string background:

$$|\mathcal{B}\rangle_z \longrightarrow \sum_{k=0}^{\infty} \frac{\mathcal{O}_{2k+1}}{(2k+1)z^{2k+1}}$$

- Cubic OSFT on N such branes $\rightarrow$ Kontsevich integral

Topological localization similar to open topological A model $\rightarrow$ Chern-Simons

Witten

- Compute $\mathcal{Z}$ in open and closed channel:

$$\mathcal{Z}^{open}(g_o, z_i) = \mathcal{Z}^{closed}\left(g_s = g_o^2, t_{2k+1} = g_s \sum \frac{1}{(2k+1){z_i}^k}\right)$$

Worldsheet theory for $(2, 1)$ model

Many formulations of topological gravity.

Careful BRST analysis necessary to define cohomological problem and handle correctly the contact term algebra

Labastida Pernici Witten, E. and H. Verlinde, Distler Nelson

Double scaling limit of a matrix model with two Grassmann coordinates agrees with topological gravity Klebanov Wilkinson

• Simplest continuum formulation: bosonic string

$c = -2$ matter $\oplus$ $c = 28$ Liouville

No subtleties arise for the open theory.

Precise treatment made possible by recent progress in Liouville field theory

Teschner, Fateev Zamolodchikov^2, $\cdots$

Strings in D=-2

$$\mathcal{S} = \frac{1}{2\pi}\int d^2z\, \epsilon_{\alpha\beta}\partial\Theta^\alpha\bar\partial\Theta^\beta + S_\phi^{c=28} + S_{bc} \quad \alpha,\beta = 1,2\,.$$

Θ^1 and Θ^2 real and Grassmann odd.

$$\Theta^1(z,\bar z)\Theta^2(0) \sim -\frac{1}{2}\log|z|^2$$

Only one *non-chiral* zero mode. Different from $\xi\eta$,

$$\eta(z) = \partial\Theta^1(z,\bar z)\,, \quad \xi(z) + \bar\xi(\bar z) = \Theta^1(z,\bar z)\,.$$

Closed string observables

$$\mathcal{O}_{2k+1} = e^{\sqrt{2}(1-k)\phi}\,\mathcal{P}_k(\partial\Theta^\alpha)\,c\bar c$$

Canonical choice of $(\frac{k(k+1)}{2}, \frac{k(k+1)}{2})$ primaries $\mathcal{P}_k$ from SL(2) invariance.
Already in the correct "picture".

Logarithmic behavior of the CFT possible way to understand contact terms.
(See Zamolodchikov)

Stable D-branes

Boundary conditions:

- Dirichlet b.c. on Θ^{α}.

- Extended D-brane in the Liouville direction with boundary interaction

$\mu_B \int_\partial e^{\phi}$ Fateev Zamolodchikov[2]

Quantum gravity interpretation

$$|\mathcal{B}\rangle_{\mu_B} \leftrightarrow \int_0^\infty e^{-\mu_B\, l} W(l) \sim \sum_{k=0}^{\infty} \frac{\mathcal{O}_{2k+1}}{\mu_B^{2k+1}}$$

$W(l)$ macroscopic loop operator Martinec Moore Seiberg Staudacher

- $\mathcal{O}_{2k+1}$ appear with correct power of μ_B in Boundary State

if $\mu_B \leftrightarrow z$ parameter in Kontsevich.

BCFT on stable branes

Spectrum on these branes:

- Θ^α: Dirichlet b.c. $\rightarrow$ a single copy of current $\partial\Theta^\alpha$, no zero modes

- Boundary (FZZT) Liouville: $\{e^{\alpha\phi}\}$

$\alpha = Q/2 + iP$ are normalizable states;

α real $\leq Q/2$ are local operators.

$(c_{Liou} \equiv 1 + 6Q^2, Q = b + 1/b, \quad b = 1/\sqrt{2})$

Bosonization Distler

$$\beta = \partial\Theta^1 e^{b\phi} \quad \gamma = \partial\Theta^2 e^{-b\phi}$$

(2,-1) $\beta\gamma$ system, (2,-1) bc system

Scalar supercharge

$$Q_S = \oint b(z)\gamma(z) = \oint b(z) e^{-\phi(z)} \partial\Theta^2(z)\,,$$

$$Q_S^2 = \{Q_B, Q_S\} = \{b_0, Q_S\} = [L_n, Q_S] = 0\,.$$

$$L_0 = \{Q_S, \cdots\}$$

Open String Field Theory

Usual OSFT action on N D-branes: Witten

$$S[\Psi] = -\frac{1}{g_o^2}\left(\frac{1}{2}\langle\Psi_{ij}, Q_B\Psi_{ji}\rangle + \frac{1}{3}\langle\Psi_{ij}, \Psi_{jk}, \Psi_{ki}\rangle\right) .$$

String field $\Psi_{ij} \in \mathcal{H}_{ij}$, open string state-space between brane i and brane j.

Topological localization

- $\mathcal{S}_{WS} = \{Q_S, \cdots\} \to$ localization on 'massless' modes
- More formal argument:

Gauge-fix action with Siegel gauge $b_0 \Psi_{ij} = 0$. Extend Ψ_{ij} to arbitrary ghost number

Q_S cohomology is one-dimensional: open string tachyon $c\, e^{b\phi}$.

OSFT action Q_S closed.

$$\Psi_{ij} = X_{ij} T_{ij} + \cdots = X_{ij} e^{b\phi} c_1 |0\rangle_{ij} + \cdots .$$

$$S[\Psi] = -\frac{1}{g_o^2} \left(\frac{1}{2} X_{ij} X_{ij} \langle T_{ij}, c_0 L_0 T_{ji} \rangle + \frac{1}{3} X_{ij} X_{jk} X_{ki} \langle T_{ij}, T_{jk}, T_{ki} \rangle \right) + Q_S(\cdots) .$$

Vacuum amplitudes containing states outside the cohomology add up to zero.

Boundary Liouville correlators

Naive computation: Liouville momentum has to add up to 3.

$\langle T_{ij}, c_0 L_0 T_{ij}\rangle = \mu_B^{(i)} + \mu_B^{(j)}$. (Needs one $\mu_B e^{\phi}$ insertion)

$\langle T_{ji}, T_{jk} * T_{ki}\rangle = 1$ (Needs no μ_B insertion)

Kontsevich model: $\mu_B = z$

Kinetic term would be naively zero (open tachyon is on-shell). Why non-zero?

Compute carefully with FZZT formulae:

$$\langle e^{\alpha\phi} e^{\alpha\phi}\rangle_{1,2} = D(\alpha, \mu_B^{(1)}, \mu_B^{(2)}, \mu_{bulk})$$

$D(\alpha, \mu_B^{(1)}, \mu_B^{(2)}, \mu_{bulk})$ has a pole as $\alpha \to 1$ that cancels against $L_0 \to 0$.

All contributions come from singular surfaces

In cell decomposition of OSFT, all propagator lengths $L_i \to \infty$

Evaluating Boundary Liouville correlators

$$D(\alpha, \mu_B^{(1)}, \mu_B^{(2)}, \mu_{bulk}) = (\frac{\pi}{\sqrt{2}}\mu_{bulk}\gamma(\frac{1}{2}))^{\frac{1}{2}-\alpha}\times$$

$$\times\frac{\Gamma_{\frac{1}{\sqrt{2}}}(\sqrt{2}\alpha - \frac{3}{\sqrt{2}})S_{\frac{1}{\sqrt{2}}}(\frac{3}{\sqrt{2}} + i\,s_1 + i\,s_2 - \sqrt{2}\alpha\,)S_{\frac{1}{\sqrt{2}}}(\frac{3}{\sqrt{2}} - i\,s_1 - i\,s_2 - \sqrt{2}\alpha)}{\Gamma_{\frac{1}{\sqrt{2}}}(\frac{3}{\sqrt{2}} - \sqrt{2}\alpha)S_{\frac{1}{\sqrt{2}}}(i\,s_1 - i\,s_2 + \sqrt{2}\alpha)S_{\frac{1}{\sqrt{2}}}(-i\,s_1 + i\,s_2 + \sqrt{2}\alpha)}$$

$$\frac{\mu_B^{(i)}}{\sqrt{\mu_{bulk}}} = \cosh\sqrt{2}\pi s_i$$

Contact terms

Open string contact terms: boundaries touch each other

Closed string contact terms: punctures touch each other

Highly nontrivial contact terms in closed string description give rise to recursion relations for amplitudes.

$$\frac{\partial}{\partial t_1}\mathcal{Z} = \mathcal{L}_{-2}\mathcal{Z} \equiv \frac{t_1^2}{2g_s^2}\mathcal{Z} + \sum_{k=0}^{\infty}(2k+3)t_{2k+3}\frac{\partial \mathcal{Z}}{\partial t_{2k+1}}$$

$$\frac{\partial}{\partial t_3}\mathcal{Z} = \mathcal{L}_{0}\mathcal{Z} \equiv \frac{1}{8}\mathcal{Z} + \sum_{k=0}^{\infty}(2k+1)t_{2k+1}\frac{\partial \mathcal{Z}}{\partial t_{2k+1}}$$

$$\frac{\partial}{\partial t_{2n+5}}\mathcal{Z} = \mathcal{L}_{2n+2}\mathcal{Z} \equiv \sum_{k=0}^{\infty}(2k+1)t_{2k+1}\frac{\partial \mathcal{Z}}{\partial t_{2k+2n+1}} + \frac{g_s^2}{2}\sum_{k=0}^{n}\frac{\partial^2 \mathcal{Z}}{\partial t_{2k+1}\partial t_{2n-2k+1}}$$

Recursion encode contact terms for $\mathcal{O}_{2n+1}$ at position of other $\mathcal{O}_{2m+1}$ and nodes of surface. Witten, Dijkgraaf Verlinde Verlinde

Non-trivial self-consistency: Virasoro algebra.

Extended recursion relations

How do boundaries affect recursion relations?

New contact terms for $\mathcal{O}_{2n+1}$

at boundaries and at nodes that pinch a boundary.

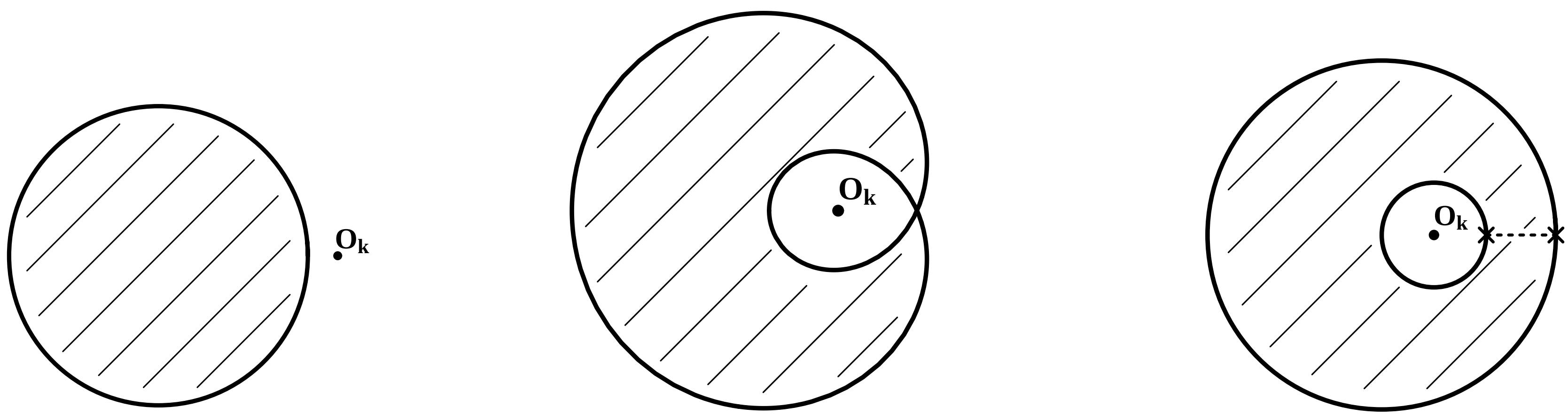
$$\frac{\partial}{\partial t_1}\mathcal{Z} = \tilde{\mathcal{L}}^{(z)}_{-2}\mathcal{Z} \equiv \mathcal{L}_{-2}\mathcal{Z} + (\frac{t_1}{z g_s} + \frac{1}{2z^2})\mathcal{Z} - \frac{1}{z}\frac{\partial \mathcal{Z}}{\partial z}$$

$$\frac{\partial}{\partial t_3}\mathcal{Z} = \tilde{\mathcal{L}}^{(z)}_{0}\mathcal{Z} \equiv \mathcal{L}_{0}\mathcal{Z} - z\frac{\partial \mathcal{Z}}{\partial z}$$

$$\frac{\partial}{\partial t_{2n+5}}\mathcal{Z} = \tilde{\mathcal{L}}^{(z)}_{2n+2}\mathcal{Z} \equiv \mathcal{L}_{2n+2}\mathcal{Z} - z^{2n+1}\frac{\partial \mathcal{Z}}{\partial z} - g_s\sum_{k=0}^{n} z^{2k+1}\frac{\partial \mathcal{Z}}{\partial t_{2n-2k+1}}$$

Unique solution:

$$\mathcal{Z}_{open+closed}(t_{2k+1}, z) = \mathcal{Z}_{closed}\left(t_{2k+1} + \frac{g_s}{(2k+1)z^{2k+1}}\right)$$


O_k
O_k
O_k

In this model, all the physics can be extracted from the open string vacuum amplitudes on an infinite number of branes

This may be the correct framework for background independence Witten

- Generalizations to other $c \leq 1$ and $\hat{c} \leq 1$

Non-zero bulk cosmological constant

Treat closed string deformation $\mu e^{2b\phi}$ perturbatively.

In OSFT, add closed string insertions with open/closed vertex. In this simple case,

$$\mathcal{Z}(g_s, \mu, z_i) = \mathcal{Z}_{closed}(g_s, \mu)\rho(Z)^{-1}\int [dX]\exp\left(\frac{1}{g_s}\mathrm{Tr}\left[-\frac{1}{2}ZX^2 + \frac{1}{6}X^3 + \mu X\right]\right)$$

Shift $X \to X - (Z^2 - 2\mu)^{\frac{1}{2}} + Z$ gives

$$\int [dX]\exp\left(\frac{1}{g_s}\mathrm{Tr}\left[-\frac{1}{2}(Z^2 - 2\mu)^{\frac{1}{2}}X^2 + \frac{1}{6}X^3\right]\right)$$

Hence $\mu_B \equiv (z^2 - 2\mu)^{\frac{1}{2}}$, which checks out.

Similarly, dilaton deformation
$t_3\mathcal{O}_3 \to$ open/closed vertex $\frac{3t_3}{g_s}\mathrm{Tr}Z^2X$.

Higher $\mathcal{O}_k$ have non-trivial contact terms $\to$ multi-trace deformations?

Generalization to $(p,1)$ (work in progress)

Kontsevich integral for $(p,1)$: One matrix with action of order $p+1$ KMMMZ

Topological OSFT $(p,1)$: $p-1$ matrices with cubic action

Example: $(\mathbf{3},\mathbf{1})$

$$\mathcal{Z}^{KMMMZ}(t) = \rho(Z)^{-1} \int [dX] \exp\left(-\frac{1}{g_o^2} \mathrm{Tr}\left[\frac{1}{4} Z^2 X^2 + \frac{1}{6} Z X^3 + \frac{1}{24} X^4 \right] \right)$$

$$\mathcal{Z}(t) \sim \int [dX][dY] \exp\left(-\frac{1}{g_o^2} \mathrm{Tr}\left[a_1 Z^2 X^2 + a_2 Z X Y + a_3 Y^2 + b_1 Z X^3 + b_2 X^2 Y \right] \right)$$

CFT: twisted $SU(2)_{p-2} \oplus \Theta_\alpha \oplus$ Liouville $\oplus\, bc \cong$ top. MM $\oplus$ top. gravity

Structure of matrix model dictated by Lioville dressing

Y can be integrated away $\rightarrow$ quartic potential: $\mathcal{Z}(t) = \mathcal{Z}^{KMMMZ}(t)$?

Conclusions

A new class of examples of exact open/closed duality

Prototype: Kontsevich model

Many possible extensions:

- (p, q) minimal models
- $c = 1$ at self-dual radius Ghoshal Mukhi Murthy
- $\hat{c} < 1$ models
- relation with critical topological strings

General lesson:

Open SFT on infinite number of branes as a universal tool for open/closed dualities