Preliminary Notes: Introduction in Topological String Theory on Calabi-Yau manifolds

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1 Motivation

Two major challenges that Superstring/M-theory faces today is the enormous number of apparently consistent solutions and the difficulty to extract detailed physical consequences in even one of them.

The hierarchy of physical properties that one needs to know ranges from basic questions about the light spectrum of particles to more sophisticated effective interaction terms due to collective effects like instantons. In the presence of supersymmetry there is a corresponding hierarchy of more and more sophisticated topological and geometric invariants of the string geometry, which have the potential to calculate the answers to these questions. The rich physical structure, which organizes this approach at the first levels, is *topological string theory*. It contains per definition part of the geometry of supergravity as its limit.

Leaving philosophical speculations about the first challenge aside, it might just reflect our erratic understanding of non-perturbative string theory. A successfully strategy to decipher these non-perturbative properties is to combine the mathematical rigidity of supergravity and topological string theory with physical consistency arguments. This bootstrap like approach revealed mirror duality, string duality, exact large N-duality and holography. These concepts enhanced the ability to grasp the physical consequences of string theory, which settled conceptual issues of quantum gravity, that are outside the range of perturbation theory. M-theory is an attempt to provide a unified description of non-perturbative string theory and is defined by various limits. Two questions are out. First whether we can develop our understanding of the topological sub sectors and the geometry of these limits sufficiently. Secondly is the mathematical structure rigid enough to predict with some ingenuity from the knowledge of the limits the non-perturbative completion.

2 Overview

A starting point for studying string theory in such a non-trivial space time geometry M is the non-linear σ model. The correlation functions, for simplicity we consider the partition function Z first, are given by a variational integral

$$Z(M) = \int \frac{\mathcal{D}h}{\text{Vol diff.weyl.}} \mathcal{D}x e^{iS(x,h,M)} \tag{1}$$

over all embeddings of the world-sheet Σ in M

$$x : \Sigma \to M \tag{2}$$

and the world-sheet metric h. The dependence of such correlation functions on the topology and geometry of M, which is treated here as a classical background, might be taken as a first step to describe stringy geometry. It is of direct practical importance as it determines the effective action in 4d for string compactifications on M. Of particular interest will be the dependence of terms in the low energy effective action on the geometric moduli of M. Understanding this depends on the geometry is a prerequisite for quantizing the latter.

However in the generic case correlation functions like (1) are far too complicated to handle. Here we want to study the exceptions. One can be found within super symmetric compactifications of critical string theory. Using diffeomorphism and Weyl invariance, maintained for the critical case in the first quantized version, the dependence on the degrees of freedom of the world-sheet metric h simplifies drastically even in the quantum theory. The world-sheet super symmetry gives rise to nilpotent operators Q, which define a theory whose physical operators are cohomology classes w.r.t. Q. It is called topological string theory. The reader might wonder how formal the expression (1) is. Certainly we have suppressed all fermionic degrees of freedom in S. The full actions will be spelled out in Sec. 4. However even if we kill some suspense let us remark that the expression for the integration over h, which is just as in the bosonic string in (1), is surprisingly accurate for our purpose. It turns out the fermions, which we need to add play merely the rôle that the ghost system plays in the bosonic string.

Physically this reduction to the topological sub sector of the theory can be thought as a *semi-classical approximation* of (1) in which the variational integral is replaced by integral over the moduli space $\mathcal{M}$ of the classical solutions $\delta S/\delta x = 0$. E.g. for the Polyakov action these are the minimal area maps. The path integral measure collapses to a measure on $\mathcal{M}$, which depends merely on the topological properties of the map (2) and on the cohomology classes of the inserted operators. This defines so an intersection theory on $\mathcal{M}$. The intersection numbers are topological invariants of the classical solutions. Examples are the Gromov-Witten invariants, which are symplectic invariants of M. σ models with $(2,2)$ world-sheet super symmetry, realized on Calabi-Yau manifolds M_6, allow for two possibilities to pick Q, leading to what is known as the A and the B topological string model[207]. Exchanging this choice underlies the mirror duality and which leads to two different ways to solve both models. The B-model approach is more effective. Open topological string theory exists as well. Preservation of at least one world-sheet Q operator restricts the boundary conditions on Calabi-Yau three folds with $SU(3)$ holonomy either to special Lagrangian branes for the A-model and holomorphic submanifolds for the B-model. It had been observed in 1992 that the open topological models are reductions of open string field theory and that this reduction leads to Chern-Simons theories on the branes [199].

The remarkable fact is that in super string theories the restriction to the classical solutions leads to exact calculations of certain low derivative terms in the effective supergravity action in 4d. This ability to perform exact calculations including non-perturbative effects is typically reflected by non-renormalization in the effective theory. For example in $N = 2$ super symmetric gauge theories the protected terms are the *kinetic terms of the moduli fi elds* $\underline{t}$, which give the exact $\underline{t}$ dependence of the *gauge couplings* as well as of the *masses of the BPS states*. Both terms are calculated by genus zero $g = 0$ topological string amplitudes. In $N = 2$ supergravity theories one obtains in addition from $g > 0$ topological string amplitudes the exact moduli dependence of the coupling of the anti-self-dual graviphoton field strength F_- to the anti-self-dual part of the Riemann curvature R_+, i.e. the coupling $\int_{M^4} \mathrm{d}^4 x F_g(\underline{t}) F_-^{2g-2} R_- \wedge R_-$ [20][8]. In $N = 1$ theories one can get the *superpotential from disk amplitudes* and the *gauge kinetic terms* from the annulus amplitudes. Higher genus open string amplitudes appear in the effective action of C deformed gauge theories [170].

Reconstruction of these exact terms in the low energy effective action of a field theory by solving the topological string theory in a suitable chosen geometry M is called geometrical engineering. In general one would like to understand emergence of nearly flat 4d space-time $M_{3,1}$ within $M_{9,1}$ dynamically. Often one considers $M_{9,1} = M_6 \times M_{3,1}$ as ansatz. In generalizations like wrapped geometries [187] or compactifications with RR/NS background fluxes on M [174], which preserve at least $N = 1$ supersymmetry one can still use topological string methods to calculate the protected terms. M_6 being compact leads to traditional compactifications including non-trivial supergravity solutions, as e.g. black hole solutions on $M_{3,1}$. The gauge sector in $M_{3,1}$ can be studied even for non-compact M_6 if gravity can be consistently decoupled. This is similar to the decoupling of bulk gravity in brane world scenarios with non-compact transversal directions.

The second class of exactly solvable examples are non-critical string theories [87][54]. Here the understanding of the infinite symmetries is much more advanced and has lead to the solvability of the string theories with $c \leq 1$ or equivalently $d \leq 2$ dimensions, including the Liouville direction, for the bosonic case. Super symmetric versions exists a well. For the non-critical case the quantization of the two dimensional metric degrees of freedom gives rise to the Liouville sector, which augments (1) in the quantum theory. The theory consist of ghost-, matter- and Liouville sector and has an nilpotent operator Q with an induced cohomological structure[202], which is strikingly similar to the one in the topological sector of the critical string. The choices of matter are (p,q) minimal models for $c < 1$ and the free boson for the $c = 1$ limiting value[132]. The infinite symmetries which underly the solvability of non-critical string are well understood. An elegant way to summarize the structure is to say that $\log(Z(\underline{t}))$ is the $\tau(\underline{t})$ function associated to a vacuum orbit in an infinite Grassmanian, which is physically described by an infinite 2d fermion system.

Major insights in $c \leq 1$ strings have been obtained via the double scaled matrix model [87][54]. The finite $N \times N$ matrix model, for which i.g. several realizations exist, provides a discretization of the string world sheet σ in terms of ribbon graphs. A vertex of valence p represents a regular p-gon in the dual discretization of Σ and it is simplest to fix $p = 3$. More importantly the dual p-gons of a graph give a discretization of the space of metrics on Σ modulo isomorphism. The continuum limit can be understood as an improving approximation of the world-sheet and its metric by graphs with an increasing number

V of the vertexes. The key intuition is that for a larger number V of p-gons the metric is approximated increasingly accurately by the deficit or surplus angels in gluing the tiles and moreover that the number of graphs which approximate a metric in a given isomorphism class becomes a good measure on the space of metrics. Therefore integrating over metrics can eventually replaced by counting contributions of the sum of graphs, just as the Feynman graph expansion of the matrix model. The continuum limit requires a regularization procedure in which one takes N to infinity while tuning the coupling(s) of the matrix model to a critical value $g \to g_c$ so that a parameter $t = N(g - g_c)^{\frac{(2-\gamma)}{2}}$ stays finite[54] [203]. The double scaling limit regularizes the total area, whose unregularized value goes like $\langle A \rangle = \langle V \rangle \sim \frac{1}{(g-g_c)}$ [23] as the number of p-gons goes to infinity. One can show[23] that a genus g contribution is suppressed with N^χ as $N \to \infty$ and enhanced with $(g - g_c)^{(2-\gamma)/2\chi}$ as $g \to g_c$, where $\chi = 2 - 2g$. The double scaling definition of t is chosen to counterbalance these effects and to get a finite all genus expansion in t.

A qualitative different relation to matrix models is provided by the Kontsevich model [203][140]. It describes the $(2, 1)$ pure 2d gravity case[1] by an hermitian matrix model whose ribbon graphs model the cell decomposition of the moduli space $\overline{M}_{g,n}$ of the world-sheet with n descendant operator $\mathcal{O}_i$ insertions. The matrix model partition function calculates correlators $\langle \mathcal{O}_1 \ldots \mathcal{O}_r \rangle$ as topological intersections numbers on $\overline{M}_{g,n}$. The cell decomposition replaces close string insertions by holes and strongly resembles the formalism of open string field theory. The couplings t_k of the operators $\mathcal{O}_k$ are given in terms of symmetric functions of the hermitian matrix eigenvalues, i.e by the Miura variables $t_k = \mathrm{tr} X^k$. Results for a given correlator $\langle \mathcal{O}_1 \ldots \mathcal{O}_r \rangle$ are exact as long as the rank N of the matrix X is large enough to provide enough independent symmetric functions for the t_k.

Exact calculations in higher dimensional topological strings have been boosted by mirror symmetry [37] and in non-critical string theory by the double scaled matrix model approach and the Kontsevich type matrix model. The subjects have never been independent as one needs to couple the topological A and B theories to worldsheet gravity to get the F_g amplitudes for $g > 1$, see [20] for the B-model. The solution of pure 2d gravity is used explicitly in the calculation of the A-model amplitudes by localization [141] together with Hodge integrals[68][135]. A more surprising link between the topological string on the conifold and the $c = 1$ string at the selfdual radius [103] has been pointed out in [89].

Two more recent developments motivate to revisit this connection. Dijkgraaf and Vafa observed in 2002 that the exact terms in the effective action of $N = 2$ and $N = 1$ supersymmetric gauge theories can be calculated also by an hermitian matrix model. Even though this has been explained in the meantime within the supersymmetric field theory framework, it is natural to relate it to topological string calculations by geometrical engineering and in fact it was discovered in this way. This leads to a matrix model descriptions of the topological string on non-compact Calabi-Yau and the quest for an unified description of the integrable structure behind topological strings in various dimensions[2].

A second motivation comes from the study of open/closed string duality. In the context of non-critical string theory the Kontsevich model has long been considered to be the simplest example of gauge theory/string duality. The gauge theory part describing the open string sector is played by the finite N-Kontsevich matrix model, while the closed string part is played by the non-critical topological string coupled to $(1, p)$ matter. Recent progress in solving the Liouville approach to critical string theory and classifying its boundary conditions revealed that the Kontsevich matrix model emerges as the action on the FZZT brane. This was anticipated from the B-model description of open string theory on local Calabi-Yau spaces[2]. It can also be shown by calculating the exact loop-operator in the double scaling limit of the matrix model[155] [109] or by doing a reduction of cubic string field theory[82] on FZZT branes.

An simple example of open/closed string duality in the case of critical topological string theory had been proven by Gopakumar and Vafa in 1999. The closed string side is played by the topological string on the non-compact Calabi-Yau geometry of two complex line bundles over the compact space $\mathbf{P}^1$ namely $E' = \mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbf{P}^1$. The topological open string geometry is reached from E' by contracting the volume t of the $\mathbf{P}^1$ and then deforming complex structure of the emerging singular geometry to the

[1] It has an extension to the coupling of 2d gravity to $(1, p)$ matter [140][205][204].

smooth cotangent bundle $E = T^*S^3$ of S^3. The latter is a Lagrangian submanifold L in E w.r.t to a natural symplectic structure on E and Witten's picture [199] of open topological string relates it to Chern-Simons theory on S^3. Exact solvability of topological Chern-Simons gauge theory on S^3 is provided by its relation to the 2d WZW model[201]. The closed topological string on E' can be solved exactly by localization [68]. This solvability on both sides provides a luxury, which is not readily available in the analogous situation in the $\mathrm{ADS}_5/\mathrm{CFT}$ string/gauge theory correspondence, namely to check explicitly that the partitions functions of gauge- and closed string theory are the same in the large N expansion of Chern-Simons theory when the volume of the $\mathbf{P}^1$ is identified with $t = N g_{CS}^2$.

Beside the partition function, which is a topological invariant of the three manifold L, Chern-Simons gauge theory is famous for calculating topological invariants associated to Wilson line expectation values along knots or links inside L. What is the topological string question answered by these quantities and what are the new parameters associated to the Wilson line ? An particular answer for the unknot in S^3 are open string amplitudes ending on a non-compact branes K which meets the $\mathbf{P}^1$ of E' in an S^1 [169]. The new parameter is the area of minimal disk ending on the S^1, which is non-contractible within K. The geometry of E' and K has a systematic generalization. E' contains the algebraic torus $T = (\mathbf{C}^*)^3$ as an open subset (one $\mathbf{C}^*$ for each line bundle and one for the $\mathbf{P}^1$). Moreover $(\mathbf{C}^*)^3$ acts on E' with the natural extension of the multiplicative action of $(\mathbf{C}^*)^3$ on itself. Varieties with this property are called toric varieties[81][167] [50][47], here in three complex dimensions. They are characterized by the degeneration of the T action, representable here as linear trivalent graphs embedded in three real dimensions. The vertices represent $\mathbf{C}^3$ patches and the graph caries the information about the transition functions. K is characterized by the property that is is a Lagrangian which is invariant under $(\mathbf{C}^*)^2 \in T$. Non-compact toric Calabi-Yau manifolds with invariant non-compact special Lagrangian branes are a simple natural class of backgrounds on which all open and closed topological string amplitudes be calculated by localization w.r.t. the torus action. The question how to understand these general amplitudes comes back to Chern-Simons gauge theory. The answer is provided by the trivalent topological vertex, which solves the problem for the open topological amplitudes among three stacks of invariant non-compact special Lagrangian branes in a $\mathbf{C}^3$ patch, and gluing rules for connecting these amplitudes on a patch to global amplitudes compatible with the global T action. As maybe expected the answer for the vertex is related to the amplitude of a link of three unknots in S^3.

The exact calculations in the topological sector of string theory have been an indispensable guide to the non-perturbative behavior of critical string theory. Virtually everything known about dualities involving strong coupling regimes is known from the analysis of the topological sub sectors of the corresponding theories. An overview over the dualities in this context is given below

Topological theories come with integrable structures, which reflect their often not immediately apparent symmetries. M-theory gives hints, but the non-perturbative formulation of string theory is illusive. Exploring possible non-perturbative completion of the topological string is a very serious chance in this context.

On various aspects of the dualities depicted here there have been recently very good lectures. In particular on the connection between matrix models and topological string in [158] and on the connection to Chern-Simons theory and aspects of open/closed duality in [159]. Older physical application of topological string theory using many of the above connections are review in [134] and newer can be found in [163]. Most of the material presented here can be studied in more detail in [105]. [197] is an introduction with the virtue of assuming very few prerequisite. An particularly important field on the borders of the material we present and yet don't reach is *categorical mirror symmetry*, see [181][125][150] and [11] for physically motivated reviews.

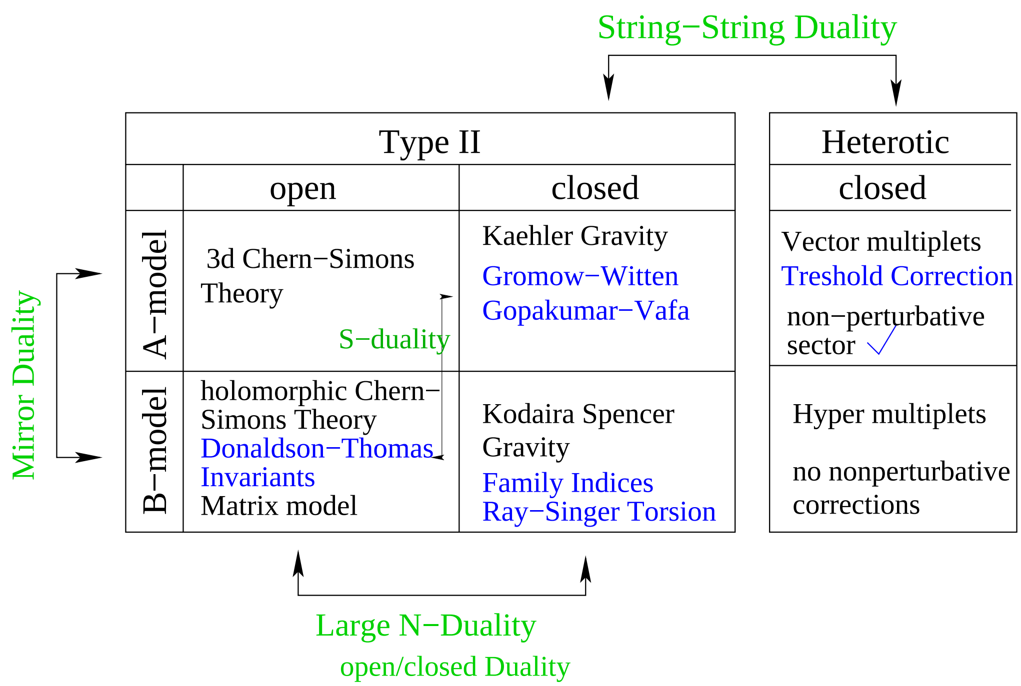

Fig. 1 Dualities relevant for the topological string of type II on backgrounds with two and heterotic string in backgrounds with four covaraint constant spinors.

3 Semi-classical approximation and super symmetric localization

Let us sketch the reduction of supersymmetric critical string theory to its topological sector. The two dimensional σ-model action $S(x,h,M) = \int_{\Sigma_g} \mathrm{d}^2\sigma \mathcal{L}(x,h,G,B,\ldots)$ depends generically on the metric G of M, the NS-two form field B on M and eventually other background fields. A possible attempt to make sense out of (1) is to expand the action around the classical solution of the equation of motion $\frac{\delta S}{\delta x}\big|_{x=x_{cl}} = 0$

$$S(x,h,M) = S(x_{cl},h,M) + \frac{(\delta x)^2}{2}\frac{\delta^2 S}{\delta^2 x}\Big|_{x=x_{cl}} + \ldots . \qquad (3)$$

The quadratic semi-classical approximation in δx in (1) leads then

$$\begin{aligned} Z(M) = & \int \frac{\mathcal{D}h}{\text{Vol diff.weyl.}} \mathcal{D}x e^{iS(x,h,M)} \\ = & \sum_{x_{cl},h_c} e^{iS(x_{cl},h_c,M)} \int \mathcal{D}\delta x e^{i\frac{(\delta x)^2}{2}\frac{\delta^2 S(x_{cl},h_c,M)}{\delta^2 x}} \\ = & \sum_{x_{cl},h_c} e^{iS(x_{cl},h_c,M)} \det{}^{-\frac{1}{2}} \frac{\delta^2 S(x_{cl},h_c,M)}{\delta^2 x} . \end{aligned} \qquad (4)$$

Here assumed that the determinant can be regularized and we have to consider all classical solutions, which are minimal embeddings of the world-sheet into M. It is useful to organize these contributions in a sum over different topological classes of such embeddings as indicated in (4). In the closed string case these classes are labeled by the genus of the domain Σ_g and the cohomology class $H^2(M,\mathbf{Z})$ of the image

$[x(\Sigma_g)]$. However depending on the case it might be that there are families of classical solutions of a given topological type parametrized by moduli of the minimal embedding and eventually the complex structure of h called h_c . In this case one has to integrate over a suitable measure over this moduli space, which is not indicated in the sums in (4). Naturally the *semi classical* approximation will be good all the configurations "localize" close to extrema of the classical action.

It is a general fact that in supersymmetric extensions of (4) there is an exact localization to classical configurations for correlation functions with a suitable fermion zero mode structure. This has its origin simply in the rules of Grassmann integration over the fermionic fields ψ_k

$$\int \psi_1 \ldots \psi_n \mathrm{d}\psi_1 \ldots \mathrm{d}\psi_n = 1, \qquad \int \psi_1 \ldots \widehat{\psi_j} \ldots \psi_n \mathrm{d}\psi_1 \ldots \mathrm{d}\psi_n = 0\,. \tag{5}$$

For a field configurations for which the supersymmetric variations do not vanish for all variations of the fermionic fields one can use the supersymmetry transformation to eliminate fermions from the action. By the second identity in (5) the fermionic measure will then produce a 0. Putting the argument around the only contributing field configurations are the ones for which the fermionic variations are *stationary*, but these are the classical configurations as we will see.

3.1 A simple supersymmetric index

This mechanism is independent of the dimension and can be demonstrated already in the 0d case, i.e. for an ordinary integral $Z = \int \mathrm{d}x \mathrm{d}\psi_1 \mathrm{d}\psi_2 e^{-S(x,\psi_1,\psi_2)}$ over the bosonic variable x and Grassmann variables ψ_1 and ψ_2. The action

$$S(x, \psi_1, \psi_2) = \frac{1}{2}(\partial h)^2 - \partial^2 h \psi_1 \psi_2, \tag{6}$$

where $h(x)$ is an arbitrary function of x. One checks easily that action $\{Q, S\} := \delta S = 0$ and measure $\delta(\mathrm{d}x\mathrm{d}\psi_1\mathrm{d}\psi_2) = 0$ are invariant under the following supersymmetric transformations

$$\begin{aligned} \delta x &= \epsilon^1 \psi_1 + \epsilon^2 \psi_2 \\ \delta\psi_1 &= \epsilon^2 \partial h \\ \delta\psi_2 &= -\epsilon^1 \partial h\,. \end{aligned} \tag{7}$$

Away from the fix points of the fermionic transformations, i.e. for $\partial h \neq 0$, we can set $\epsilon^1 = \epsilon^2 = -\frac{\psi_1}{\partial h}$ and use the supersymmetry transformation to eliminate the first fermion, i.e. with $\hat{x} = x + \delta x$ and $\hat{\psi}_i = \psi_i + \delta\psi_i$, $i = 1, 2$ one gets $S(\hat{x}, 0, \hat{\psi}_2) = S(x, \psi_1, \psi_2)$. So in the hatted variables there is no $\hat{\psi}_1$ to "soak up" the $\mathrm{d}\hat{\psi}$ integration and the integral vanishes. To be more explicit we transform the integration measure also to the hatted variables. Since the transformation is singular we consider a nearby transformation $\epsilon^2 = (\alpha(x) - 1)\frac{\psi_1}{\partial h}$, $\epsilon^1 = -\frac{\psi_1}{\partial h}$ and send $\alpha \to 0$ after transforming the integral. Note that $\int \psi \mathrm{d}\psi = 1$ is invariant under $\psi \to \hat{\psi} = \alpha(x)\psi$, therefore $\mathrm{d}\hat{\psi} = \frac{1}{\alpha}\mathrm{d}\psi$. In the transformed integral one finds beside terms which go to 0 with α only a term which is total derivative w.r.t. $\mathrm{d}x$ integral and vanishes at the boundary.

Since the integral gets contributions only from the critical points of $h'(x_c) = 0$, we can collect the contributions near those points by considering $h(x) = h(x_c) + \frac{\kappa_c}{2}(x - x_c)^2$, with $\kappa_c = h''(x_c)$, which yields a Gaussian integration. The partition function

$$\begin{aligned} Z = \quad & \frac{1}{\sqrt{2\pi}} \int \mathrm{d}x\mathrm{d}\psi_1\mathrm{d}\psi_2 e^{-S(x,\psi_1,\psi_2)} = \sum_{x=x_c} \frac{1}{\sqrt{2\pi}} \int \mathrm{d}x\mathrm{d}\psi_1\mathrm{d}\psi_2\, e^{-\frac{1}{2}\kappa_c^2(x-x_c)^2 + \kappa_c\psi_1\psi_2} \\ & = \sum_{x_c} \frac{h''(x_c)}{|h''(x_c)|}\,. \end{aligned} \tag{8}$$

becomes a primitive version of a supersymmetric index. It counts sum of zeros of $h'(x)$ weighted with $+1$ (-1) for positive (negative) slope at $h'(x_c)$. If $h'(x)$ is continuous a $+1$ zero of $h'(x)$ can only disappear together with a -1 zero under deformations of $h'(x)$, which leave the behavior of $h'(x)$ for $|x| \to \infty$ invariant, see Fig. 2. That means that Z is an invariant under such deformations and can be thought as a topological invariant of $h(x)$. This idea extends to interesting indices, see Secs. 9.4 and 9.5. We can

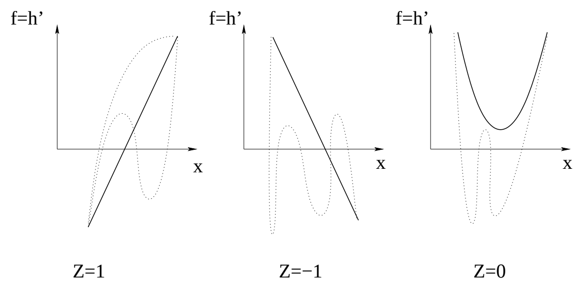

Fig. 2 Deformation invariance of the simple index

interprete (8) by defining $D = \begin{pmatrix} 0 & -\partial^2 h \\ \partial^2 h & 0 \end{pmatrix}$ and the fermionic integral definition of the Pfaffian $\mathrm{Pf}(D) = \int \prod_k \mathrm{d}\psi_k \exp\left(-\frac{1}{2}\psi_i D_{ij} \psi_j\right)$ as well as usual bosonic Gaussian integration as the expression

$$Z = \sum_{x_c} \frac{\mathrm{Pf}(D)}{\sqrt{\mathrm{Det}(D)}} \,. \tag{9}$$

We might further interprete (8) by defining $g(z) = \sqrt{h''(z)^2}$ and $f(z) = h'(z)$ and the meromorphic differential

$$\Omega = \frac{1}{2\pi i} \frac{g \mathrm{d}z}{f} = \frac{1}{2\pi i} \frac{g \mathrm{d}z}{\frac{\partial h}{\partial z}} \,, \tag{10}$$

which we want to integrate over $\mathbf{P}^1$. It has a pole of first order at infinity and Z is the residuum at this of this pole which can be $0, \pm 1$. We can express now Z as the follwing residuum expression

$$Z = \int_\Gamma \Omega = \sum_{x_c} \frac{|h''(x_c)|}{h''(x_c)} \tag{11}$$

where Γ encircles all finite critical point and we take always the positive branch of the root. In this formalism the critical points are the analogs of the Calabi-Yau manifold. It is a far shot, but conceptually true, that the solution the B-model by the period integrals can be viewed as generalization of this example from zero to three complex dimensions in which in particular (10) is identified with (266,267) and (280) the analog of the first integration to Z. A model, almost as simple as the above, with a manifold and a holomorphic vector bundle was used in [17] to proof the vanishing of instanton contributions to $N = 1$ superpotentials in $(0, 2)$ compactifications.

An important lesson from these simple examples is that the fermionic integral and the fermionic symmetries decide crucically about the contribution of the expression. Less subtle then the above argument, which involves supersymmetry, is the following general consideration. An operator D, e.g. the Dirac operator, usually pairs the fermions $\bar\psi D \psi$ in the action. If this operator has zero eigenvalues, some fermions disappear from the action and the integral becomes zero as above. Fortunatly fermions are geometrical non trivial sections and their zero modes are captured by "easy" cohomological information of the geometry much like the zero modes of the Laplacian count harmonic forms, which are related to cohomology. The Atiyah-Singer index theorem links a net count of the zero modes to topological invariants, which are often quite easy to evaluate. This idea is made more explicite in Sec. 9.4 and 9.5.

4 Supersymmetric nonlinear σ-models

Essential features of the $0d$ topological toy model carry over to super symmetric σ-models and other supersymmetric theories. A $1d$ (supersymmetric) σ-model is simply a $1d$ field theory associated to a manifold M such that the fields are coordinates (and supercoordinates) of M, which depend only on one variable. It is natural to think this one variable as the time and the whole setup as (supersymmetric) quantum mechanics on M. In $2d$ dimensional σ models, the case relevant to string theory, the coordinates (and supercoordinates) of M depend on two variables the WS coordinates of the string and σ-model fields can be viewed as a map $x : \Sigma \to M$ from the worldsheet Σ to the targetspace M.

As in the 0d toy case we search in these models for field configurations which are fixpoints under some super symmetry transformation. The super symmetry generators become nilpotent operators Q on the Hilbert space of the filed theory. The cohomology of Q is a natural structure to extract topological invariants of the classical bosonic configuration space. In more interesting situations indices can occur, which are invariant under some deformations, but are *family indices* w.r.t. others. Physically the family indices can be particular correlation functions. Their dependence on certain geometrical deformation parameters, e.g. of the target space metric, can often be exactly calculated e.g. in an all genus string loop expansion. This is the main physical benefit from topological theories. Apart form this more interesting geometry there is only one *new conceptual issue* in the 2d case and that are potential *anomalies* of the $2d$ quantum field theory on the WS.

The original references for the following are [206][146] and especially [207]. We have adopted the conventions from the review [105]. There is a well known dictionary between properties of the worldsheet theory and properties of M. In particular if M is a Kähler manifold the σ-model will have $(2,2)$ worldsheet supersymmetry [216]. The inverse statement is not quite true, i.e. one can construct more general geometric backgrounds that allow for $(2,2)$ worldsheet supersymmetric σ-models[83].

In order to have superconformal invariance M has to be a Calabi-Yau manifold. A Calabi-Yau manifold is Kähler manifold with vanishing first Chern class of its tangentbundle $c_1(TM) = 0$. This is equivalent to the statement that there exists a hermitean metric g for which the Ricci curvature vanishes $R_{i\bar{\jmath}} = 0$. This in turn is equivalent to the statement that the holomomy group of M is contained in $SU(3)$. We call a Calabi-Yau threefold a manifold where the holonomy is the full $SU(3)$ (or a least $SU(2) \times Z_2$), which implies that there are exactly two covariant constant spinors on M. This leads to $N = 2$ supergravity theories in 4d for the compacification of type II on M. Many of the above facts and concepts are reviewed in detail in Sec. 9. We will start the discussion of the symmetries of the actions at the classical level and comment then on the potential anomalies and their cancellation.

4.1 $N = (1,1)$ nonlinear σ-model

Let us first treat the $N = (1,1)$ case. For this case the target space needs to have just a Riemannian metric. We parametrize the map $x : \Sigma \to M$ by x^I, where $I \ldots, d$ where d is the real dimension of M. The worldsheet is parametrized by $z, \bar{z}$, hence x is given in local coordinates as $x^I(z,\bar{z})$ The fields of the σ model have the following transformation properties under worldsheet and targetspace reparametrizations. With K and $\bar{K}$ the canonical and anti-canonical bundle of Σ and TM the complexified tangentbundle of M one has WS-fermions which transform as $\psi_+^I \in \Gamma(\bar{K}^{\frac{1}{2}} \otimes x^*(TM))$ and $\psi_-^I \in \Gamma(K^{\frac{1}{2}} \otimes x^*(TM))$, where Γ denotes sections of the indicated bundles. The Lagrangian of the non-linear 2d σ-model is then given by

$$L = 2t \int_\Sigma \mathrm{d}^2 z \left(\frac{1}{2} g_{IJ}(x) \partial_z x^I \partial_{\bar{z}} x^J + \frac{i}{2} g_{IJ} \psi_-^I D_z \psi_-^J + \frac{i}{2} g_{IJ} \psi_+^I D_{\bar{z}} \psi_+^J + \frac{1}{4} R_{IJKL} \psi_+^I \psi_+^J \psi_-^K \psi_-^L \right). \tag{12}$$

The covariant derivatives $D_{\bar z}$ (D_z) are obtained using the pullback of the Levi-Civita connection from M as

$$D_{\bar z}\psi^I_+ = \frac{\partial}{\partial \bar z}\psi^I_+ + \frac{\partial x^J}{\partial \bar z}\Gamma^I_{JK}\psi^K_+ \tag{13}$$

and R_{IJKL} is the Riemann-Tensor of M. Here we assumed a flat world-sheet or a local trivialization of $K^{\frac{1}{2}}$, so that no spin connection appears in (13). Soon global properties of $K^{\frac{1}{2}}$ and $\bar K^{\frac{1}{2}}$ become all important.

With Grassmann valued supersymmetry parameters $\epsilon_- \in \Gamma(K^{-\frac{1}{2}})$ and $\epsilon_+ \in \Gamma(\bar K^{-\frac{1}{2}})$ one checks at the classical level the following supersymmetry transformation

$$\begin{aligned} \delta x^I =& \ -\epsilon_-\psi^I_+ + \epsilon_+\psi^I_- \\ \delta\psi^I_+ =& \ i\epsilon_-\partial x^I + \epsilon_+\psi^K_-\Gamma^I_{KM}\psi^M_+ \\ \delta\psi^I_- =& \ -i\epsilon_+\partial x^I + \epsilon_-\psi^K_+\Gamma^I_{KM}\psi^M_- \ . \end{aligned} \tag{14}$$

These equations (14) are quite similar to (7) and we would like to define nilpotent operators from the supersymmetry transformations. The obstruction is that there are no global trivial sections of $K^{\frac{-1}{2}}$ or $\bar K^{-\frac{1}{2}}$ unless $g = 1$. This means that there no global supersymmetry transformations on the worldsheet unless[2] $g = 1$.

In the case of the worldsheet beeing a torus one can chose globally defined sections $\epsilon_- \in \Gamma(K^{-\frac{1}{2}})$ and $\epsilon_+ \in \Gamma(\bar K^{-\frac{1}{2}})$ to obtain globally defined supersymmetry generators $Q^2_- = 0$ and $Q^2_+ = 0$ on the Hilbert space $\mathcal{H}$. E.g. we can chose $\epsilon_\pm$ both to be in trivial sections of $K^{-\frac{1}{2}}$ and $\bar K^{-\frac{1}{2}}$ respectively. In view of (14) we have to chose corresponding trivializations for $\psi^I_+ \in \Gamma(\bar K^{\frac{1}{2}} \otimes x^*(TM))$ and $\psi^I_- \in \Gamma(K^{\frac{1}{2}} \otimes x^*(TM))$ and this simply means that the fermions will have periodic boundary conditions on T^2. These boundary conditions are called *twisted* boundary conditions. Q_- and Q_+ are globally defined and $Q_+|\Psi\rangle = Q_-|\Psi\rangle = 0$ for $\Psi \in \mathcal{H}$ forces the cohomological states to be in the $E = 0$ super symmetric ground state of the Hamiltonian [208]

$$H = \frac{1}{2}\{Q_+, Q_-\} = \frac{1}{2}(\mathrm{d}\mathrm{d}^* + \mathrm{d}^*\mathrm{d}) \ . \tag{15}$$

Generically the non-trivial information in the double twisted model is the *Witten index*. It is simplest written in the operator formalism

$$\chi(M) = \mathrm{Tr}(-1)^F q^{H_+}\bar q^{H_-} = \mathrm{Tr}(-1)^F, \tag{16}$$

where $F = F_+ + F_-$ and F_+/F_- count the left/right moving fermion numbers so that $\{(-)^{F_\pm}, Q_\pm\} = 0$ while $[(-)^{F_\mp}, Q_\pm] = 0$. The σ model cohohomology is equivalent to cohomology of M, much in the same way as we will made explicit in Sec. 6.1 and 8.1. Since (162) is the Laplacian and the fermion number, measured by $(-1)^F$, corresponds to the form degree, the Witten index is equal to the Euler number $\chi(M)$ of M [208]. The insertion of $(-1)^F$ kills the information about the time evolution and spatial exitation of the string. The latter fact reduces the model to constant maps, i.e. supersymmetric quantum mechanics on M, i.e. the index can also be obtained starting with a $1d$ supersymmtric σ model on M. The consideration that leads to the index is referred to as *quantizing the zero mode sector*. If further global quantum numbers are present one can get slightly finer information then just the Euler number, by inserting the corresponding charge operator in the trace. These ideas play a rôle in extracting BPS numbers for instance associated to branes see Sec. 6.16.

[2] The quest for covariant constant spinors is familiar on the target space in order to obtain spacetime supersymmetric compactifi cations. It requires restricted holonomies, see section 9.9, which is equivalent to the familiar $c_1(TM) = 0$ condition for $N = 2$ ($N = 1$) II (heterotic) compactifi cations 6d internal manifolds.

Much more detailed information survives in the string context if one choses only ϵ_+ to be in a trivial section. The corresponding index is called the *elliptic genus*[3]

$$\mathcal{E}(M) = \mathrm{Tr}(-1)^{F_+} q^{H_+} \bar{q}^{H_-} = \mathrm{Tr}(-1)^{F_+} \bar{q}^{H_-} \ . \tag{17}$$

Here only the left moving states are forced in the left moving groundstate. The trace over the right moving states explores information which goes far beyond cohomological information of M. It can be defined for 2d supersymmeric field theories and is conformally invariant even if the underlying field theory is not [211]. It requires $(-)^{F_+}$ not to be anomalous, which is essentially equivalent to M being spin [213]. It carries information, which is robust under certain deformations. In the case of the σ model on M $\mathcal{E}(M)$ is the Dirac index of the loop space of M [209, 210]. This index varies with the volume parameters of M, but is independent of the complex structure of M and is the first example of the promised family indices. There are further simple refinements possible, if as below in the $N = (2,2)$ theories F_- comes from an $U(1)_L$ current $F_- = \oint J_L$. If the latter is not anomalous one can insert $(-1)^{\theta F_-}$ in the trace in (17) and even if the $U(1)_L$ is broken to Z_K (17) with $\exp(\frac{i\pi}{k} F_-)$ inserted is still an index. A theme of the lecture is to explore more sophisticated family indices mainly in the $N = (2,2)$ context and even at genus one there are further refinements such as (323).

4.2 Compactifications with $N = (2,2)$ world sheet supersymmetry

The additional structure that allows to define more general family indices for the $(2,2)$ worldsheet theories are right and left $U(1)_{R/L}$ symmetries, so called R-symmetries. Since the nilpotent Q operators are derived from the supersymmetry transformations and since there are no covariant constant spinors for world sheets of genus $g \neq 1$ there will be no well defined supersymmetry operators on general Σ_g without further modifications. For the topological theory to make sense at all genus g we "change" the transformation properties of the fields, so that the supersymmetry transformation becomes a scalar operator on the world sheet. This modification is implemented by twisting the world sheet Lorentz group either by the vector $U(1)_V = U(1)_L + U(1)_R$ or the axial $U(1)_A = U(1)_L - U(1)_R$ symmetry. To do this we first gauge the R-symmetries. Then we combine the $U(1)$ gauge connection with the spin connection to a twisted world sheet spin connection. Contrary to the $U(1)_V$ the $U(1)_A$ current develops an quantum anomaly proportional to $\int_\Sigma x^*(c_1(TM))$. Therefore the B model, which is obtained by twisting with the $U(1)_A$ connection, is only well defined on Calabi-Yau manifolds ($c_1(TM) = 0$), while the A model, which is obtained by twisting with the $U(1)_V$ connection can be considered on any Kähler manifold.

4.3 The $(2,2)$ non-linear σ-model

Let us now study this mechanism in the Kähler case, which has at the classical level a $N = (2,2)$ supersymmetry and hence the necessary $U(1)$ symmetries. The action is given by

$$S = 2t \int_\Sigma \mathrm{d}^2 z \left(-g_{i\bar{j}} \partial_\mu x^i \partial^\mu x^{\bar{j}} + i g_{\bar{i}\,i} \psi_-^{\bar{i}} D_z \psi_-^i + i g_{\bar{i}\,i} \psi_+^{\bar{i}} D_{\bar{z}} \psi_+^i + R_{i\bar{i}j\bar{j}} \psi_+^i \psi_+^{\bar{i}} \psi_-^j \psi_-^{\bar{j}} \right) . \tag{18}$$

Here we have split the index I into i and $\bar{i}$ according to the Kähler decomposition of the CY metric. Such a metric can locally be written as $g_{i\bar{j}} = \partial_i \partial_{\bar{j}} K(x^i, x^{\bar{i}})$ and its Levi-Civita connection in Kähler geometry is pure in the indices $\Gamma^i_{jk} = g^{i\bar{j}} \partial_j g_{k,\bar{j}}$ as discussed in more detail in Sec. 9.2. On a non-flat Riemann surface Σ one has the connection

$$\begin{aligned} D_{\bar{z}} \psi_+^i &= \partial_{\bar{z}} \psi_+^i + \tfrac{i}{2} \omega_{\bar{z}} \psi_+^i + \Gamma^i_{kl} \partial_{\bar{z}} x^k \psi_+^l \\ D_z \psi_-^i &= \partial_z \psi_-^i - \tfrac{i}{2} \omega_z \psi_+^i + \Gamma^i_{kl} \partial_z x^k \psi_-^l \ , \end{aligned} \tag{19}$$

[3] Unfortunately there many notations common to distinguish the left- and right moving sectors in this context unbarred/barred for euclidean worldsheets, R/L, $+/-$ and without tilde/with tilde are maybe most often used.

where ω_z and $\omega_{\bar{z}}$ are the components of the spin connection of Σ.

In superfield formalism one can can write $L = 2t\int \mathrm{d}\theta^4 K(\mathbf{X}^i, \bar{\mathbf{X}}^{\bar{\imath}})$, where the chiral field $\mathbf{X}^i$ has components $x^i, \psi^i_\pm, F^i$. F^i is an auxiliary field that has has no kinetic terms and can be eliminated from the action by its equation of motion $F = \Gamma^i_{ij}\psi^j_+\psi^k_-$. This offshell superfield formalism is particularly useful when one couples a holomorphic superpotential $W(x^i)$ to the action, which only possible for non-compact target spaces M. This formalism is worked out in detail including the off-shell supersymmetry transformations in [146] and reviewed in [105]. For notational brevity we restrict ourselves to the onshell formalism.

Classically there are now twice as many super symmetries, one set for the holomorphic and one set for the antiholomorphic space time indices. They are generated by $\epsilon_+ \in \Gamma(K^{\frac{1}{2}})$, $\epsilon_- \in \Gamma(\bar{K}^{\frac{1}{2}})$ and $\bar{\epsilon}_\pm$. The latter are sections of the same bundles but have opposite charges under $U(1)_A$ and $U(1)_V$

$$\begin{aligned}
\delta x^i &= -\epsilon_-\psi^i_+ + \epsilon_+\psi^i_- \\
\delta x^{\bar{i}} &= \bar{\epsilon}_-\psi^{\bar{i}}_+ - \bar{\epsilon}_+\psi^{\bar{i}}_- \\
\delta\psi^i_+ &= 2i\bar{\epsilon}_-\partial_+ x^i + \epsilon_+\psi^j_+\Gamma^i_{jm}\psi^m_- \\
\delta\psi^{\bar{i}}_+ &= -2i\epsilon_-\partial_+ x^{\bar{i}} + \bar{\epsilon}_+\psi^{\bar{j}}_-\Gamma^{\bar{i}}_{\bar{j}\bar{m}}\psi^{\bar{m}}_+ \\
\delta\psi^i_- &= -2i\bar{\epsilon}_+\partial_- x^i + \epsilon_-\psi^j_+\Gamma^i_{jm}\psi^m_- \\
\delta\psi^{\bar{i}}_- &= 2i\epsilon_+\partial_- x^{\bar{i}} + \bar{\epsilon}_-\psi^{\bar{j}}_-\Gamma^{\bar{i}}_{\bar{j}\bar{m}}\psi^{\bar{m}}_+ \,.
\end{aligned} \tag{20}$$

The relation between the existence of two supersymmetries and the decomposition of the exterior derivative on Kähler manifolds into a holomorphic and antiholomorphic derivative $\mathrm{d} = \bar{\partial} + \partial$, which gives rise to the Hodge decomposition of cohomology groups into $H^{p,q}(M)$, has been discussed first by [216]. The fields x^i, $x^{\bar{\imath}}$, $\psi^i_\pm$ and $\psi^{\bar{i}}_\pm$ transform as before under WS transformations. W.r.t. the spacetime transformations one has now simply a splitting of $TM_{\mathbf{C}}$ into $T^{1,0}M \oplus T^{0,1}M$ with i referring to $T^{1,0}M$ and $\bar{\imath}$ referring to $T^{0,1}M$, so e.g. $\psi^i_+ \in \Gamma(\bar{K}^{\frac{1}{2}} \otimes x^*(T^{1,0}M))$ e.t.c. All transformation properties are summarized in table 1.

The action of the $U(1)_V$ and $U(1)_A$ are conveniently formulated in superfield formalism, i.e. expand any field in Grassmann valued $\theta^+, \theta^-, \bar{\theta}^+, \bar{\theta}^-$ complex fermionic spinor coordinates on which complex conjugation is given by $(\theta^\pm)^* = \bar{\theta}^\pm$. The WS Lorentz transformation acts on $t = x^0$ and $s = x^1$ (with $(1,1)$ signature) and on spinors as

$$\begin{aligned}
\begin{pmatrix} x^0 \\ x^1 \end{pmatrix} &\to \begin{pmatrix} \cosh\gamma & \sinh\gamma \\ \sinh\gamma & \cosh\gamma \end{pmatrix}\begin{pmatrix} x^0 \\ x^1 \end{pmatrix} \\
\theta^\pm &\to e^{\pm\frac{\gamma}{2}}\theta^\pm \\
\bar{\theta}^\pm &\to e^{\pm\frac{\gamma}{2}}\bar{\theta}^\pm
\end{aligned} \tag{21}$$

Since the fermionic variables anticommute w.r.t. to each other the Taylor expansion in them contains only 2^4 terms

$$\Phi(x,\theta^\pm,\bar{\theta}^\pm) = x(t,s) + \theta^+\psi_+(t,s) + \theta^-\psi_-(t,s) + \bar{\theta}^+\bar{\psi}_+(t,s) + \bar{\theta}^-\bar{\psi}_-(t,s) + \theta^+\theta^- A_{+-}s,t + \dots \tag{22}$$

In this sense one can think superspace as a thin space in the fermionic directions, which contains no second order derivative information in a given fermionic direction. The relation to calculus with differential forms is very obvious. The action of the vector $U(1)_V$ and axial $U(1)_A$ symmetries on all component fields is induced from

$$\begin{aligned}
e^{i\alpha F_V} &: \quad \Phi(x,\theta^\pm,\bar{\theta}^\pm) \mapsto e^{i\alpha q_V}\,\Phi(x, e^{-i\alpha}\theta^\pm, e^{i\alpha}\bar{\theta}^\pm) \\
e^{i\beta F_A} &: \quad \Phi(x,\theta^\pm,\bar{\theta}^\pm) \mapsto e^{i\beta q_A}\,\Phi(x, e^{\mp i\beta}\theta^\pm, e^{\pm i\beta}\bar{\theta}^\pm) \,.
\end{aligned} \tag{23}$$

Let us denote now the four supersymmetry operators corresponding to $\epsilon^{\pm}$ and $\bar{\epsilon}^{\pm}$ transformations $Q_{\mp}$ and $\bar{Q}_{\mp}$ respectively. A general supersymmetry transformation is then generated by the operator

$$\hat{\delta} = i\epsilon_+ Q_- - i\epsilon_- Q_+ - i\bar{\epsilon}_- \bar{Q}_- + i\bar{\epsilon}_+ \bar{Q}_+ \,, \tag{24}$$

where $(Q^{\pm})^{\dagger} = \bar{Q}_{\pm}$ and $\hat{\delta}^{\dagger} = -\hat{\delta}$.

More generally for any infinitesimal field transformation $\delta_Q \phi$ we will denote the infinitesimal transformation on the field operator $\delta\mathcal{O}_\phi$ by $\delta_Q \mathcal{O}_\phi = [Q, \mathcal{O}_\phi]_{\pm}$, where Q is the corresponding generating operator. Let M be the generator of two dimensional Lorentz rotations $SO(1,1)$. It is convenient to make the Wick rotation $x^0 = -ix^2$ and we call $M_E = iM$ the generator of the compact Euclidean rotation group $U(1)_E$. Beside the supersymmetry generators one has on the WS H the generator of (euclidean) time translations, P generator of translations. Furthermore there are the R-charge operators associated to the $U(1)_V$ and $U(1)_A$ currents called F_V and F_A. These generators fulfill the algebra

$$\begin{aligned}
Q_+^2 = \quad & Q_-^2 = \bar{Q}_+^2 = \bar{Q}_-^2 = 0, \\
\{Q_{\pm}, \bar{Q}_{\pm}\} \quad & = H \pm P, \qquad \{\bar{Q}_+, \bar{Q}_-\} = \{Q_+, Q_-\} = \{Q_-, \bar{Q}_+\} = \{Q_+, \bar{Q}_-\} = 0, \\
[M_E, Q_{\mp}] \quad & = \mp Q_{\pm}, \qquad [M_E, \bar{Q}_{\pm}] = \mp \bar{Q}_{\pm}, \\
[F_V, Q_{\pm}] \quad & = -Q_{\pm}, \qquad [F_V, \bar{Q}_{\pm}] = \bar{Q}_{\pm}, \\
[F_A, Q_{\pm}] \quad & = \mp Q_{\pm}, \quad [F_V, \bar{Q}_{\pm}] = \pm \bar{Q}_{\pm},
\end{aligned} \tag{25}$$

It becomes soon important that $Q_{\pm}$ and $\bar{Q}_{\pm}$ have opposite charges under the R symmetry groups. As already stated F_A is present at the quantum only for Calabi-Yau manifolds, the conformal case, while F_V is generically present. See [146] for a further discussion of this algebra.

5 Twisting the $N = (2,2)$ theories and cohomological field theories

Twisting amounts to a modification the Euclidean rotation group $U(1)_E$ by a generator of the global $U(1)$ R-symmetry groups and define the new generator of the Euclidean rotation group $U(1)_{E'}$ as $M'_E = M_E + R$. As explained our goal is to make some of fermionic Q operators scalar w.r.t. M'_E, so that they are well defined on all genus world-sheets. These "scalar" operators can then be used to define a cohomological theory on an arbitrary Riemann surface. The term twisting is familiar in the orbifold context, where it means to modify the boundary conditions of a field along cycles of the worldsheet by an element g of a global symmetry group G, e.g. for the torus with a A cycle of length 2π a field is periodically identified by $\phi(x + 2\pi) = g\phi(x)$. The analogy is appropriate since also in the above case we change the boundary conditions of some fermionic fields to become periodic. We encountered such twisting already in the discussion of Witten index and the elliptic genus.

Here the twisting is implemented by *gauging* the $U(1)$-R symmetry group and adding the corresponding gauge connection A^R_μ to the spin connection, so that the transformation property of the spinor fields depend now on their R charge. In important consequence of gauging the $U(1)$-R symmetry is that the gauge field modifies the energy momentum tensor, see (29). Since we are dealing with a 2d quantum field theory this program of gauging the R symmetry might be obstructed by anomalies. The potentially dangerous terms in the action are the fermion kinectic terms $ig_{\bar{i}\,i}\psi^{\bar{i}}_- D_z \psi^i_- + ig_{\bar{i}\,i}\psi^{\bar{i}}_+ D_{\bar{z}} \psi^i_+$ in (18). As explained in Sec. 9.4 (and is wellknown from the standard model) the vector $U(1)_V$ will never be anomalous. The anomaly density for the axial current is calculated also in Sec. 9.4 and from (19) we see that we have a Dirac operator on Σ_g coupled to a connection of a bundle, which is the pullback by x of the holomorphic tangent bundle to Σ written as $x^*(T^{0,1}M)$. The Atiyah-Singer index theorem (380) for the *twisted* spin complex gives us then the anwer that the axial $U(1)_A$ current violation is

$$\int_\Sigma \partial_\mu j^\mu_A = 2\int_\Sigma c_1(x^*(T^{1,0}M)) = 2\int_\Sigma x^*(c_1(T^{1,0}M)) = 2[C]\cdot c_1(TM)\,. \tag{26}$$

	Section before wisting	Section $(+)$ twist	Section $(-)$ twist
x	$x^*(TM)$	$x^*(TM)$	$x^*(TM)$
ψ^i_-	$x^*(T^{1,0})\otimes K^{\frac{1}{2}}$	$x^*(T^{1,0})$	$x^*(T^{1,0})\otimes K$
$\bar\psi^{\bar\imath}_-$	$x^*(T^{0,1})\otimes K^{\frac{1}{2}}$	$x^*(T^{0,1})\otimes K$	$x^*(T^{0,1})$
ψ^i_+	$x^*(T^{1,0})\otimes \bar K^{\frac{1}{2}}$	$x^*(T^{1,0})$	$x^*(T^{1,0})\otimes \bar K$
$\bar\psi^{\bar\imath}_+$	$x^*(T^{0,1})\otimes \bar K^{\frac{1}{2}}$	$x^*(T^{0,1})\otimes \bar K$	$x^*(T^{0,1})$

Table 1 Space time transformation of the non linear σ-model fi elds after $+$ and $-$ twist. Classically and in non-anomalous theories one can chose the twisting on the left movers $\psi^i_-, \psi^{\bar\imath}_-$ and the right movers $\psi^i_+, \psi^{\bar\imath}_+$ independently.

	Before Twisting				A twist $(-,+)$			B twist $(+,+)$		
	$U(1)_V$	$U(1)_A$	$U(1)_E$	$spin$		$U(1)'_E$	$spin$		$U(1)'_E$	$spin$
x	0	0	0	$\mathbf{1}_C$	x	0	$\mathbf{1}_C$	x	0	$\mathbf{1}_C$
ψ^i_-	-1	1	1	$K^{\frac{1}{2}}$	χ^i	0	$\mathbf{1}_C$	ρ^i_z	2	K
$\bar\psi^{\bar\imath}_+$	1	1	-1	$\bar K^{\frac{1}{2}}$	$\chi^{\bar\imath}$	0	$\mathbf{1}_C$	$-\frac{1}{2}(\theta^{\bar\imath}+\eta^{\bar\imath})$	0	$\mathbf{1}_C$
$\bar\psi^{\bar\imath}_-$	1	-1	1	$K^{\frac{1}{2}}$	$\rho^{\bar\imath}_z$	2	K	$\frac{1}{2}(\theta^{\bar\imath}-\eta^{\bar\imath})$	0	$\mathbf{1}_C$
ψ^i_+	-1	-1	-1	$\bar K^{\frac{1}{2}}$	$\rho^i_{\bar z}$	-2	$\bar K$	$\rho^i_{\bar z}$	-2	$\bar K$

Table 2 Space time transformation of the non linear σ-model fi elds and charges after A and B twist. We also indicate the names of the fi elds in the A and B model.

This breaks the $U(1)_A$ symmetry generically to a Z_2. For a discussion of the $U(1)_A$ anomaly in the linear σ-model context see [213].

The most important consequence of the above result is that on a Calabi-Yau manifold where $c_1(TM)=0$ we can twist by the $U(1)_A$ and the $U(1)_V$ symmetry as both are *anomaly free*. In the $(2,2)$ theory we have therefore two fundamentally different possibilities to twist

$$\begin{array}{lll} \mathrm{A-Twist:} & M_{E'} & = M_E + F_V \\ \mathrm{B-Twist:} & M_{E'} & = M_E + F_A\,. \end{array} \tag{27}$$

The tables below record how the twisting changes the WS transformation properties of the fields. We do this first for the the so $+$ and the $-$ twist first. In the above notation of table 1 the A twist corresponds to a $(-,+)$ twist, i.e. to a combination of the $(-)$ twist on $\psi_-,\bar\psi_-$ and the $(+)$-twist on $\psi_+,\bar\psi_+$, while the B twist is $(+,+)$ twist, i.e. a combination of the $(+)$ twist on $\psi_-,\bar\psi_-$ and the $(+)$-twist on $\psi_+,\bar\psi_+$. There are the possibilities of an $(+,-)$ twist and an $(-,-)$ twist making $\bar Q_A$ and $\bar Q_B$ nilpotent operators. They lead to the definition of conjugated cohomological sectors and considered for them self to no new theories. However as explained in Sec. 5.5 the combined geometry of the sectors conjugated to each other leads to an interesting geometry, the so called tt^* geometry.

The effects on the fields and the supersymmetry transformation can be summarized in the tables 2 and 3 respectively.

As it is clear from the table 3 and (25) the following combinations

$$\begin{aligned} Q_A &= Q_- + \bar Q_+ \\ Q_B &= \bar Q_- + \bar Q_+ \end{aligned} \tag{28}$$

are now scalar, nilpotent operators which can be used to define two different cohomological theories, the topological A- and the topological B-model respectively. Mirror symmetry exchanges the $-$ twist with

	Before Twisting				A − twist		B − twist	
	$U(1)_V$	$U(1)_A$	$U(1)_E$	$spin$	$U(1)'_E$	$spin$	$U(1)'_E$	$spin$
Q_-	-1	1	1	$K^{\frac{1}{2}}$	0	$\mathbf{1}_C$	2	K
$\bar{Q}_+$	1	1	-1	$\bar{K}^{\frac{1}{2}}$	0	$\mathbf{1}_C$	0	$\mathbf{1}_C$
$\bar{Q}_-$	1	-1	1	$K^{\frac{1}{2}}$	2	K	0	$\mathbf{1}_C$
Q_+	-1	-1	-1	$\bar{K}^{\frac{1}{2}}$	-2	$\bar{K}$	-2	$\bar{K}$

Table 3 Space time transformation of the supersummetry generators after the A and B twist

the $+$ twist on the $\psi_-, \bar{\psi}_-$ side. Even before twisting Q_A and Q_B define cohomological theories on the plane the torus, where covariantly constant spinors exist. One can also choose to twist only the say $\psi_-, \bar{\psi}_-$ side. The indices of so called half-twisted models are the closest analogs of the elliptic genus (17) at higher genus [207][212]. This indices are shared between the A and the B model and contain information about the couplings of $\mathbf{27}\overline{\mathbf{27}}$ in the heterotic string with standard embedding.

If we denote the gauge current, which corresponds to the gauge variations δA^R_μ by J^R_μ. It will modify the energy momentum tensor to

$$\hat{T}_{\mu\nu} = T_{\mu\nu} + \frac{1}{4}\left(\epsilon^\lambda_\mu \partial_\lambda J^R_\nu + \epsilon^\lambda_\nu \partial_\lambda J^R_\mu\right) \ . \tag{29}$$

In the action of the gauged theory of covaraint theory the world sheet there is a coupling

$$\Delta S = \int_\Sigma J^\mu \omega_\mu = \frac{1}{2}\int_\Sigma J\bar{\omega} + \bar{J}\omega = \frac{1}{2}\int_\Sigma R\phi + \text{total der.}\ , \tag{30}$$

to the spin connection ω. In the second equality we bosonized the $U(1)_R$ current $\partial\phi = J$ and integrated partially. Contact terms of operators with the this expression will play a rôle in determining properties of the correlation functions.

5.1 Generalities on the physical observables

One calls an operator *a chiral* operator or (c, c) operator ϕ if

$$[Q_B, \phi] = 0 \ . \tag{31}$$

Chiral and twisted chiral superfields play an important rôle in formulating the general $(2, 2)$ worldsheet theory, see [213]. The lowest component ϕ of chiral superfield Φ obeys $[\bar{Q}_\pm, \phi] = 0$ and is a hence a chiral operator. An operator ϕ is called *twisted chiral* or (a, c) if

$$[Q_A, \phi] = 0 \ . \tag{32}$$

The lowest component v of a twisted chiral superfield Σ obeys $[\bar{Q}_+, v] = [Q_-, v] = 0$ and is hence a twisted chiral operator. $[\bar{Q}_-, \phi_-] = 0$ and $[Q_-, \phi_-] = 0$ define left chiral- and antichiral operators while $[\bar{Q}_+, \phi_+] = 0$ and $[Q_+, \phi_+] = 0$ define right chiral- and antichiral operators.

The key concept is now to define a cohomological theory whose observables are the equivalence classes $[\phi]$ of Q *closed* operators. To be closed the operators have to fulfill $[Q, \phi] = 0$ and the equivalence relation is as usually up to *exact* operators $\mathcal{E} = [Q, \Lambda]_\pm$, i.e.

$$\phi \sim \phi + [Q, \Lambda]_\pm \ . \tag{33}$$

If the vacuum is annihilated by Q, which is the case if Q comes from a unbroken symmetry as above, then the correlation function of the Q closed operators does not depend on the representative of the class

$$\begin{aligned} \langle \phi_1 \ldots (\phi_k + \{Q, \Lambda\}) \ldots \phi_n \rangle = & \langle \phi_1 \ldots \phi_n \rangle \pm \langle 0 | \phi_1, \ldots \phi_{k-1} \Lambda \phi_{k+1} \ldots \phi_n Q | 0 \rangle \\ & \pm \langle 0 | Q \phi_1, \ldots \phi_{k-1} \Lambda \phi_{k+1} \ldots \phi_n | 0 \rangle \\ = & \langle \phi_1 \ldots \phi_n \rangle \end{aligned} \tag{34}$$

Above the $\pm$ signs are uncorrelated and the two terms vanish independently if the vacuum is Q invariant. The analogy of the definition of topological correlators with cohomological intersections $\int_M \omega_1 \wedge \ldots \wedge (\omega_k + \mathrm{d}\lambda) \wedge \ldots \wedge \omega_n = \int_M \omega_1 \wedge \ldots \wedge \omega_k \wedge \ldots \wedge \omega_n$ is not just formal in the case of the $(2,2)$-sigma model as we will see.

An important property of these operators is that they form position independent rings. Using the algebra (25), the properties of the twisted chiral operators and $[\{A,B\},C] = \{[A,C],B\} + \{A,[B,C]\}$ it is easy to see that e.g.

$$\begin{aligned} \tfrac{i}{2}\left(\tfrac{\partial}{\partial x^0} + \tfrac{\partial}{\partial x^1}\right)\phi &= [(H+P),\phi] = [\{Q_+,\bar{Q}_+\},\phi] = \ldots = \{Q_B,[Q_+,\phi]\} \\ \tfrac{i}{2}\left(\tfrac{\partial}{\partial x^0} - \tfrac{\partial}{\partial x^1}\right)\phi &= [(H-P),\phi] = [\{Q_-,\bar{Q}_-\},\phi] = \ldots = \{Q_B,[Q_-,\phi]\} \end{aligned} \tag{35}$$

and similar for the A model. Combining (34) and (35) one sees that the correlation functions of the twisted chiral operators do not depend on the position of the insertions of the operators, which is also true for the chiral operators. The ring structure comes from the operator product expansion. It is obvious respects the symmetry that the OPE of two (twisted) chiral fields is (twisted) chiral again and by (35) position independent. One defines the structure constants of the ring in a basis of the ring ϕ_k as

$$\phi_i \phi_j = C_{ij}^k \phi_k + [Q,\Lambda]_\pm \,, \tag{36}$$

i.e. identifying an element on the right hand side up to exacts term. The ring satisfies the usual associativity $C_{jl}^m C_{ik}^l = C_{lk}^m C_{ij}^l$. The unit $\phi_0 = 1$ is always (twisted) chiral, so $C_{0j}^k = C_{j0}^k = \delta_j^k$.

The position independence (35) and its realization on p-form operators can be formulated in a covariant way as the so called *descend equations*, see [56] for a review. If $\mathcal{O}^{(0)} = \phi$ is a Q closed position independent 0-form operator, one can define the following non-local n-form operators

$$\begin{aligned} 0 &= [Q,\mathcal{O}^{(0)}] \\ \mathrm{d}\mathcal{O}^{(0)} &= \{Q,\mathcal{O}^{(1)}\} \\ \mathrm{d}\mathcal{O}^{(1)} &= [Q,\mathcal{O}^{(2)}] \\ \mathrm{d}\mathcal{O}^{(2)} &= 0\,. \end{aligned} \tag{37}$$

Using (35) and the corresponding relation for the A-model one can find the descend operators explicitly noting that $Q_-\mathrm{d}z$ ($\bar{Q}_-\mathrm{d}z$) and $Q_+\mathrm{d}\bar{z}$ ($\bar{Q}_+\mathrm{d}\bar{z}$) are covariant combinations

$$\begin{aligned} &\mathrm{A-mod.} \quad \mathcal{O}_A^{(1)} = i\mathrm{d}z[\bar{Q}_-,\mathcal{O}_A^{(0)}] - i\mathrm{d}\bar{z}[Q_+,\mathcal{O}_A^{(0)}], \quad \mathcal{O}_A^{(2)} = \mathrm{d}z\mathrm{d}\bar{z}\{Q_+,[\bar{Q}_-,\mathcal{O}_A^{(0)}]\}, \\ &\mathrm{B-mod.} \quad \mathcal{O}_B^{(1)} = i\mathrm{d}z[Q_-,\mathcal{O}^{(0)}] - i\mathrm{d}\bar{z}[Q_+,\mathcal{O}_B^{(0)}], \quad \mathcal{O}_B^{(2)} = \mathrm{d}z\mathrm{d}\bar{z}\{Q_+,[Q_-,\mathcal{O}_B^{(0)}]\}\,. \end{aligned} \tag{38}$$

The descend equations truncate, because of the anti symmetrization in the world-sheet indices. The $\bar{Q}_B$ and $\bar{Q}_A$ operators define the (a,a) and (c,a) ring states which we call $\bar{\mathcal{O}}_B^{(0)}$ and $\bar{\mathcal{O}}_A^{(0)}$ respectively. Their descendants $\bar{\mathcal{O}}_B^{(1,2)}$ and $\bar{\mathcal{O}}_A^{(1,2)}$ are defined as in (38) with the barred and unbarred Q operators exchanged. As an easy exercise one checks that $\mathcal{O}_B^{(2)}$ ($\bar{\mathcal{O}}_B^{(2)}$) and $\mathcal{O}_A^{(2)}$ $\bar{\mathcal{O}}_A^{(2)}$ are $\bar{Q}_B$ (Q_B) and $\bar{Q}_A$ (Q_A) exact.

The significance of the descend p-form operators is that one can integrate them over closed p-cycles C_p of the WS (or more general the topological field theory space-time) to obtain *non-local operators* $\mathcal{O}(C_p) = \int_{C_p} \mathcal{O}^{(p)}$, which are automatically Q closed, because of Stokes theorem $[Q,\mathcal{O}(C_p)]_\pm = \int_{C_p}[Q,\mathcal{O}^{(p)}]_\pm = \int_{C_p} \mathrm{d}\mathcal{O}^{(p-1)} = \int_{\partial C_p} \mathcal{O}^{(p-1)} = 0$. Reversed use of Stokes theorem shows that the topological equivalence class of $\mathcal{O}(C_p)$ depends only the homology class of C_p. For a $p-1$ chain S with $C_p - C_p' = \partial S$ the difference $\mathcal{O}(C_p) - \mathcal{O}(C_p') = \int_{\partial S} \mathcal{O}^{(p)} = \int_S \mathrm{d}\mathcal{O}^{(p)} = [Q, \int_S \mathcal{O}^{(p+1)}]_\pm$ is Q exact.

5.2 A first look at the metric (in)dependence and topological string theory

In a topological theory the correlation functions are not only formally position independent, but decouple formally from variations of the worldsheet metric $h^{\mu\nu}$. Classically the energy momentum tensor $T_{\mu\nu} = \frac{1}{\sqrt{h}}\frac{\delta S}{\delta h^{\mu\nu}}$ is the generator of those variations. From the first order variation of the weight factor e^S one gets a dependence of a correlation function on metric variations $\delta h^{\mu\nu}$

$$\delta_h \langle \mathcal{O} \rangle_g = \langle \mathcal{O} \int_{\Sigma_g} \sqrt{h} \mathrm{d}^2\sigma \delta h^{\mu\nu} T_{\mu\nu} \rangle_g . \tag{39}$$

In a topological theory $\delta_h\langle\mathcal{O}\rangle_g = 0$ does not require that $T_{\mu\nu} = 0$, but in virtue of (34) that it is exact

$$T_{\mu\nu} = \{Q, G_{\mu\nu}\} . \tag{40}$$

This structure ensures general covariance or *topological invariance.* It plays a key role in covariant quantization of string theory, where $Q^2 = 0$ is the BRST operator and the part of $G_{\mu\nu}$ is played by the antighost field $b_{\mu\nu}$. It is also is the starting point of closed string field theory formulations [199]. One can have topological invariance independently of conformal invariance and also independently of the decoupling between ghost and matter sector [199]. For instance the A model relies on this structure and can be defined on Kähler manifolds on which the σ model is not conformally invariant.

In string theory we integrate the world-sheet metric h of Σ_g over all possible choices $\mathcal{H}_g$. Some review references for the following short account of the metric dependence are [**?**][74][55][173] from the physical and [119] from the mathematical perspective. Classically the integral over h is invariant under diffeomorphism and Weyl- and conformal transformations of the metric

$$\tilde{h}_{ab}(\tilde{\sigma}) = \exp[2\omega(\sigma)] \frac{\partial\sigma^c}{\partial\tilde{\sigma}_a}\frac{\partial\sigma^d}{\partial\tilde{\sigma}_b} h_{cd} . \tag{41}$$

These “gauge” invariances are present at quantum level in critical string theory, which requires an anomaly cancellation for the latter. The integral over the metric hence contains huge gauge orbits over the diffeomorphism- and the Weyl group, which we we divide from the path integral measure and consider

$$\mathcal{M}_g = \mathrm{LGT}\backslash\mathcal{H}_g/(\mathrm{diff}_0 \times \mathrm{Weyl})_g = \mathrm{LGT}\backslash\mathcal{T}_g . \tag{42}$$

Large gauge transformations (LGT) refer to discrete diffeomorphism of Σ_g not connected to the identity the so called *mapping class group* $\mathrm{LGT} = \frac{\mathrm{diff}}{\mathrm{diff}_0}$, which does not affect the dimension or other local properties of $\mathcal{M}_g$. Focussing on the latter means considering the Teichmüller space $\mathcal{T}_g = \mathcal{H}_g/(\mathrm{diff}_0 \times \mathrm{Weyl})$. Locally near a reference metric h^0_{ab} we can lineralize the problem and once this is done it is easy to see the key property that this moduli space is *finite dimensional.* Infinitesimal Weyl and diffeomorphism transformations are read of from (41)

$$\begin{aligned} \tilde{\delta} h_{ab} &= 2\delta\omega h_{ab} - \nabla_a\delta\sigma_b - \nabla_b\delta\sigma_a \\ &= (2\delta\omega - \nabla_c\delta\sigma^c)h_{ab} - 2(P_1\delta\sigma)_{ab} \end{aligned} \tag{43}$$

with $(P_1\delta\sigma)_{ab} = \frac{1}{2}(\nabla_a\delta\sigma_b + \nabla_b\delta\sigma_a - h_{ab}\nabla_c\delta\sigma^c)$. The scalar product for the linearized metric deformations $\delta^i h_{ab}$ near $h^{(0)}_{ab}$ is

$$G^{ij} = \langle \delta^i h_{ab} | \delta^j h_{ab} \rangle = \int_\Sigma \mathrm{d}^2\sigma\sqrt{h}\, \delta^i h_{ab}\, \delta^j h^{ab} , \tag{44}$$

where $\delta^i h^{ab} := h^{(0)ac}h^{(0)bd}\delta^i h_{cd}$ is compatible with the first order approximation. It has a straightforward generalization for other tensors on Σ transforming in $(\otimes^q_{i=1} T\Sigma) \otimes (\otimes^p_{i=1} T^*\Sigma)$ and allows us to define the adjoint of linear operators such as P_1, see Sec. 9.4. Locally $\mathcal{T}_g$ is parametrized by the linear changes δh_{ab} of the metric, which are orthogonal to $\tilde{\delta} h_{ab}$ of (43), i.e. $0 = \langle \delta h_{ab} | \tilde{\delta} h_{ab} \rangle = \langle \delta h_{ab} | (2\delta\omega - \nabla \cdot \delta\sigma) h_{ab} \rangle - 2\langle \delta h_{ab} | (P_1 \delta\sigma)_{ab} \rangle = \langle h^{ab} \delta h_{ab} | (2\delta\omega - \nabla \cdot \delta)\sigma \rangle - 2\langle (P_1^\dagger \delta h)_b | \delta\sigma_a \rangle$. Up to a small subtlety (dependence), which we discuss below, the free variaton of $\delta\sigma_a$ and $(2\delta\omega - \nabla \cdot \delta\sigma)$ span $T^*\Sigma$ and the space of functions on Σ so that the required orthogonality enforces the conditions

$$h^{ab} \delta h_{ab} = 0, \qquad (P_1^\dagger \delta h)_b = 0 \ . \tag{45}$$

The first is tracelessness of δh_{ab} and in a hermitian gauge choice $h^0_{z\bar{z}}$ we see in Sec. (**??**) that the second means holomorphicity of δh_{ab}. I.e. $\delta h_{zz}(z) = \phi(z)_{zz}$ are components of *holomorphic quadratic differentials*. Holomorphicity of a quadratic differentials in one complex dimension is equivalent to harmonicity and the spectrum of the Laplacian is finite on compact Σ, which establishes the *key property*.

It is easy to connect this to the discussion in Sec. 8.2. If we pick a metric $h^0_{z\bar{z}}$ we can define from ϕ^* the components of the so called Beltrami differentials $\mu_{\bar{z}}^{z} = h^{\bar{z}z} \phi^*_{\bar{z}\bar{z}}$. Holomorphicity of ϕ implies that $\mu_{\bar{z}}^{z} \mathrm{d}\bar{z} \frac{\partial}{\partial z} \in H^1(T\Sigma)$ is a harmonic representatives. Sec.8.2 uses Čech-cohomology to ignore trivial changes of the metric by complex reparametrizations, which relates by (334) to the gauge condition $(P_1^\dagger \delta h)_b = 0$. To summarize can span the tangent space $T\mathcal{M}$ of the complex moduli space by $\mu^{k\ z}_{\bar{z}}(z) \mathrm{d}\bar{z} \frac{\partial}{\partial z}$ and the cotantgent space $T^*\mathcal{M}$ by $\phi^{(k)}_{zz} \mathrm{d}z \mathrm{d}z$ with $k = 1, \ldots, h^1(T\Sigma)$. For the a hermitian choice $h_{z\bar{z}}$ of the metric the pairing (44) becomes a Kähler metric $G^{i\bar{\jmath}} = \int_\Sigma \mathrm{d}^2 z (h^{z\bar{z}})^2 \phi^i \phi^{*\ \bar{\jmath}}$ called the *Weil-Peterson metric*.

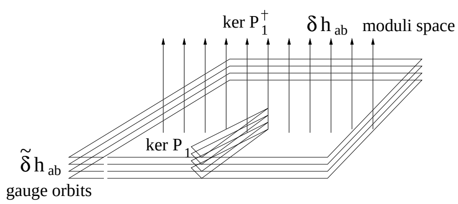

Fig. 3 Schematic of the objects in the linearisation of the metric variations

Let us come to the small subtlety mentioned obove. If $\delta\sigma^a$ in is the kernel of P_1, i.e. $(P_1 \delta\sigma)_{ab} = 0$ we may pick a $\delta\omega$ so that $\langle \delta h_{ab} | \tilde{\delta} h_{ab} \rangle = 0$, without restricting δh_{ab}. Such vector fields $\delta\sigma_a$ in the kernel of P_1 are elements of $H^0(T\Sigma)$, appropriatly called *conformal Killing* fields, as they don't change the conformal class of h_{ab}. So appart from restricting changes of the metric to complex structure changes only, which is the main effect of the divison by the gauge group, we have to subtract these null vectors because they appear in the numerator of (42). Hence the *expected dimension* of $\mathcal{M}_g$ is $h^1(T\Sigma) - h^0(T\Sigma)$, which we calculate in Sec. (9.3) by *Hirzebruch-Riemann-Roch* (365) to be $3g - 3$.

To avoid the peculiarities of $h^0(T\Sigma) \neq 0$ (3 and 1 for $g = 0$ and $g = 1$) consider $g > 1$ and let $z^a =: m^a$, $a = 1, \ldots, 3g-3$ the complex structure variables of Σ. We can describe then a first oder deformation of the metric modulo Weyl and diffeomorphisms as $\int_\Sigma \mathrm{d}^2\sigma \sqrt{h} \tilde{\delta} h^{ab} T_{ab} = \int_\Sigma \mathrm{d}^2 z \mu^{(a)\ z}_{\bar{z}} \delta m^a T_{zz} + \bar{\mu}^{a\ \bar{z}}_{z} \delta \bar{m}^a \bar{T}_{\bar{z}\bar{z}}$ and if we insert that in (39) we conclude that

$$\frac{\partial}{\partial m^a} \langle \mathcal{O} \rangle_g = \langle \mathcal{O} \int_\Sigma \mathrm{d}^2 z \mu^{a\ z}_{\bar{z}} T_{zz} \rangle_g =: \langle \mathcal{O} T^a \rangle_g \tag{46}$$

and similarly $\frac{\partial}{\partial \bar{m}^a} = \langle \mathcal{O} \bar{T}^a \rangle_g$. Eq. 40 is strictly true, so the argument that cohomological states and the vacuum are Q closed would make *topological string theory* completely metric independent and therefore *trivial*! However the argument involving the invariance of the vacuum fails, because the measure on the moduli space of higher genus Riemann surfaces, which is part of the vacuum definition is not Q closed. It is a real $6g - 6$ form μ_g for surfaces of $g > 1$ and the argument fails in a very specific way. If we act with Q on it, it gives an exact form, as we will see in detail in Sec. 8.13. This is like a descend equation, but with exterior derivative in the moduli space direction. By Stokes or rather Dolbeaults theorem the contribution to the integral can then only come from the *boundary of* $\mathcal{M}_g$, which represents degenerate Riemann surfaces. If the vacuum is not Q closed we cannot trust the argument about position independence either. In the moduli space $\mathcal{M}_{g,n}$ with insertion of n operators the codimension one locus, where two operators coincide is part of the boundary components. Its contributions has to be taken into account by so called *contact terms*. Most of what topological string theory is about is organizing the contributions of these boundaries. The questions which boundaries do give contributions leads to the stable compactifications on $\overline{\mathcal{M}}_{g,n}$ in which only the boundary components are included, which are in complex codimension one. These facts will govern the coupling of the A and the B-model to WS gravity as discussed in Sec. 6.2 and 8.13.

This section sketched the leap that one can take in topological string theory from a hopeless looking path integral to essentially a combinatorial problem. The linear approximations to the moduli space of Σ_g scratched the surface of this subject by one ϵ to be exact. We have not etablished global properties including existence. We will say more about that for Calabi-Yau manifold in Sec. 8.3 and leave the reader in the case of Riemann surfaces with the literature [119].

5.3 A first look at the deformation space

What is of importance is that integrals of the two form operators $\int_\Sigma \mathcal{O}_i^{(2)}$ defined in the Sec. 5.1 can be added to the topological action as deformations

$$S = \int_\Sigma \mathrm{d}z^2 \mathcal{L}_0 + \sum_{i=1}^r t^i \int_\Sigma \mathcal{O}_i^{(2)} \ . \tag{47}$$

After the A twist we can define zero form operators $\mathcal{O}^{(0)}_{w_{i\bar\jmath}} = w_{i\bar\jmath}\chi^i\chi^{\bar\jmath}$, which have $(U(1)_V, U(1)_A)$ charges $(0,2)$, see Tab. 2 . This charge is offset by $Q_+, \bar Q_-$ in (38), as seen from table (3) so that $\mathcal{O}^{(2)}_{w_{i\bar\jmath}}$ is neutral. As we shall see these operators are associated to to elements in $H^{1,1}(M)$ (108,109). Similarly the operators associated to elements in $A \in H^1(M,TM)$ (218) in the B-model $\mathcal{O}^{(0)}_A = w^i_{\bar\jmath}\eta^{\bar\jmath}\theta_i$ have $(U(1)_V, U(1)_A)$ charge $(2,0)$ which is offset by Q_+, Q_- so that $\mathcal{O}^{(2)}_A$ in (38) is neutral. Derivatives w.r.t. to t^i bring down such operators in the correlation functions and neutrality implies that arbitrary derivatives do no violate any selection rule. Generically this extends the theory to a family of theories. In the above discussion we omitted the consideration of $w_{ij}\chi^i\chi^j \leftrightarrow H^{2,0}(M)$ in the A-model and bi-vectors $w^{ij}\theta_i\theta_j \leftrightarrow H^0(M,\Lambda^2 TM)$ as these cohomology groups are trivial on manifolds with strict $SU(3)$ holonomy[4]. Perturbations w.r.t. the full set of operators have been considered in [207][16].

It is interesting to recover this first order condition of the CFT from the spacetime point of view, see [34, 33], where we use the linearization approach from the last section now for the space time moduli. We know that the geometrical background has to be Calabi-Yau manifold to allow for a conformal field theory [5]. The exactly marginal deformations $\mathcal{O}^{(1,1)}$ must correspond hence to first order deformations of the geometry, which preserve the Calabi-Yau condition. I.e. to deformations of the background metric $g_{\mu\nu} + \delta g_{\mu,\nu}$ (and B-field $b_{\mu\nu} + \delta b_{\mu\nu}$), which do not change the Calabi-Yau condition[6] $R_{\mu\nu}(g) = 0$, i.e.

$$R_{\mu\nu}(g + \delta g) = 0 \ . \tag{48}$$

In analyzing this equation we have to eliminate the δg, which come from coordinate transformations. Coordinate transformations or equivalently diffeomorphism of M are generated by vectors fields V^μ, compare

[4] A slight modification of the twisting procedure makes the descend operators to these fields neutral [126]

[5] There is an interesting extension of these considerations for non-conformal $N = (2,2)$ σ-models involving massive (non-marginal) deformations.

[6] Strictly speaking one should ask for perturbations, which leave the Ricci-form $\mathcal{R}$ in the $c_1(M) = 0$ cohomology class. Though the representatives of the deformations in the cohomology classes would be different, the counting would be the same, see Sec. 9.8.

Sec. 8.2. An actual change of the metric $\delta g_{\mu\nu}$ is orthogonal to diffeomorphism generated by the vector field in the following sense $\int \sqrt{g}\delta g^{\mu\nu}(\nabla_\mu V_\nu + \nabla_\nu V_\mu)\mathrm{d}^m x = 0$, which is equivalent to the gauge condition $\nabla^\mu \delta g_{\mu\nu} = 0$, compare (44) and (45). Expanding with this constraint (48) to linear order around $R(g) = 0$ one gets

$$\nabla^\rho \nabla_\rho \delta g_{\mu\nu} - 2R_\mu{}^\kappa{}_\nu{}^\sigma \delta g_{\kappa\sigma} = 0 \tag{49}$$

Using the splitting of a Kähler metric in holomorphic and holomorphic indices one can analyze $\delta g_{i\bar{\jmath}}$, and δg_{ij} separately. Note that $\delta g_{i\bar{\jmath}}$ is real, while δg_{ij} with $\overline{\delta g_{ij}} = \delta g_{\bar{\imath}\bar{\jmath}}$ is complex. From (352) it follows that $\delta g_{i\bar{\jmath}}$ is Δ_d harmonic and $\delta g^i = \delta g^i_{\bar{\jmath}} \mathrm{d}z^{\bar{\jmath}} = g^{i\bar{k}} \delta g_{\bar{k}\bar{\jmath}} \mathrm{d}z^{\bar{\jmath}}$ is $\Delta_{\bar{\partial}}$ harmonic. In other words the first order deformations factorize and correspond to elements in $H^{1,1}(M)$ and $H^1(M, TM)$ respectively. These are also among the deformations of the A- and B-model as mentioned above and further discussed in the following Sec. 6.1 and 8.1.

Let us first discuss the two moduli space associated to $H^{1,1}(M)$. In a basis of $(1,1)$-forms $w^{(k)}_{(1,1)}$, we expand a Kähler form

$$\omega = \sum_{k=1}^{h^{11}} t_k \omega^{(k)}_{(1,1)} \tag{50}$$

in terms of the real Kähler parameters $t_k > 0$. The range of t_k is bounded by the inequalities, which ensure positivity of the volumes of curves C, divisors D and M, i.e.

$$\int_C \omega > 0, \quad \int_D \omega \wedge \omega > 0, \qquad \int_M \omega \wedge \omega \wedge \omega > 0\,. \tag{51}$$

These conditions describe a real cone in $\mathbf{R}_+^{h^{1,1}}$, which is called the *K¨ahler cone*. The parameters t_k are identified with the areas of dual curves C_k to $w^{(k)}_{(1,1)}$, which shrink to zero area at the boundaries of the Kähler cones[7]. In the σ-model (106) it is natural to complexify the parameter t_k to $t_k^\sigma = \int_{C_k}(\omega - iB)$ by adding the integral of the antisymmetric tensor field $B \in H^{1,1}(M)$ to t_k. Moreover due to mirror symmetry one has a natural choice of the complex parametrization of the complexified Kähler moduli space $\mathcal{M}_K$, simply the complex structure parameters of the mirror t_k^m[8]

As it is clear from the fact that the deformations $\delta g_{ij}, \delta g_{\bar{\imath}\bar{\jmath}}$ change the $(i,\bar{\imath})$ type of the metric, the moduli space $H^1(M, TM)$ is associated to complex structure deformations. It is fair to say that most of what we know about the moduli space of $(2,2)$ theories comes from the theory of complex structure deformations. In particular it can be shown that the first order deformations of the complex structures elevate to finite deformations. This more thoroughly discussed in the Sec. 8.2 and 8.3.

Let us conclude the description of emerging picture of the deformation spaces. We have found that the $U(1)_{A/V}$ neutral world sheet two form operators $\mathcal{O}^{(2)}_{W_{(1,1)}}$ with $W_{1,1} \in H^{1,1}(M, Z)$ and $\mathcal{O}^{(2)}_A$ with $A \in H^1(M, TM)$ correspond geometrically to complexified Kähler and complex structure deformations of the Calabi-Yau metric and are expected to be exactly marginal from the CFT point of view. In the low energy effective action of type II A/B string theory these marginal deformations arise as vacuum expectation of complex scalar fields labeling the vacuum manifold of the N=2 supergravity in 4d. The general structure of this vacuum manifold for abelian gauge groups $U(1)^{\#V}$ and $U(1)^{\#H}$ is that it is *locally* of the form $\mathcal{M}_{2\#V} \times \mathcal{Q}_{4\#H}$, where $\mathcal{M}$ is a complex special Kähler manifold for the scalar fields in the vector multiplets[51][52][49][73] and $\mathcal{Q}$ is a quaternionic manifold [41] for the scalar fields in the hypermultiplets. The subscripts indicate the real dimension of the moduli space. Its relation to the

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[7] At the boundary of the K¨ahler also a divisor may collapse. In this case t_k is still the area of a curve C_k in D.

[8] As a corollary all singularities of $\mathcal{M}_K$ occur at complex codimension one and the cone structure disappears completely.

perturbative sector of the II A/B string compactifications on a Calabi-Yau 3 fold M is as follows

$$\mathcal{M}^{IIA}_{tot}(M) = \mathcal{M}^{IIA}_{2h^{1,1}(M)} \times \mathcal{Q}^{IIA}_{4(h^{2,1}(M)+1)} \qquad \mathcal{M}^{IIB}_{tot}(W) = \mathcal{M}^{IIB}_{2h^{2,1}(W)} \times \mathcal{Q}^{IIB}_{4(h^{1,1}(W)+1)} \,. \quad (52)$$

One very far reaching definition of the mirror conjecture is that type IIA and type IIB compatifications are completely identically if M and W are mirror pairs. This in particular implies $\mathcal{M}^{IIA}_{tot}(M) = \mathcal{M}^{IIA}_{tot}(W)$. The best studied object is $\mathcal{M}^{IIB}_{2h^{2,1}(W)}$ since it is literally the complex moduli space of W. The enhancement of the Calabi-Yau metric moduli space from the complex to the quaternionic space $\mathcal{Q}$ of Kähler multiplets is due to the moduli of Ramond forms. The additional quaternionic dimension in $\mathcal{Q}$ comes from the universal dilation, whose scalar components (S, C) contain in particular the type II dilation S.

5.4 Conformal Field Theory point of view

A most remarkable fact is that for all 145 Calabi-Yau threefolds defined in weighted projective space subject to the constraint (403) and for which the defining polynomial is of Fermat type

$$P = \sum_{i=1}^{5} a_i x^{m_i} \quad (53)$$

with $m_i w_i = d$, $\forall i$ and $\sum_{i=1}^5 w_i = d$ there is a well founded conjecture for an exact conformal field theory description, which captures the full perturbative sector and not just the topological part of it. The CFT description is based on an orbifold of tensor products of minimal $N = 2$ super conformal field theories found by Gepner [84]. The description is valid only at one point in complex structure and complexified Kähler structure moduli space the so called *Gepner point.* In the complex moduli space the constraint (53) literally describes this special point. In the complexified Kähler moduli the point can also be described by (53) after dividing by phase symmetry groups such as (265,274), which identifies (53) with the mirror manifold. It is far away from the large volume limit.

The purpose of the present section is to describe the topological sub sectors in CFT language and to link them to the full perturbative spectrum of the string.

As it is well known [173] Vol. II $N = 2$ supergravity and $N = 1$ heterotic string $E_8 \times E_8$ string compactifications with standard embedding require and $N = (2, 2)$ supersymmetry. Only a $N = (1, 1)$ symmetry is gauged. The $N = 2$ chiral part of a superconformal algebra on the worldsheet has beside the chiral component of energy momentum tensor[9] $T(z) = \sum_{n\in Z} \frac{L_n}{z^{n+2}}$ with conformal dimension and $U(1)$ charge $(h, Q) = (2, 0)$ an $U(1)$ current $J(z) = \sum_{n\in Z} \frac{J_n}{z^{n+1}}$ with $(h, Q) = (1, 0)$ and two super currents $G^{\pm} = \sum_{r\in Z\pm\nu} \frac{G^{\pm}_r}{z^{r+\frac{3}{2}}}$ with $(h, Q) = (\frac{3}{2}, \pm 1)$. The shift ν can take arbitrary real values. The short distance

[9] The standard notation in CFT is quite different then the one common in the discussion of σ models that we used in Sec. 4. One uses in CFT $z = \sigma^1 + i\sigma^2$ and $\bar{z} = \sigma^1 + i\sigma^2$ where $\sigma^2 = i\sigma^0$ is the euclidean time. Correspondingly one indicates the left moving sector which carried a $+$ index in Sec. 4 by quantities without bar and the right moving carrying before $-$ with quantities with bar. Moreover the unbarred or barred super charges are now distinguished by $-$ and $+$ respectively, e.g. $Q_+ \leftrightarrow G^-_0$, $\bar{Q}_+ \leftrightarrow G^+_0$, $Q_- \leftrightarrow \bar{G}^-_0$ and $\bar{Q}_- \leftrightarrow \bar{G}^+$.

operator expansion is

$$\begin{aligned}
T(z)T(0) &\sim \tfrac{c}{2z^4} + \tfrac{2}{z^2}T(0) + \tfrac{1}{z}\partial T(0), \\
T(z)G^{\pm}(0) &\sim \tfrac{3}{2z^2}G^{\pm}(0) + \tfrac{1}{z}\partial G^{\pm}(0), \\
T(z)J(0) &\sim \tfrac{1}{z^2}J(0) + \tfrac{1}{z}\partial J(0), \\
G^{+}(z)G^{-}(0) &\sim \tfrac{2c}{3z^3} + \tfrac{2}{z^2}J(0) + \tfrac{2}{z}T(0) + \tfrac{1}{z}\partial J(0), \\
G^{+}(z)G^{+}(0) &\sim G^{-}(z)G^{-}(0) \sim 0, \\
J(z)G^{\pm}(0) &\sim \pm\tfrac{1}{z}G^{\pm}(0), \\
J(z)J(0) &\sim \tfrac{c}{3z^2},
\end{aligned} \tag{54}$$

Let us recapitulate the standard procedure in 2d QFT which recovers the algebra of charge operators from an operator algebra such as (54). To the operator $A(z)$ we assign charge operators $A_\xi = \oint_{C_0} \mathrm{d}z\, \xi(z)A(z)$, where C_0 is a contour around the origin 0 and $\oint_{C_0} \mathrm{d}z := \int_{C_0} \frac{\mathrm{d}z}{2\pi i}$. In particular for $\xi(z) = z^{n+h(A)-1}$ the charges are the modes A_n of $A(z)$. The transformation of the operator $B(w)$ under (δ_{A_ξ}) is generated by the commutator with A_ξ. In radial time ordering the commutator is given by the following contour integrals

$$\begin{aligned}
(\delta_{A_\xi})B(w) &= [A_\xi, B(w)] = \oint_{\substack{C_0 \\ |z|>|w|}} \mathrm{d}z\, \xi(z)A(z)B(w) - \oint_{\substack{C_0 \\ |z|<|w|}} \mathrm{d}z\, \xi(z)A(z)B(w) \\
&= \oint_{C_w} \mathrm{d}z\, \xi(z)A(z)B(w)\ ,
\end{aligned} \tag{55}$$

see Fig. 4. The spatial transformations δ_ξ corresponding to conformal transformations[10] $z \to z + \xi(z)$

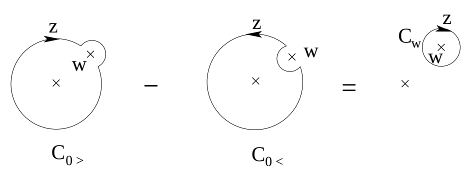

Fig. 4

are generated by $T(z)$, i.e. $\delta_\xi = \delta_{T_\xi}$. One can integrate (55) with $\oint_{C'_{w=0}} \mathrm{d}w\ z^{m+h(B)-1}$ to recover as residuum the mode algebra from

$$\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n} + \tfrac{c}{12}m(m^2-1)\delta_{m,-n}, \\
[L_m, G^{\pm}_r] &= \left(\tfrac{m}{2} - r\right) G^{\pm}_{m+r}, \\
[L_m, J_n] &= -nJ_{m+n}, \\
\{G^{+}_r G^{-}_s\} &= 2L_{r+s} + (r-s)J_{r+s} + \tfrac{c}{3}\left(r^2 - \tfrac{1}{4}\right)\delta_{r,-s}, \\
\{G^{+}_r, G^{+}_s\} &= \{G^{-}_r, G^{-}_s\} = 0, \\
[J_n, G^{\pm}_r] &= \pm G^{\pm}_{r+n}, \\
[J_m, J_n] &= \tfrac{c}{3}\delta_{m,-n},
\end{aligned} \tag{56}$$

10 These are holomorphic in 2d.

with $L_n^\dagger = L_{-n}$, $J_n^\dagger = J_n$ and $(G_r^\pm)^\dagger = G_{-r}^\mp$. In case that the $N = (2,2)$ CFT theory is the internal part of a string compactification it must have $c = \bar{c} = 9$ to cancel the Weyl anomaly. It represents the internal manifold M. In fact $d := \dim C(M) = \frac{c}{3}$. The generalized GSO projection restricts the internal $U(1)$ charges to odd integer values for space time bosons and half integer values for space time fermions, see [84, 173] for more details.

If we consider now the $(+,-)$ twisting[11] [66][56]

$$\hat{T}(z) = T(z) \pm' \frac{1}{2}\partial J(z) \qquad \to \qquad \hat{L}_0 = L_0 \pm' \frac{1}{2} J_0 \tag{57}$$

then the modifications of (54) occur in the following short distance expansions

$$\begin{aligned} \hat{T}(z)\hat{T}(0) &\sim \tfrac{2}{z^2}\hat{T}(0) + \tfrac{1}{z}\partial\hat{T}(0) \\ \hat{T}(z)G^\pm(0) &\sim \tfrac{3\pm'\mp 1}{2z^2}G^\pm(0) + \tfrac{1}{z}\partial G^\pm(0) \\ \hat{T}(z)J(0) &\sim \tfrac{1}{z^2}J(0) + \tfrac{1}{z^2}\partial J(0) \mp' \tfrac{c}{3z^3}, \\ G^+(z)G^-(0) &\sim \tfrac{2c}{3z^3} + \tfrac{2}{z}J(0) + \tfrac{2}{z}\hat{T}(0) + \tfrac{1\mp'1}{z}\partial J(0). \end{aligned} \tag{58}$$

Let us point out the salient features of the operator product expansions in (58)

- Since the central term in the first OPE vanishes no ghost system is required to quantize the world sheet theory.
- By the second OPE either G^+ (+-twist) or G^- ($-$-twist) become a spin one currents, so either $Q = G_0^+ = \oint G^+$ or $G_0^- = \oint G^-$ becomes conformal, i.e. scalars that are defined on every genus world sheet. The opposite super currents G^- (+-twist) or G^+ ($-$-twist), become spin 2 fields.
- The above conformal zero modes are recognized as building blocks for nilpotent operators $Q_{A/B}$. $Q_A = G_0^+ + \bar{G}_0^-$ in the case of the $(+,-)$ twist defining the (c,a) twisted chiral ring as cohomology. $Q_B = G_0^+ + \bar{G}_0^+$ for the $(+,+)$ twist defining the (c,c) chiral ring. The relation to geometry of M is[12] for the A-model $Q_A \leftrightarrow$ d and the for the B-model $Q_B \leftrightarrow \bar{\partial}$ as discussed in more detail in the Sec. 6.1, 8.1.
- The third OPE shows that $J(z)$ has an anomalous transformation. By arguments familiar from the BRST quantization of the bosonic string this gives rise to an anomaly in the divergence of the current, see (383) for a derivation, which can be covariantly written as

  $$\int \nabla^\mu J_\mu = -\int \frac{d}{2\pi}\sqrt{h}R = -d\int c_1(\Sigma_g) = d(2g-2). \tag{59}$$

  For $d = \frac{c}{3} = 3$ this comes precisely with the same anomalous coefficient -3 as the ghost current in the BRST quantization of the bosonic string $j_g = -:bc:$, see [173]. Integration the anomaly in the divergence of the current leads to a $U(1)$-charge violation of $d(2g-2)$ on a genus g Riemann surface.
- The last OPE finally is like the one between the BRST current and the b ghost. Integration around a contour to isolate G_0^+, yields for the $+$ twist

  $$\{Q, G^-(z)\} = T(z)\ , \tag{60}$$

  which echos the main equation $\{Q_{BRST}, b(z)\} = T^{g+m}(z)$ in the BRST quantization of the bosonic string. We have seen already that G^- has $(h,Q) = (2,-1)$, which are precisely the conformal dimension and ghost charges the $b(z)$ ghost.

[11] $\pm'$ marked by a prime are correlated in (29,58).

[12] For Calabi-Yau manifolds this identifi cations can be viewed as convention and is reversed in [20].

To summarize we have for the $(+,+)$ twist [20] exactly the same structure as in the bosonic string if we identify

$$(G^{+}(z), J(z), T(z), G^{-}(z)) \leftrightarrow (J_{BRST}(z), j_g = - :bc:(z), T^{m+g}(z), b(z)) \quad (61)$$

and similar for the anti chiral half. This implies also $Q_B \leftrightarrow Q_{BRST}$ and the ghost number becomes $U(1)_A$ charge.

The degenerate ground states in the Ramond-Ramond sector fulfill [151]

$$G_0^{\pm}|\psi\rangle = 0. \quad (62)$$

These Ramond-Ramond ground states have by (56)

$$h = \frac{c}{24} = \frac{3}{8}. \quad (63)$$

An operator $\mathcal{O}$ with charge Q in the theory can be decomposed into a part $\hat{\mathcal{O}}$ which is neutral under the $U(1)$ current and a charge carrying part, i.e. $\mathcal{O} = \hat{\mathcal{O}} e^{iQ\sqrt{\frac{3}{c}}\phi}$, where we bosonize the current as $J = \sqrt{\frac{c}{3}}\partial\phi$ [178, 151]. Hence there is a natural operation, which shifts the $U(1)$ charge of every operator $e^{iQ\sqrt{\frac{3}{c}}\phi} \to e^{i(Q-a)\sqrt{\frac{3}{c}}\phi}$. It is easy to see that this operation induces a family of algebra automorphisms known as *spectral fbw* [178]

$$\begin{aligned} L_n \to L_n' &= L_n + aJ_n + \tfrac{1}{6}a^2 c\delta_{n,0} \\ J_n \to J_n' &= J_n + \tfrac{1}{3}ac\delta_{n,0} \\ G_r^{\pm} \to (G_r^{\pm})' &= G_{r\mp a}^{\pm} \, . \end{aligned} \quad (64)$$

The Ramond ground states are related by (64) with $a = \pm' \frac{1}{2}$ to states in the NS sector with

$$G_r^{\pm}|\psi\rangle = L_n|\psi\rangle = J_n|\psi\rangle = 0, \qquad r > 0, \quad n > 0, \quad \text{and } G_{-\frac{1}{2}}^{\pm'}|\psi\rangle = 0 \quad (65)$$

Only the $\pm'$ in (65) correlates with the one in $a = \pm'\frac{1}{2}$ and one has $(+)$ for chiral and (-1) for anti chiral states. It is easy to see that (65,56) imply

$$h = \pm\frac{1}{2}Q, \qquad |Q| \leq \frac{c}{3} = d \, . \quad (66)$$

Massless space-time scalars are have $(Q, \bar{Q}) = (\pm 1, \pm 1)$. The states in the chiral- and anti chiral rings with this property are related to the cohomology of M. The (c,c) ring corresponds to $H^{2,1}(M)$ and the (c,a) ring corresponds[13] to $H^{1,1}(M)$. The above spectral flow operators with $a = \pm\frac{1}{2}$ relate space time superpartners with each other and are identified with internal part of the spacetime susy operators [84].

The main point in Gepners construction is to identify the internal $c = \bar{c} = 9$ theory with an orbifold of a tensor product of minimal $(2,2)$ superconformal field theories. The factor theories are constructed as cosets of supersymmetric, WZW models, see [131] for a general discussion. WZW models and cosets are an important source of rational CFT beyond $c > 1$. In the simplest case based on a $(SU(2) \times U(1))/U(1)$ coset the central charge is

$$c_k = \frac{3k}{k+3}, \qquad k \in N \, . \quad (67)$$

[13] The (a,a) and (a,c) rings correspond to conjugated fi elds and contain no independent information.

Primary states $|l,q,s\rangle$ of the algebra (54) are labeled in the minimal models by integers which have the following *standard range*[14]

$$\begin{array}{l} 0 \le l \le k, \\ 0 \le |q-s| \le l \\ s = \left\{ \begin{array}{ll} 0,2 & \mathrm{Neveu-Schwarz-sector} \\ \pm 1 & \mathrm{Ramond-sector} \end{array} \right\} \\ l+q+s = 0 \bmod 2 \end{array} \tag{68}$$

and have conformal dimension and charge

$$h = \frac{l(l+2)}{4(k+2)-q^2} + \frac{s^2}{8}, \qquad Q = -\frac{q}{k+2} + \frac{s}{2}. \tag{69}$$

Above we discussed only the right moving part of the theory. There is a remarkable $A-D-E$ classification, behind the question how to combine the $\chi_{l,q,s}$ and $\chi_{\bar{l},q,s}$ characters to a modular invariant one loop partition function [31]. Note that above only $l \neq \bar{l}$. That is because all possible shifts of q,s w.r.t. $\bar{q},\bar{s}$ are obtainable in a separate step by orbifold constructions w.r.t. to simple current symmetries. The simplest way to get a modular invariant theory is to start with a left right symmetric theory with states $|l,q,s;l,q,s\rangle$, this corresponds to the A-series. Considering only this series there are 145 possibilities to build a tensor product theory with $\bar{c} = c = \sum_{i=1}^5 c_{k_i} = 9$. Note that at most one k_j is allowed to be zero, because of the $c=9$ condition. This is the same number as $c_1(T_M)=0$ Fermat hypersurfaces in WCP^4, i.e. with $\sum_{i=1} w_i = d$, see Sec. 9.10. In fact identifying $m_i = d/w_i = k_i+2$ it is easy to see that both enumerations lead to the same diophantic problem. The simplest possibility is $k_i = 3$ for $i=1,\ldots,5$. This leads to $d=5$, $w_i=1$, $i=1,\ldots 5$, the Quintic in $\mathbf{P}^4$. Gepners orbifold construction divides the symmetric tensor product by a symmetry group which is generically the subgroup $G = \mathbf{Z}_{least\ com.\ mult.\{k_i\}} \times (\mathbf{Z}_2)^{r+1}$ among the group generated by the simple currents and constructs a modular invariant orbifold. The effect is that the factor theories and the space-time part are either all in the NS-NS sector or all in the R-R sector and that the charges in the internal NS-NS sector become odd integers [84, 85]. It then easy to see that states in (c,c) ring from the invariant sector[15] of the orbifold are of the form $\bigotimes_i |l_i,l_i,0;l_i,l_i,0\rangle$. For the tensor product model that corresponds to the quintic this leads in view of (68) to 101 elements. The counting is the same that leads to the 101 independent complex structure deformations under Eq. (264), which are identified with elements in $H^{2,1}(M)$. All states in the (a,c) ring are from the twisted sector. They are more complicated to count but one checks that they yield the number of independent elements in $H^{1,1}(M)$. It is also straightforward to identify the orbifold action, like e.g. (265,274), that leads to the mirrors W of the manifolds M in (53) in the conformal field theory context and to check that it indeed exchanges the (c,c) with (c,a) ring [96, 76]. A fascinating idea has been to use Cardy states [176] to classify D-branes as boundary conditions in the rational CFT at the Gepner-point and compare with geometric pictures of D-branes [28] in particular the triangulated category of coherent sheaves over M for the B-branes or the category of special Lagrangian submanifolds of M for the A-branes respectively.

5.5 tt^* equations, special geometry and contact terms

The tt^* equations describe the geometry of the ground states of $N=(2,2)$ two dimensional theories. The construction does not require necessarily conformal invariance, but rather the following structure. A nilpotent operator Q and its adjoint $Q^\dagger$

$$\{Q,Q^\dagger\} = H \tag{70}$$

[14] For the orbifold procedure the following equivalences are important $q \sim q \bmod 2(k+2)$, $s = s \bmod 4$ and $|l,q,s;\bar{l},\bar{q},\bar{s}\rangle \sim |k-l,q,s;k-\bar{l},\bar{q}+k+2,\bar{s}+2\rangle$.

[15] In general there might be (c,c) states in the twisted sectors but for the smooth hypersurfaces, such as the quintic, there are none.

and a conserved fermion number. Q and its adjoint $Q^\dagger$ define rings of cohomological operators $\mathcal{R}$ and $\mathcal{R}^*$ respectively. The advantage of the approach is that it derives the relevant geometry with minimal assumptions. E.g. special Kähler geometry follows just with an additional requirement on integral charge conservation for the A-model the B-model and even the more exotic cases introduced in [83]. To make contact with the previous sections this can be realized as

$$Q = \begin{cases} Q_A & = Q_- + \bar{Q}_+, \quad \mathcal{R} = (a,c) \\ Q_B & = \bar{Q}_- + \bar{Q}_+, \quad \mathcal{R} = (c,c) \end{cases} \qquad Q^\dagger = \begin{cases} Q_A^\dagger & = \bar{Q}_- + Q_+, \quad \mathcal{R}^* = (c,a) \\ Q_B^\dagger & = Q_- + Q_+, \quad \mathcal{R}^* = (a,a) \end{cases} \tag{71}$$

As explained we have to twist the theories by identifying the corresponding A^R gauge connection with the spin connection. Since only the fermion number must be conserved [44] one needs only a Z_2 anomaly free subgroup of the $U(1)_R$-currents. The tt^* geometry is applicable to $N=(2,2)$ 2d field theories with marginal (conformal) but also relevant (non-conformal) deformations. While these theories might not have a geometrical target space realization, it is still[16] useful to think of a formal correspondence to the deRham (Dolbeault) cohomology on a manifold M with $(Q, Q^\dagger, H) \sim (\mathrm{d}, \mathrm{d}^*, \Delta)$

The Ramond-Ramond *vacuum states*, compare (62), are defined by

$$Q|\alpha\rangle = Q^\dagger|\alpha\rangle = 0\,. \tag{72}$$

Such states play the rôle of harmonic forms. We call the space of vacua $\mathcal{H}$. The *operator state correspondence* of 2d QFT associates to every operator $\phi \in \mathcal{R}$ acting on a any vacuum state α a state $|\phi\rangle = \phi|\alpha\rangle$. In order to avoid too many indices we call the zero-form operators $\mathcal{O}^{(0)} = \phi$ and the two form operators $\mathcal{O}^{(2)} = \mathcal{O}$. Since $|\phi\rangle_\alpha = \phi|\alpha\rangle$ is closed, Hodge decomposition (348) applies $|\phi\rangle_\alpha = |\phi_0\rangle_\alpha + Q|\phi_-\rangle_\alpha + Q^\dagger|\phi_+\rangle_\alpha$ and by that we get a map

$$\Pi_h : |\phi\rangle_\alpha \mapsto |\phi_0\rangle_\alpha \tag{73}$$

from $\mathcal{R}$ to $\mathcal{H}$. If α is fixed and as will soon see there is *preferred choice* we can find a canonical map from the ring $\mathcal{R}$ to the Ramond-Ramond groundstates. Moreover every $\phi \in \mathcal{R}$ induces a map

$$\Phi : |\alpha\rangle \mapsto |\phi_0\rangle_\alpha \tag{74}$$

from $\mathcal{H}$ to $\mathcal{H}$. Everything we said from Eq. (72) on, could have been said verbatim for the conjugated sector defined by $Q^\dagger$. In particular we get for the same choice of α a second basis of $\mathcal{H}$, which we call $|\bar{\imath}\rangle$, $\bar{\jmath} = 1, \ldots, r$. If one has unbroken $U(1)_{R/L}$ symmetries as in Sec. 5.4 one could single out $|\alpha\rangle$ as the lowest charge state in the Ramond-Ramond groundstate.

The following path integral argument requires only conserved fermion number. In the operator approach[7][56] to 2d field theory one defines a state the Hilbert space H of 2d theory by the path integral over a half sphere HS^2 bounding an S^1. Parametrize the S^1 by θ and denote the fields generically by $\phi(\theta)$. The path integral is a functional of the boundary field configuration $\phi(\theta) \in L^2$ on the S^1 and defines a state $|\phi\rangle$ in H as in (76). Anti periodic boundary conditions for fermionic states on contractible loops as S^1 on HS^2 are the natural boundary conditions in the path integral so that (76) does not yield periodic Ramond-Ramond states in H. However the connection A^R_μ of the gauged $U(1)$ R-symmetry couples to the fermion number with charge $\frac{1}{2}$, i.e. acts like a spin connection ω_μ. When one transports the fermion along the S^1, the connection is integrated to a Wilson loop phase rotation acting on the fermionic state as

$$e^{\pi i \oint_{S^1} \omega \mathrm{d}x} = e^{\pi i \int_{HS^2} \mathrm{d}\omega} = e^{\pi i \int_{HS^2} \frac{R}{2\pi i}\sqrt{h}} = e^{\pi i \int_{HS^2} c_1(T)} = -1, \tag{75}$$

which rectifies the periodicity. A projection to the Ramond-Ramond groundstates at the boundary can now be achieved by attaching a cylinder of length T to HS^2, see Fig. 5. Call the combined surface $H_T S^2$. The

[16] For σ model on M this formal correspondence becomes an actual correspondence.

"evolution" of a state $|\phi\rangle$ defined by the original boundary S^1 of HS^2 to the far boundary is described by $e^{-HT}|\phi\rangle$. If the length T of the cylinder goes to infinity only the groundstates in $\mathcal{H}$ survive, because they have 0 as energy eigenvalue of H, cff (63).

After this preparation we can define the path integral version of a projector (73)

$$|i\rangle = \lim_{T\to\infty} \int \mathcal{D}\phi e^{-\int_{H_T S^2} L(\phi)} \phi_i = \Pi_p(\phi_i) \ . \tag{76}$$

The $T\to\infty$ limit makes the projector only sensitive to cohomological information of ring states $\phi \in \mathcal{R}$ or $\bar\phi \in \mathcal{R}^*$. Exact pieces have non-zero energy and are completely suppressed. Note that $\Pi(\mathbf{1}) = |0\rangle$ defines a *preferred vacuum* state. We call the image of a basis $\phi_i \in \mathcal{R}$, $i = 0,\ldots,r$ with $\Phi_0 = \mathbf{1}$ in $\mathcal{H}$ the *topological basis* $|i\rangle = \Pi_p(\phi_i)$. By the operator state correspondence we can also represent the rings (36) on the vacuum states

$$\phi_i|j\rangle = C_{ij}^k|k\rangle \tag{77}$$

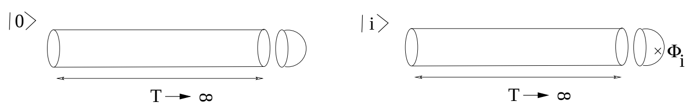

Fig. 5 Path integral projectors to the Ramond-Ramond ground states $\mathcal{H}$

The path integral (76) with insertions of $\bar\phi_i \in \mathcal{R}^*$ defines the *anti-topological basis* $|\bar\imath\rangle = \Pi_p(\bar\phi_i)$. The two basis of $\mathcal{H}$ namely $|i\rangle$ and $|\bar\imath\rangle$ must be related by a linear transformation, the real structure,

$$|i\rangle = M_i^{\bar\imath}|\bar\imath\rangle \ . \tag{78}$$

The CPT theorem of the 2d field theory states that the effect of complex conjugating all expressions in (76) sends $|i\rangle \to |\bar\imath\rangle$, i.e. $|\bar\imath\rangle = M_{\bar\imath}^j|j\rangle$ which implies $MM^* = 1$. One has a topological bilinear pairing

$$\langle i|j\rangle = \eta_{ij} \tag{79}$$

and an hermitian bilinear pairing called the tt^* metric

$$\langle\bar\imath|j\rangle = g_{\bar\imath j} \ , \tag{80}$$

which are in an obvious way related by the real structure

$$g^{\bar l i}\eta_{ij} = M_j^{\bar l} \ . \tag{81}$$

Note that $\langle i| \neq (|i\rangle)^\dagger$. Both bilinear pairings can be defined by the path integral as in Fig. 6. These objects are topological to different extend. Changing the representative of the Q cohomology class $|i\rangle \mapsto |i\rangle + Q|\lambda\rangle$ or $\langle j| \mapsto \langle j| + \langle\lambda|Q$ will do nothing in $\langle i|j\rangle$ as $|j\rangle$ and $\langle i|$ are Q closed. Due to (35) the pairing η_{ij} is independent of the position. That is true for all length/diameter ratios of the cylinder, i.e. the cylinder is not needed at all in the definition. For the pairing $g_{\bar\imath j}$ with $\langle\bar\imath| \mapsto \langle\bar\imath| + \langle\lambda|Q^\dagger$ and $|i\rangle \mapsto |i\rangle + Q|\lambda\rangle$ the argument does not apply as $|j\rangle$ is not $Q^\dagger$ and $\langle\bar\imath|$ not Q closed. However from (70) and $Q|\lambda\rangle \neq 0$ ($\langle\lambda|Q^\dagger \neq 0$) follows that these exact states have positive energy. The only states with zero energy are R-R vacua. I.e. in the case of $g_{\bar\imath,j}$ we need the $T\to\infty$ limit to define a topological quantity.

Locally the tangent space of the (t,t^*) moduli space is spanned by elements from $\mathcal{R}(t)$ and $\mathcal{R}^*(t^*)$. It is clear that the pairing η_{ij} depends only on the t moduli. Moreover one shows that as metric it is completely

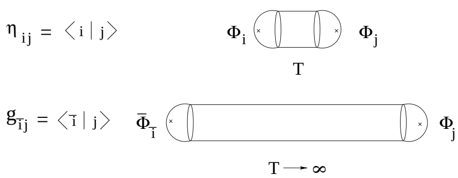

Fig. 6 Path integral representation of the topological pairing η_{ij} and the topological-antitopological pairing $g_{\bar{\imath}j}$.

flat, i.e. all components of the curvature tensor vanish similar as in $d < 1$ strings [60]. One can therefore find coordinates which make the metric η_{ij} constant. This defines the moduli dependent basis of $\mathcal{R}$. As it is clear from the construction of the basis $|i\rangle$ and $|\bar{\imath}\rangle$ via the projection of moduli dependent elements in the rings $\mathcal{R}$ and $\mathcal{R}^*$ they will depend on the moduli $\underline{m} = (\underline{t}, \underline{t}^*)$. In the Landau-Ginzburg approach [193] η_{ij} is explicitly defined in terms of the Landau Ginzburg superpotential as

$$\eta_{ij} = \mathrm{Res}[\phi_i \phi_j] = \frac{1}{(2\pi i)^n} \int_\Gamma \frac{\phi(X) \mathrm{d}X^1 \wedge \ldots \wedge \mathrm{d}X^n}{\partial_1 W \ldots \partial_n W} = \sum_{\mathrm{d}W} \phi(X) \det{}^{-1}[\partial_i \partial_j W] \, . \tag{82}$$

Another approach to define η_{ij} is via the supersymmetric Schroedinger equation [42]. We will not dwell deeper into the derivation of (82), except for remarking that it is a zero dimensional analog of the Griffith residuum expressions (266,280) used in Sec. 8.7 to define the periods, with the identification $W = P$.

The tt^* equations describe how the vacuum states in $\mathcal{H}$ vary over the moduli space parametrized by $\underline{m}$. One calls the corresponding bundle also $\mathcal{H}$. Let e_γ be a basis, i.e. a section in $\mathcal{H}$, and denote its connection

$$A^\alpha_{\beta\gamma} = g^{\alpha\kappa} \langle e_\kappa | \partial_\beta | e_\gamma \rangle . \tag{83}$$

If the basis of $\mathcal{H}$ changes by a "gauge" transformation $|e_\gamma\rangle \mapsto |e'_\gamma\rangle = \Lambda_{\gamma\delta} |e_\delta\rangle$ then the connection undergoes a gauge transformation $A \mapsto \Lambda^{-1} A \Lambda + \Lambda^{-1} \mathrm{d}\Lambda$. Let us consider the perturbation

$$S = \int_\Sigma \mathrm{d}^2 z \mathcal{L}_0 + \sum_i t^i \int_\Sigma \mathrm{d}^2 z \mathcal{O}_i + \sum_{\bar{\imath}} \bar{t}^{\bar{\imath}} \int_\Sigma \mathrm{d}^2 z \bar{\mathcal{O}}_i \, , \tag{84}$$

where the two-form descendants are called $\mathcal{O}_i := \mathcal{O}_i^{(2)}$. It is easy to show that the following mixed indices of this connection vanish in the holomorphic basis. Consider e.g. $A^i_{\bar{\imath}j}$ using (81) we can write $A^i_{\bar{\imath}j} = g^{i\bar{k}} \langle \bar{k} | \partial_{\bar{\imath}} | j \rangle = \eta^{ik} \langle k | \partial_{\bar{\imath}} | j \rangle$. By (38) we can write $\int_\Sigma \bar{\mathcal{O}}_{\bar{\imath}} = [Q, \Lambda]$ and since ϕ_j is Q closed we can write $\partial_{\bar{\imath}} | j \rangle = \Pi_h([Q, \Lambda] \phi_j) = Q \Pi_h(\Lambda \phi_j) = Q(\Lambda | j \rangle)$. Since $\langle k | Q = 0$ is closed this expression vanishes

$$A^i_{\bar{\imath}j} = 0 \, . \tag{85}$$

Similarly one shows that $A^i_{k\bar{\jmath}} = \eta^{il} \langle l | \partial_k | \bar{\jmath} \rangle = 0$.

The metric connection is characterized by

$$0 = D_k g_{i\bar{\jmath}} = \partial_k g_{i\bar{\jmath}} - (\partial_k \langle i |) | \bar{\jmath} \rangle - \langle i | \partial_{\bar{k}} | \bar{\jmath} \rangle = (\partial_k \langle i |) | \bar{\jmath} \rangle \, . \tag{86}$$

From this and the $\bar{D}_{\bar{k}}$ derivative, we get formulas for A^j_{km} and $A^{\bar{\jmath}}_{\bar{k}\bar{m}}$

$$A^j_{km} = g^{j\bar{\jmath}} \partial_k g_{m\bar{\jmath}}, \qquad A^{\bar{\jmath}}_{\bar{k}\bar{m}} = g^{m\bar{\jmath}} \partial_{\bar{k}} g_{m\bar{m}} \, . \tag{87}$$

as hermitian connection of g. Indeed the topological basis $|i\rangle$ and the anti-topological basis $|\bar{\imath}\rangle$ form holomorphic and antiholomorphic sections of the vacuum bundle over the moduli $\underline{m}$ and one gets the vanishing of the following components of the curvature

$$[D_i, D_j] = [\bar{D}_{\bar{\imath}}, \bar{D}_{\bar{\jmath}}] = 0 \, . \tag{88}$$

The most important relation comes from analyzing the $[D_i, \bar{D}_{\bar{\imath}}]$ curvature term. Let us do this for definiteness for the B model. Since the twisting (58) is so that $\bar{Q}_+(z) \sim G^+(z)$ and $\bar{Q}_-(z) \sim \bar{G}^+(z)$ have dimension one, we can define

$$\bar{Q}_+ = \oint \mathrm{d}z G^+(z), \qquad \bar{Q}_- = \oint \mathrm{d}z \bar{G}^+(z) \, . \tag{89}$$

Here we adopt the notation to use the CFT conventions for the twisted currents. The commutators and anticommutators in the definition of the descendants (38) can be represented by (55) as

$$\begin{aligned} \mathcal{O}_i: \;\; &= \mathcal{O}^{(2)}_i = \{Q_+, [Q_-, \phi_i(u)]\} \sim \oint_{C_u} \mathrm{d}z G^-(z) \oint_{C'_u} \mathrm{d}w \bar{G}^-(w) \bar{\phi}_{\bar{\imath}}(u) \, , \\ \bar{\mathcal{O}}_{\bar{\imath}}: \;\; &= \bar{\mathcal{O}}^{(2)}_{\bar{\imath}} = \{\bar{Q}_+, [\bar{Q}_-, \bar{\phi}_{\bar{\imath}}(u)]\} \sim \oint_{C_u} \mathrm{d}z G^+(z) \oint_{C'_u} \mathrm{d}w \bar{G}^+(w) \bar{\phi}_{\bar{\imath}}(u) \end{aligned} \tag{90}$$

We calculate $[D_i, \bar{D}_{\bar{\imath}}]$ in $|l\rangle$ basis i.e.

$$\begin{aligned} [D_i, \bar{D}_{\bar{\jmath}}]^l_k = \;& \partial_i A^l_{\bar{\jmath}\;k} - \partial_{\bar{\jmath}} A^l_{i\;k} = \eta^{lp}[(\partial_i \langle p|)\bar{\partial}_{\bar{\jmath}}|k\rangle - (\bar{\partial}_{\bar{\jmath}}\langle p|)\partial_i|k\rangle] \\ = \;& \eta^{lp}\Pi\left(\phi_p \textstyle\int_{HS^2_L}\{Q_+,[Q_-,\phi_i]\}\right)\Pi\left(\textstyle\int_{HS^2_R}\{\bar{Q}_+,[\bar{Q}_-,\bar{\phi}_{\bar{\jmath}}]\}\phi_k\right) \\ & -\eta^{lp}\Pi\left(\phi_p \textstyle\int_{HS^2_L}\{\bar{Q}_+,[\bar{Q}_-,\bar{\phi}_{\bar{\jmath}}]\}\right)\Pi\left(\textstyle\int_{HS^2_R}\{Q_+,[Q_-,\phi_i]\}\phi_k\right) \\ = \;& \eta^{lp}\left[\Pi\left(\phi_p \textstyle\int_{HS^2_L}\partial\bar{\partial}\phi_i\right)\Pi\left(\textstyle\int_{HS^2_R}\bar{\phi}_{\bar{\jmath}}\phi_k\right) - \Pi\left(\phi_p \textstyle\int_{HS^2_L}\bar{\phi}_{\bar{\jmath}}\right)\Pi\left((\textstyle\int_{HS^2_R}\partial\bar{\partial}\phi_i)\phi_k\right)\right] \\ = \;& \eta^{lp}\left[\Pi\left(\phi_p \textstyle\int_{HS^2_L}\bar{\phi}_{\bar{\jmath}}\right)\Pi\left(\textstyle\int_{C_R}(\partial_{\tau_2}\phi_i)\phi_k\right) - \Pi\left(\phi_p \textstyle\oint_{C_L}\partial_{\tau_2}\phi_i\right)\Pi\left((\textstyle\int_{HS^2_R}\bar{\phi}_{\bar{\jmath}})\phi_k\right)\right] \\ = \;& \eta^{lp}\left[\Pi\left(\phi_p \textstyle\int_{HS^2_L}\bar{\phi}_{\bar{\jmath}}\right)\Pi\left((\textstyle\oint_\Gamma H(z)\oint_{C_R}\phi_i)\phi_k\right) - \Pi\left(\phi_p \textstyle\oint_\Gamma H(z)\int_{C_L}\phi_i\right)\Pi\left((\textstyle\int_{HS^2_R}\bar{\phi}_{\bar{\jmath}})\phi_k\right)\right] \end{aligned} \tag{91}$$

the contours of $G^-(z), \bar{G}^-(z)\; G^+(z), \bar{G}^+(z)$ are as in Fig. (7). Moreover we consider operators ϕ in the (c, c) and $\bar{\phi}$ in the (a, a) ring, e.g. ϕ is $\bar{Q}_+$ and $\bar{Q}_-$ closed. In the language of current algebras that means that the short distance expansion of $\phi(v)$ with $\bar{Q}_+(z) \sim G^+(z)$ and $\bar{Q}_-(w) \sim \bar{G}^+(z)$ has no pole and $\phi(v)$ can be ignored when deforming Γ_z and Γ_w. The contours e.g. of the term in the third line can be deformed as in fig. 7 and the contours of $G^-(z), \bar{G}^-(z)$ encircling $G^+(z), \bar{G}^+(z)$ give the L_{-1} and $\bar{L}_{-1}$ acting as ∂ and $\bar{\partial}$ derivatives on ϕ_i by (35). Similar manipulations apply to the term in the second line of (91). Applying Gauss's law in both terms gives the integral over the normal derivative $-\partial_{\tau_2}$. The minus sign is due to the orientation of τ_2. The normal direction is "time" evolution by H, i.e. $\partial_{\tau_2} = \partial_n \phi_i = [H, \phi_i]$, which is used in the last line of (91), where $H(z)$ is integrated around ϕ_i From now on we exploit the

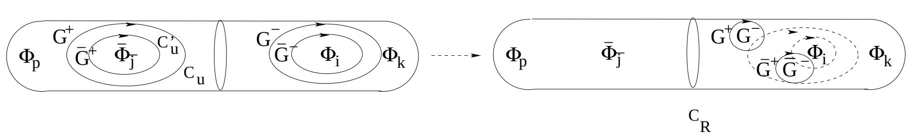

Fig. 7 Contour manipulation on Σ in the evaluation of $[D_i, \bar{D}_{\bar{\jmath}}]^l_k$.

topological nature of the theory and take ordered limits of Σ

$$\text{first}: \quad T_R, T_L \to \infty, \qquad \text{second}: \quad T \to \infty \tag{92}$$

as depicted Fig.8. The tubes are all normalized to have perimeter 1. Elongation T_R and T_L projects ϕ_p

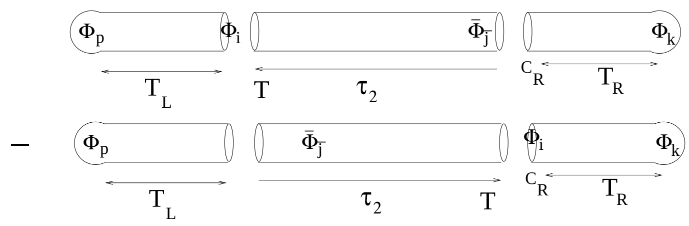

Fig. 8 Limits taking in the evaluation of $[D_i,\bar D_{\bar\jmath}]_k^l$.

and ϕ_k to the Ramond-Ramond vacuum state $\langle p|$ and $|k\rangle$ respectively. The procedure of the limits is a prescription how to deal with short distance singularities and the only such issue in topological field theory are *contact terms* see (100) and (142).

The action of H on these states yields zero. The two terms in the last line of (91) are transformed into each other by exchanging the left- and right infinity. We discuss the $-\Pi\left(\phi_p\int_{HS_L^2}\bar\phi_{\bar\jmath}\right)\Pi\left((\oint_\Gamma H(z)\oint_{C_R}\phi_i)\phi_k\right)$ explicitly. Vanishing of $H|k\rangle$ means that H may considered as acting on the full state $\Pi\left((\oint_{C_R}\phi_i)\phi_k\right)$. In Hilbert space notation is denoted as $H|(\oint_{C_R}\phi_i)|k\rangle$ and similar $\Pi\left(\phi_p\int_{HS_L^2}\bar\phi_{\bar\jmath}\right)$ as $\langle p|\int_{HS_L^2}\bar\phi_{\bar\jmath}|$. We can move the H integral to the left and since ϕ_p is projected to the groundstate the non-vanishing contribution comes from its action on $\int_{HS_L^2}\bar\phi_{\bar\jmath}$. If the insertion of $\bar\phi_{\bar\jmath}$ is on the most left part in fig (8) it will also be projected to the groundstate in the $T\to\infty$ limit and annihilated by H. Therefore is remains to consider the contribution from integral over the middle tubus whose length is parametrized by T. This integral is $\int_{Tu}\bar\phi_{\bar\jmath}=\int_0^T \mathrm{d}\tau_2\oint_{C_L}\mathrm{d}\theta\bar\phi_{\bar\jmath}$. H creates τ_2 translations, so $[H,\bar\phi_{\bar\jmath}]=-\partial_{\tau_2}\bar\phi_{\bar\jmath}$ and the integration over τ_2 becomes trivial. Note that only the lower boundary $\tau_2=0$ contributes. The upper boundaries, where $\bar\phi_{\bar\jmath}$ is near ϕ_i in both contributions see Fig. 8, cancels. Therefore

$$\begin{aligned}
[D_i,\bar D_{\bar\jmath}]_k^l = &\ \eta^{lp}\lim_{T_{L/R}\to\infty}\left[\Pi\left(\phi_p\int_{HS_L^2}\bar\phi_{\bar\jmath}\right)\Pi\left((\oint_\Gamma H\oint_{C_R}\phi_i)\phi_k\right)-\Pi\left(\phi_p\oint_\Gamma H\int_{C_L}\phi_i\right)\Pi\left(\int_{HS_R^2}\bar\phi_{\bar\jmath}\phi_k\right)\right]\\
= &\ \eta^{lp}[\langle p|\left(\int_{Tu}\bar\phi_{\bar\jmath}\right)H\left(\oint_{C_R}\phi_i\right)|k\rangle-\langle p|\left(\int_{C_L}\phi_i\right)H\left(\int_{Tu}\bar\phi_{\bar\jmath}\right)|k\rangle]\\
= &\ \eta^{lp}\lim_{T\to\infty}[\langle p|\left(\oint_{C_L}\bar\phi_{\bar\jmath}\right)e^{-HT}\left(\oint_{C_R}\phi_i\right)|k\rangle-\langle p|\left(\int_{C_L}\right)\phi_i e^{-HT}\left(\oint_{C_R}\bar\phi_{\bar\jmath}\right)|k\rangle]\\
= &\ (\bar C_{\bar\jmath}C_i)_k^l-(C_i\bar C_{\bar\jmath})_k^l=-[C_i,\bar C_{\bar\jmath}]_k^l
\end{aligned}\tag{93}$$

This is the main identity within the tt^* equations. The others are easier to derive and all are summarized below in the topological basis

$$\begin{aligned}
[D_i,\bar D_{\bar\jmath}] = &\ -[C_i,\bar C_{\bar\jmath}]\\
[D_i,D_j]=[\bar D_{\bar\imath},\bar D_{\bar\jmath}] = &\ [D_i,\bar C_{\bar\jmath}]=[\bar D_{\bar\imath},C_j]=0\\
D_iC_j=D_jC_i &\qquad \bar D_{\bar\imath}\bar C_{\bar\jmath}=\bar D_{\bar\jmath}\bar C_{\bar\imath}
\end{aligned}\tag{94}$$

We can now define a *flat* $[\nabla_i,\nabla_j]=[\nabla_i,\bar\nabla_{\bar\jmath}]=[\bar\nabla_{\bar\imath},\bar\nabla_{\bar\jmath}]=0$ *connection*

$$\nabla_i=D_i+\alpha C_i,\qquad \bar\nabla_{\bar\jmath}=\bar D_{\bar\jmath}+\alpha^{-1}\bar C_{\bar\jmath}\,. \tag{95}$$

The sections of the vacuum bundle are identified with the periods in the Calabi-Yau σ model context. The above *flat connection* goes by the name *Gauss Manin connection* in this context, see Sec. 8.5. Since it is flat it seems that the theory is *trivial*! However flat connections can still have *monodromies*, over non simply connected manifolds, see Fig. 33,34, which are the essential data of our theories. Where do these monodromies come from? The key is that (52), which is based on a local consideration of the tangent spaces of metric deformations at a generic point of the moduli space fails at singular degenerations of the space time Calabi-Yau manifold. At these loci charged Ramond-Ramond states become light, the simplest example is the charged *black hole* at the conifold [184], which sits in a hyper multiplet. In the presence of massless charged states the supergravity argument for the factorization (52) into hyper- and vector multiplets does not apply either. In fact the logarithm in third period that produces the monodromy M_1 in (284) can be interpreted as the one loop correction of the vector multiplet gauge coupling due to the massless hypermultiplet. An intriguing experimentally verifiable occurance of mondromies of flat connections is the *Berry Phase* in quantum mechanics [18] see [162] for a review.

The tt^* equations describe the essence of the WS super symmetry constraints on the topological correlators. These equations have in general to be supplemented with information about the structure constants C^l_{ij} and boundary conditions. But already with some $U(1)$ i.e. R symmetry charge constraints they become powerful. E.g. for $d < 1$ (66) implies $|Q| < 1$ moreover these theories are rational and have finitely many chiral primaries in this charge range. We assign to the t^i of say the (c, c) ring (84) the weight $w_i = (1 - Q_i) > 0$. The last equation (94) called *associativity* guarantees the existence of a potential $\mathcal{F}$ with $C_{ijk} = D_i D_j D_k \mathcal{F}$. As discussed one can chose flat coordinates, which we call for convenience also t^i such that $C_{ijk} = \partial_i\partial_j\partial_k\mathcal{F}$ Charge conservation implies that $\mathcal{F}$ is homogeneous of degree 2 in the weights w_i of the t^i, i.e. a finite polynomial and *associativity* determines its coefficients up to an overall normalization. These constraints imply indeed that there is a completely solvable discrete infinite set of $d < 1$ $N = (2, 2)$ theories with an ADE classification. For $d \geq 1$ there are zero and negative weight t^i and this simple way of approaching the problem loses its grip.

However if $d \in \mathbf{Z}$ and the R charges are also integer, we expect from Sec. 5.4 that beside world-sheet super symmetry also space-time super symmetry constraints the correlators. Let us show that (94) implies for the Calabi-Yau σ models on threefolds $d = 3$ and odd integer R charges *special K¨ahler geometry*. In the holomorphic basis we use (85) to write $[D_i, \bar{D}_{\bar{\jmath}}]^k_l = -\bar{\partial}_{\bar{\jmath}} A^k_{il} = -[C_i, \bar{C}_{\bar{\jmath}}]$. With $(C^k_{il})^\dagger = \bar{C}^{\bar{l}}_{\bar{\imath}\bar{k}}$ and hence $C^k_{\bar{\jmath}m} = g^{k\bar{k}} C^{\bar{m}}_{\bar{\jmath}\bar{k}} g_{\bar{m}m}$ we write

$$\bar{\partial}_{\bar{\jmath}} A^k_{il} = [C_i, \bar{C}_{\bar{\jmath}}]^k_l = [C_i, g^{-1} C^\dagger_j g]^k_l \ . \tag{96}$$

In the case of Calabi-Yau σ model the R charge conservation law forbids many correlators, see sections 6.1 and 8.1. In particular $g_{0\bar{k}} = g^{0\bar{k}} = 0$ for $\bar{k} \neq \bar{0}$ and $C^k_{i0} = \delta^k_i$ and $C^{\bar{k}}_{\bar{\imath}\bar{0}} = \delta^{\bar{k}}_{\bar{\imath}}$. If we specialize (96) to $k = l = 0$ we can write

$$\begin{aligned} \bar{\partial}_{\bar{\jmath}} A^0_{i0} = \bar{\partial}_{\bar{\jmath}}(g^{0\bar{k}} \partial_i g_{0,\bar{k}}) = &\quad [C_i, g^{-1}(C_j)^\dagger g]^0_0 \\ \bar{\partial}_{\bar{\jmath}} \partial_i \log(g_{0\bar{0}}) = &\quad -g^{0\bar{0}} C^{\bar{k}}_{\bar{\jmath}\bar{0}} g_{\bar{k}i} \\ = &\quad -\frac{g_{\bar{\jmath}i}}{g_{0\bar{0}}} \ . \end{aligned} \tag{97}$$

As follows from the identification (214,215) in the B-model and (232) or Serre duality (394) the vacuum states $|0\rangle$ and $|\bar{0}\rangle$ are associated to the holomorphic $(n, 0)$ and anti-hololomorhic $(0, n)$ forms. In particular

$$e^{-K} = i \int_M \Omega \wedge \bar{\Omega} = \langle \bar{0} | 0 \rangle \tag{98}$$

and comparing (256, 257) with (97,98) we identify the Weil-Peterson metric with a sub-block of the tt^* metric

$$G_{i\bar{\jmath}} = g_{i\bar{\jmath}} e^K \ . \tag{99}$$

In (93) we have related the curvature of of $g_{i\bar{\jmath}}$ to a bilinear in the 3-point functions and with (99) this becomes the special geometry relation (262). In other words tt^* in genus 0 implies special Kähler geometry, but the main virtue of the formalism is that it generalized readily special Kähler geometry to higher genus. This will become essential to solve the B-model.

It is worth mentioning the closely related *contact term approach* to the definition of the connection (86), see e.g. [145] for a short introduction. It does use conformal invariance and restricts the analysis to exactly marginal ring operators. If the operators are exactly marginal for all values of $t = \{t, \bar{t}\}$ of marginal perturbation parameters as (84) then the most general short distance expansion in the basis e_γ of them is

$$\mathcal{O}_\alpha(z)\mathcal{O}_\beta(0) \sim \frac{G_{\alpha\beta}}{|z|^4} + \Gamma^\gamma_{\alpha\beta}\delta^2(z)\mathcal{O}_\gamma(0) \ . \tag{100}$$

Clearly this expansion is compatible with dimensional analysis, $\delta^2(z) = \frac{\partial}{\partial z}\frac{1}{\bar{z}}$. Marginality implies in first order in t that $\int \mathrm{d}^2 z \langle \mathcal{O}_\alpha(z)\mathcal{O}_\beta(1)\mathcal{O}_\gamma(0)\rangle$ gets only contributions from $z = 1$ and $z = 0$, which explains that only the δ-function appears on the right of (100) in this order. Exact marginality means that scale independence, i.e. vanishing β functions, are maintained to all orders in t. To next order follows the closing on exactly marginal operators, as opposed to arbitrary $(1, 1)$ operators, on the right in (100). The *Zamolodchikov metric* is defined as the sphere correlator

$$G_{\alpha\beta} = \langle \mathcal{O}_\alpha(1)\mathcal{O}_\alpha(0)\rangle \tag{101}$$

and because of conformal invariance it does not require a limit as in the tt^* case. Taking the derivatives with respect to perturbations one gets

$$\frac{\partial G_{\alpha\beta}}{\partial t_\gamma} = \int \mathrm{d}^2 z \langle \mathcal{O}_\alpha(z)\mathcal{O}_\beta(1)\mathcal{O}_\beta(0)\rangle = \Gamma^\delta_{\alpha\gamma}G_{\delta\beta} + \Gamma^\delta_{\gamma\beta}G_{\delta\alpha} \ , \tag{102}$$

which establishes $\Gamma^\delta_{\alpha\gamma}$ as connection of the *Zamolodchikov metric*. So far the discussion of the contact terms has been about a general ansatz and in particular all $\Gamma^\delta_{\alpha\gamma}$ could have been zero. However [95] observed first that in order to ensure marginality in superconformal theories with non trivial triple couplings C^k_{ik} the contact terms have to be present, which is of course required to get (94). The virtue of the $tt*$ equations is to generalize this analysis to all ring states replacing $\Gamma^\delta_{\alpha\beta}$ with $A^\delta_{\alpha\beta}$ and non-conformal theories.

As an exercise one may derive the special geometry relation in $N = (2, 2)$ SCFT using the contact term approach as a specializing of the derivation of the tt^* equations. The decomposition of α, β into $j\bar{\jmath}$ comes from the possibility of picking the holomorphic basis in $N = (2, 2)$ WS theories. Of course the real challenge is to understand the occurrence of the monodromies, which we identified as the data of the theory, which however requires to understand the spacetime Ramond-Ramond states.

5.6 Surgery

As we have seen in the Sec. 5.2 the integral over the metric and positions of insertions points, i.e. the measure on $\overline{\mathcal{M}}_{g,n}$ in topological string, induces a specific dependence on the former data because the measure is not Q-invariant[17], which results in $Q|vac\rangle \neq 0$.

In contrast one can define form theories, such as Chern-Simons theory, in which the Lagrangian is simply metric independent[201], see [159] for a review. These theories are topological without any need to reduce to cohomological sectors and said to be of Schwarz-type, while the ones which need a nilpotent symmetry operator to define a metric independent cohomology of states are called of cohomological- or Witten-type. We can consider 2d cohomological field theories, e.g. topological gauge theories on Riemann surfaces, where we do not integrate over the metric and $Q|vac\rangle = 0$ is maintained. By definition correlation functions in such theories are then topological invariants of the defining geometry, e.g. of three manifold-, knot- and link invariants in the case of $3d$ Chern-Simons theory and of Riemann-surfaces with gauge bundles in the second example.

[17] Somestimes this is called an anomaly.

It is a very remarkable fact that all topological types of manifolds[18] in dim 2,3 can be obtained by surgery operations from primitive building blocks. This is wellknown in the case of Riemann surfaces we start by cutting holes – S^1 boundaries – into two spheres S^2 and glue them together along the boundaries. This procedure of cutting and gluing can be iterated and will obviously construct Riemann surfaces of arbitrary genus, i.e. all topological types. Similarly oriented 3 manifolds can be obtained by starting with two solid tori T^2, i.e. 3 manifolds with boundaries, and glue them together along the T^2. In this procedure one has a freedom to identify the T^2 boundaries up to a $SL(2,\mathbf{Z})$ identification of the $(a,b),(a',b')$ cycles along the two T^2. This procedure can also be iterated by cutting out solid toric from 3 manifolds and glueing along them along the T^2 boundaries with the $SL(2,\mathbf{Z})$ freedom mentioned above. This is surgery operation or rather it inverse is known as Heegaard splitting.

For physical theories on these geometries a very natural question is how the correlation functions itself behave under the surgery operations. This can be addressed already in non-topological theories, if the gluing is compatible with the addtional structure that is needed to define the theory. A wellknown example is Segals operator approach to 2d conformal field theory, where the gluing is defined over a strip broadening the S^1, so that the conformal structure extends to all components. The key properties of the operator formalism needed here are sketched in Sec. 5.5. In the conformal case the above strip is conformally equivalent to the infinite cylinders and does not imply a projection to the groundstate and $|i\rangle$ in (76) is a state in the Hilbert space $\mathcal{H}$ of the conformal field theory. The half disk in (76) can be replaced by any genus Riemann surface (eventually with insertions) bounding the S^1. The gluing of correlators over two boundaries is simply described by inserting a complete set of states $\sum_{ij}|i\rangle\eta_{ij}\langle j|$ at the boundary. In view of the operator state correpondence of 2d field theories we can also write this as $\sum_{ij}\phi^i\eta_{ij}\phi^j$, where the ϕ^i are inserted in the corresponding correlations functions. The inverse of the gluing process is provided by splitting all higher genus Riemann surfaces into pants and caps where operators are inserted. It is obvious that all correlators can be reduced by this procedure to the two point- $\eta_{ij}=\langle\phi_i\phi_j\rangle_0=\langle i|j\rangle$ and the threepoint correlator $c_{ijk}=\langle\phi_i\phi_j\phi_k\rangle_0=\langle i|\phi_j|k\rangle=c^l_{jk}\eta_{il}$ on the sphere, cff. (77, 79). Three basic steps are depicted in Fig. 9.

Very important consistency conditions such as the *associativity* in the splitting of the fourpoint function (first case in Fig. 9) result simpliy from the fact that the geometrical surgery is *not unique* while the physical amplitudes have to be *unique*. In particular in CFT this provides important relations among the conformal blocks of admissible theories. As explained below Fig. (6) the splitting factors through the decomposition of the Hilbert space $\mathcal{H}$ into $\mathcal{H}=\mathcal{H}_0\oplus\mathcal{H}_{E>0}$ cohohomological non-trivial and trivial states. I.e. in cohomological theories the insertions $\sum_{ij}\phi^i\eta_{ij}\phi^j$ depends only the cohomological class $[\phi^i]$ of the operators ϕ^i and the multiplicity of the representatives can be absorbed in the definition of η^{ij}. Simple topological theories can be solved by the consistency conditions i n a bootstrap approach, see Sec. 5.5 Eq. (95) cff.

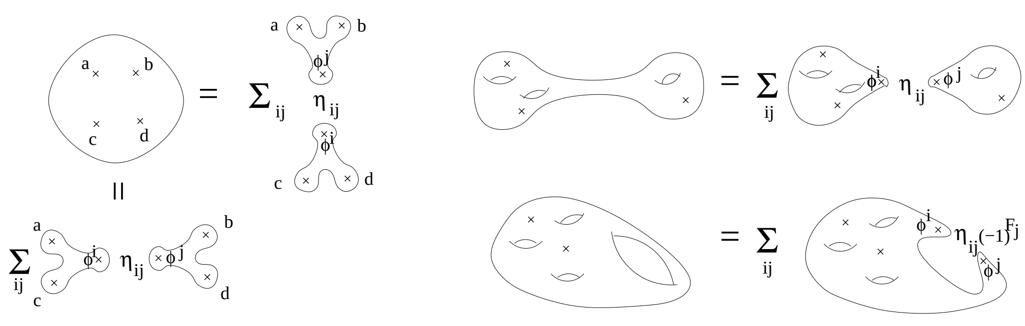

Fig. 9 Graphical representation of three principal splitting procedures.

[18] For simplicty we assume that they are oriented in the following.

We give the formulas at the level of the corellators for the three basic splitting procedures below

$$\langle \phi_a \phi_b \phi_c \phi_d \rangle_0 = c_{abi} \eta^{ij} c_{jcd} = c_{aci} \eta^{ij} c_{jbd}, \tag{103}$$

$$\langle \phi_{a_1} \ldots \phi_{a_n} \rangle_g = \sum_{ij} \langle \phi_{a_1} \ldots \phi_{a_r} \phi_i \rangle_h \eta^{ij} \langle \phi_j \phi_{a_{r+1}} \ldots \phi_{a_n} \rangle_{g-h}, \tag{104}$$

$$\langle \phi_{a_1} \ldots \phi_{a_n} \rangle_g = \sum_{ij} \eta^{ij} (-1)^{F_i} \langle \phi_i \phi_j \phi_{a_1} \ldots \phi_{a_n} \phi_i \rangle_{g-1} . \tag{105}$$

The $(-1)^{F_j}$ factor that occurs for fermions in the cut loop corresponds to the familiar -1 loop factor for fermions in the field theory limit. Its occurance in string perturbation theory is explained in [173].

In Chern-Simons theory with gauge group G the relevant Hilbert space $\mathcal{H}_0$ is spanned by the conformal blocks of the WZW model with gauge group G [201]. The above mentioned $SL(2, \mathbf{Z})$ action is the usual modular transformation realized on these blocks. The pictures are essentially the one in Fig. 9, with the difference that we glue over T^2 boundary conditions, that the rôle of the insertions is played by Wilson loops and that the basic object with boundaries is the filled T^2 instead of the punctured sphere (disk). In effect the surgery procedure provides formulas that express all correlations functions for link invariants on arbitrary 3 manifolds with help of the S, T modular transformations on the WZW characters in terms of the basic link invariants on T^2 [201].

It should be emphasized that the pictures in Fig. 9 are closly related to string loop expansions, but are conceptually different. In the theories where the identities apply, there is no need to integrate over the metric, let alone summing over topologies. On the other hand in string theory there is an expansion over the genus, but the identities are modified due to contact and boundary terms.

Nevertheless the above surgery approach plays an important rôle in the calculation of topological string amplitudes. Obviously a surgery procedure in non-trivial higher dimensional space-time geometry would be a very important step towards summing over space-time topologies as required by quantum gravity. In the last two years notable progress has been made in developing such space-time surgery for non-compact Calabi-Yau manifolds. Let us list some typical situations

- In the local Calabi-Yau geometry $\mathcal{O}(k_1) \oplus \mathcal{O}(k_2) \to \Sigma_g$ with $k_1 + k_2 = 2g - 2$ the geometry is described by line bundels over Σ_g. In this case [29] provide a surgery description on Σ_g, compatible with the glueing of the line bundles, that solves the theory. Remarkably there are parameters in the theory, which interpolate between 2d Yang-Mills theory and the topological string on the local CY geometry.

- The topolological vertex provides a surgery prescribtion, which solves the open and closed topological on any non-compact toric CY manifold [1], see also Sec. 7.

- The mirror of geometry of the vertex is given by 3 punctured sphere with a specified sympletic structure[2]. Surgery of Riemann surfaces compatible with the sympletic structure, i.e. up to W^∞ transformations, provides the general amplitudes[2].

6 The topological A-model

As explained in Sec. 5 only the $U(1)_A$ symmetry is at risk to become anomalous. The A model, which is obtained by twisting the spin connection with the gauged the $U(1)_V$ vector symmetry symmetry, can be defined for all geometries that allow for an σ-model in which Q_A can be defined as in 5. It does not require conformal invariance and exists in particular on any Kähler manifold. In fact only a symplectic structure and a compatible almost structure will be required below[19].

[19] More general realizations e.g. in the setting of [83] are possible.

6.1 A model without worldsheet gravity

In this section we want to describe the operators and correlation functions of topological A topological and their relation to the geometry of the target space M. We call the anticommuting scalars from table 2 $\chi^i := \psi^i_-$ and $\chi^{\bar\imath} := \bar\psi^{\bar\imath}_+$ and the one forms i.e. sections of K and $\bar K$ are denoted by $\rho^{\bar\imath}_z = \bar\psi^{\bar\imath}_-$ and $\rho_{\bar z} := \psi^i_+$. The action is then

$$L = 2t\int \mathrm{d}^2 z\left(g_{i\bar\jmath}\partial_\nu x^i\partial^\nu x^{\bar\jmath} + i\epsilon^{\mu\nu}b_{i\bar\jmath}\partial_\mu x^i\partial_\nu x^{\bar\jmath} - ig_{i\bar\jmath}\rho^{\bar\jmath}_z D_{\bar z}\chi^i + ig_{i\bar\jmath}\rho^i_{\bar z}D_z\chi^{\bar\jmath} - \frac{1}{2}R_{i\bar k j\bar l}\rho^i_{\bar z}\chi^j\rho^{\bar k}_z\chi^{\bar l}\right)\,, \tag{106}$$

where we added the term involving the antisymmetric 2-form $b_{i\bar\jmath}\in H_2(M,Z)$, which plays an important rôle in the bosonic sector of the topological A model. The relevant fermionic symmetry $\delta = \bar\epsilon_-\bar Q_+ + \epsilon_+ Q_-$ acts by

$$\begin{array}{llll} \delta x^i & = \epsilon_+\chi^i, & \delta x^{\bar\imath} & = \bar\epsilon_-\chi^{\bar\imath} \\ \delta\rho^i_{\bar z} & = 2i\bar\epsilon_-\partial_{\bar z}x^i + \epsilon_+\Gamma^i_{jk}\rho^j_{\bar z}\chi^k, & \partial\chi^{\bar\imath} & = 0 \\ \delta\chi^i & = 0, & \delta\rho^{\bar\imath}_z & = -2i\bar\epsilon_+\partial_z x^{\bar\imath} + \bar\epsilon_-\Gamma^{\bar\imath}_{\bar\jmath\bar k}\rho^{\bar k}_z\chi^{\bar\jmath} \end{array} \tag{107}$$

with $\delta^2 = 0$. There is a fixpoint of δ on fermionic zero mode configuration when x^i a holomorphic map $x:\Sigma_g\to M$, i.e. $\partial_z\bar x^{\bar\jmath} = \partial_{\bar z}x^i = 0$, on which the path integral will localize by the fermionic zero mode integration as in Sec. 3.1, so that the bosonic integration reduced to a integration over the moduli space $\mathcal{M}$ of such holomorphic maps[20]. This moduli space $\mathcal{M} = \mathcal{M}_{g,n}(M,\beta,x)$ is labeled by the following topological data: the genus g of $\Sigma_{g,n}$, the number of marked points n on $\Sigma_{g,n}$ as well as the cycles in M that they map to (this indicated by the argument x in $\mathcal{M}_{g,n}(M,\beta,x)$) and the homology class $\beta = [x_*(\Sigma_g)]\in H_2(M,Z)$ of the image of $\Sigma_{g,n}$ in M. For genus zero we must chose three marked points to stabilize the moduli space, see Sec. 6.3.

The 0-form correlation observables are combinations of $x^i, x^{\bar\imath}$ and $\chi^i,\chi^{\bar\imath}$ the latter anticommutating operators can be identified with the forms on M, i.e $\chi^i\leftrightarrow \mathrm{d}x^i$ and $\chi^{\bar\imath}\leftrightarrow\mathrm{d}x^{\bar\imath}$ One checks now using the super symmetry transformations that under this correspondence Q_- and $\bar Q_+$ are identified with the exterior derivatives of Dolbault cohomology ∂ and $\bar\partial$. Since then $Q = Q_- + \bar Q_+$ is identified with the deRham operator $\mathrm{d} = \partial+\bar\partial$ one can summarize the correspondence between the BRST cohomology of the Q_A and the deRham cohomology of M as follows. For each form

$$W = w_{I_1,\ldots,I_n}(x)\mathrm{d}x^{I_1}\wedge\ldots\wedge\mathrm{d}x^{I_n} \tag{108}$$

on M there is a topological operator

$$\mathcal{O}^{(0)}_{W(P)} = w_{I_1,\ldots,I_n}(x)\chi^{I_1}\ldots\chi^{I_n}(P) \tag{109}$$

of the A-model and the operation of Q_A is identified with the exterior derivative

$$\{Q_A,\mathcal{O}_W\} = -\mathcal{O}_{\mathrm{d}W}\,, \tag{110}$$

where the form degree n of W is identified with the ghost number of $\mathcal{O}_W$, since χ has ghost number $+1$.

The action can be written as

$$S = it\int_\Sigma \mathrm{d}^2 z\{Q,V\} + t\int_\Sigma x^*(\omega),\qquad \text{with}\qquad V = g_{i\bar\jmath}\left(\rho^{\bar\imath}_z\partial_{\bar z}x^j + \partial_z x^{\bar\imath}\rho^j_{\bar z}\right) \tag{111}$$

[20] In considering only $Q_A = \bar Q_+ + Q_-$, i.e. setting $\epsilon_+ = \bar\epsilon_-$ one neglects structure, which would give information about the individual cohomology groups of $\mathcal{M}$.

and

$$\int_{\Sigma} x^*(\omega) = \int_{\Sigma} \mathrm{d}^2 z \left(\partial_z x^i \partial_{\bar z} x^{\bar\jmath} g_{i\bar\jmath} - \partial_{\bar z} x^i \partial_z x^{\bar\jmath} g_{i\bar\jmath}\right) = \omega\cdot\beta \geq 0, \tag{112}$$

where ω is the Kähler form $\omega = i g_{i\bar\jmath}\mathrm{d}x^i \mathrm{d}\wedge x^{\bar\jmath}$ and β is the cohomology class $[x(\Sigma)]$ of the image of Σ. The positivity holds if ω is in the Kählercone. If the antisymmetric tensor field is B is non-zero we replace ω by a complexified Kähler form $\omega_c = \omega + iB = (b_{i\bar\jmath} + i g_{i\bar\jmath})\mathrm{d}x^i \mathrm{d}\wedge x^{\bar\jmath}$.

The correlation function of physical operators

$$\langle \prod_{i=1}^n \mathcal{O}_i\rangle_\beta = e^{-it\beta\cdot\omega}\int_{\mathcal{M}_\beta} \mathcal{D}x\mathcal{D}\chi\mathcal{D}\rho e^{-it\{Q,\int V\}}\prod_{i=1}^n \mathcal{O}_i \tag{113}$$

depends on the metric of M only via the Kähler class ω (or on the complexified Kählerclass ω_c). Other metric dependence in particular on the complex structure of M as well as on Σ_g is contained in V. However this dependencies appears only as a Q exact expression in (113) and decouples by (34). Moreover taking the derivative w.r.t. t implies by (34) that the second factor in (113) is independent of t and the correlation can be calculated for ω in the Kählercone for $\mathrm{Re}t > 0$ in limit of infinite t i.e. at the classical minimum of the action. This is another way to understand the supersymmetric localization. If we write

$$\begin{aligned} S_B &= \textstyle\int_\Sigma g_{i\bar\jmath}\left(\partial_z x^i \partial_{\bar z} x^{\bar\jmath} + \partial_{\bar z} x^i \partial_z x^{\bar\jmath}\right) \\ &= 2\textstyle\int_\Sigma g_{i\bar\jmath}\partial_{\bar z} x^i \partial_z x^{\bar\jmath} + \int_\Sigma x^*(\omega) \end{aligned} \tag{114}$$

then the second term in the second line depends only of the class of the map, so it is obvious that the minimum is taken for holomorphic maps $\partial_{\bar z} x^i = \partial_z x^{\bar\jmath} = 0$. This equation requires to specify a holomorphic structure j on Σ_g and one J on M. For fixed J and fixed j there will no maps for $g > 0$. Only if we couple the theory to gravity and integrate over j we have a chance to get contributions from integrals over moduli spaces $\mathcal{M}_{g,n}(M,\beta,x)$ of an infinite series of holomorphic maps. I.e. the path integral collapses to these integrals over $\mathcal{M}_{g,n}(M,\beta,x)$, which are finite- in fact in many case zero-dimensional.

Let us discuss the selection rules for $g = 0$ correlators $\langle\prod_{k=1}^n \mathcal{O}_{W_k}\rangle_\beta$. We note from table 2 and the identification of χ^i and $\chi^{\bar\imath}$ that χ^i has charge $q_l = -1$ and $q_r = 0$ under the left and right $U(1)_{l/r}$ respectively, while $\chi^{\bar\imath}$ has $q_l = 0$ and $q_r = 1$. Because of the splitting of the tangent bundle of $M = T^{(1,0)} \oplus T^{(0,1)}$ we can associate to $\mathcal{O}_{W_k}$ an element in the Dolbeault cohomology group $H^{(p_k,q_k)}$. Since the vector $U(1)_V$ is unbroken in the quantum theory we get a vector charge conservation constraint $q_V = \sum_{k=1}^n p_k - \sum_{k=1}^n q_k = 0$. For the classical axial charge we would get naively $q_A = \sum_{k=1}^n p_k + \sum_{k=1}^n q_k = 0$. However the $U(1)_A$ is anomalous. Form the kinetic terms of ρ and χ we see that its anomaly is given by the index of the *twisted Dolbeault complex* associated to $D_z, D_{\bar z}$ on Σ (378), which is calculated by the Hirzebruch-Riemann-Roch theorem as explained at the end of Sec. 9.3 to be

$$\begin{aligned} q_A = &\quad \#(\chi 0 \text{ modes}) - \#(\rho\, 0 \text{ modes}) = 2(h^0(x^*(TM)) - h^1(x^*(TM))) \\ = &\quad 2\int_\Sigma \mathrm{ch}(x^*(TM^{(1,0)}))\mathrm{td}(T\Sigma) = 2(c_1(TM)\cdot\beta + \dim_C M\ (1-g)) \ . \end{aligned} \tag{115}$$

Combining the two charge constraints we get

$$\sum_{k=1}^n q_k = \sum_{k=1}^n p_k = c_1(TM)\cdot\beta + \dim_C M(1-g) \ . \tag{116}$$

In particular for $g = 0$ we can have on a Calabi-Yau threefold a non-vanishing coupling $\langle\mathcal{O}_{W_i}(P_i)$ $\mathcal{O}_{W_j}(P_j)\mathcal{O}_{W_k}(P_k)\rangle$, where all W_l are $(1,1)$-forms. We associate a divisor $D_k \in H_4(M)$ to each $W^{(k)}_{(1,1)}$

with the two nondegenerate pairings $\int_M W^{(k)}_{(1,1)} \wedge W_{(l)\ (2,2)} = \delta^k_l$ and $\int_{D_i} W_{(j)\ (2,2)} = \delta^i_j$. One always find a representative of $W^{(k)}_{(1,1)}$ that has δ-function support on D_k. This implies that the point P_k in $\mathcal{O}_{W^k_{(1,1)}}(P_k)$ maps to D_k. With β denoting the cohomology class of the image $[C := x(\Sigma)]$ of the worldsheet in M we can write the product $\beta \cdot \omega = 2\pi \sum_{k=1}^{h^{1,1}} t_k d_k$, where $d_k = C \cap D_k$ is the number of intersections of C with D_k or the degree of C w.r.t. D_k. The map with $d_k = 0\ \forall k$ is special. It is the constant map that maps the worldsheet, for $g = 0$ the three punctured sphere $\Sigma_{0,3}$, to a point in M. For the constant genus zero map the path integral collapses hence to the intersection number of $D_i \cap D_j \cap D_k$. We define $q_k = e^{-2\pi i t_k}$. The general genus zero correlation function is then given by[21] is

$$C_{ijk}(t) = \langle \mathcal{O}_{W_i} \mathcal{O}_{W_j} \mathcal{O}_{W_k} \rangle = D_i \cap D_j \cap D_k + \sum_{\{d_i\} \neq \{0\}} r^{g=0}_{\{d_i\}} \prod_{i=1}^{h^{1,1}} q_i^{d_i}\ , \tag{117}$$

where the $r^{g=0}_{\{d_i\}}$ are the result of the integration over $\mathcal{M}_{0,3}(M, \beta, x)$. They are called genus zero Gromow-Witten invariants.

This deformed intersection (117) is a piece of the structure known as *quantum cohomology ring* of M. It is a deformation of the classical cohomology ring on M by the parameters q_k. One needs in general the deformations of all pairings $[m] : H^{\otimes n} \to H$ indexed by $m \in H^*(\mathcal{M}_{0,n+1})$, see [156] and [46] for a review, which we can be provided on the mirror side. Note that the relation to classical intersections in the limit picks a natural normalization of the operators $\mathcal{O}_W$ and of their two-point functions.

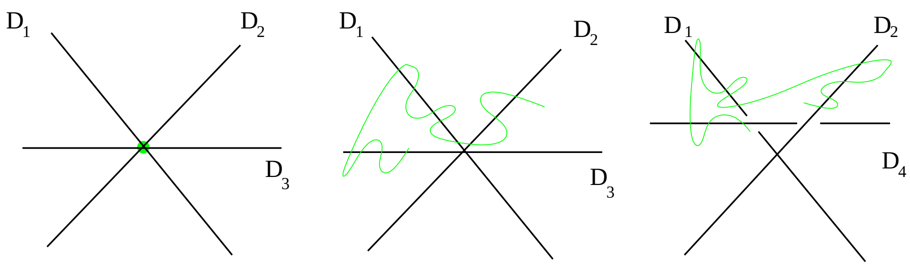

Fig. 10 This figure shows instanton corrections to the coupling C_{123} with $D_1 \cap D_2 \cap D_3 = O(1)$ and C_{124} with $D_1 \cap D_2 \cap D_4 = 0$. From the left to the right we pictured an instanton of degree 0 contributing of $O(1)$ to C_{123}, an instanton of degree $d_1 = 5, d_2 = 3, d_3 = 4$ contributing $\sim q_1^5 q_2^3 q_3^4$ to C_{123} and an instanton of degree $d_1 = 5, d_2 = 4, d_4 = 3$ contributing $\sim q_1^5 q_2^4 q_4^3$ to C_{124}. Roughly speaking for large radii second the coupling C_{124} is expected to be exponentially supressed against the first C_{123}. The precise statement depends on the growth of $r^{g=0}_{\{d_i\}}$. Such collective effects of the intantons can be analyzed best in the B-model.

One collective effect of the instantons corrections is that structure functions $C_{ijk}(t)$ behaves smoothly at singularities in codimension two in M as for instance through flop transitions [213][10].

We note from table 2 and 3 and from (38) that the $U(1)_V$ as well as the $U(1)_A$ charge of the operator $\mathcal{O}^{(2)}_{W_j}$ vanishes. In view of (47) this means that non-vanishing derivatives of $C_{jkl}(t)$ such as

$$\frac{\partial}{\partial t^i} \langle \mathcal{O}_{w^j} \mathcal{O}_{w^k} \mathcal{O}_{w^l} \rangle \Big|_{t^i=0} = \langle \mathcal{O}_{w^j} \mathcal{O}_{w^k} \mathcal{O}_{w^l} \int_\Sigma \mathcal{O}^{(2)}_{w^i} \rangle \tag{118}$$

do exist according to the selection rules. This non-vanishing correlators signal that a non-trivial deformation family exist, but do not contain new information once $c_{jkl}(t)$ is known as function after summing up

[21] We abbreviate $\prod_{i=1}^{h^{1,1}} q_i^{d_i} = q^\beta$ in the following.

all intantons or easier from a B-model calculation. By $SL(2,\mathbf{C})$ invariance on S^2 there is a symmetry between fixing any three of the $\{i,j,k,l\}$ points and integrating over the fourth. This implies that

$$\partial_i C_{jkl}(t) = \partial_j C_{ikl}(t) \tag{119}$$

which is the integrability condition for the existence of a function $\mathcal{F}^{(0)}(t)$ with the property that

$$C_{ijk}(\underline{t}) = \partial_i \partial_j \partial_k \mathcal{F}^{(0)}(t) \ , \tag{120}$$

where we defined $\partial_i = \frac{\partial}{\partial t^i}$. This is in perfect accordance with facts concerning $\mathcal{F}(t)$ from the analysis of the vector moduli space of $N=2$ supergravity in 4d, which is identified in type IIA compactifications with complexified Kähler moduli space. This facts can also be established in the complex structure deformation space, see Sec. (8.5), which again is identified by mirror symmetry with the complexified Kähler moduli space of the A-model. We should finally note that eqs. (118-120) are not written covariantly, but rather in special coordinates. Covariant derivatives are discussed in the B-model section.

6.2 Coupling the A model to worldsheet gravity

While we have prepared our topological theories by the twist to make sense on any genus Riemann surface, we have ignored the degrees of freedom of the worldsheet metric in our discussion so far. As explained in Sec. 5.2 in string perturbation theory one has to integrate over the complex structure of the worldsheet and the position of the insertion points, in other words over the moduli space of Riemann surfaces with n insertion of operators $\mathcal{M}_{g,n}$. We have rightfully ignored that in the genus zero correlator (117), because fixing three points kills the $SL(2,\mathbf{C})$ invariance of S^2, which has no complex structure deformations, so that $\mathcal{M}_{0,3} = point$. Despite the fact that (116) predicts a nontrivial zero point function for $g=1$, without integrating over the complex structure of Σ the answer for the correlation function $\mathcal{F}^{(1)}$ would be generically vanishing. As an intuitive example consider maps from $\Sigma = T^2$ to $M = T^2$, allowed by the selection rule (116). If we fix the complex structure of Σ and M there would be, by definition of inequivalent complex structures, no holomorphic maps unless we hit with the complex structure parameter τ_Σ the one of τ_M. Including all multicoverings [19] the answer $\mathcal{F}^{(1)} = -\log(\eta(\tau_M))\delta(\tau_M - \tau_\Sigma)$ begs to be integrated over τ_Σ as it is natural in string theory. For higher genus (116) predicts vanishing of the correlation functions. That means if we fix the world-sheet metric there are just no holomorphic maps from a genus $g>1$ Riemann surface to M.

6.3 Topological gravity

The simplest example of string theory where integration over the the moduli space discussed in Sec. 5.2 is required is pure topological gravity. This is an good warm up example in which M is replaced by a point. It plays a pivotal rôle for the A- as well as for the B-model coupling to gravity. The calculation of the expected dimension (367) was for smooth curves, which represent an open top dimensional subset of the moduli space of all curves. In order to integrate of $\mathcal{M}_g$ we need some compactification of $\mathcal{M}_g$. Including nodal curves, but so that the the automorphism group, which is finite for smooth curves of $g>1$, stays finite is called the stable *Deligne-Mumford compactification* $\overline{\mathcal{M}_g}$. Genus zero curves have a $SL(2,\mathbf{C})$ automorphism and $g=1$ curves an $z \to z+c$ automorphism. These can be killed either by a puncture or the position of a node. Because of the former fact it is convenient to extend the discussion right away to punctured Riemann surfaces. Inserting a so called *puncture operator* $\mathbf{1}$ at the point $x \in \Sigma$ in the path integral means that we want to restrict the diffeomorphism group in (42) to a subgroup which preserves that point x. We call the moduli space with n punctures $\mathcal{M}_{g,n}$. Its dimension is enhanced by n complex dimensions relative to $\mathcal{M}_g$. Intuitively one may picture the movement of the point as additional dimension of $\mathcal{M}_{g,n}$. The more accurate picture is complementary. The restriction of the diffeomorphism group by the part, which moves the point in the denominator of (42) enhances the dimension.

Let us call punctures and an ordinary double points (nodes) *special points* of Σ. The *Deligne-Mumford compactifi cation* $\overline{\mathcal{M}_{g,n}}$ is the appropriate compactification to define good measures on $\overline{\mathcal{M}_{g,n}}$ in topological string theory. [202, 195]. It allows the above special points under the condition that they do not meet. The further conditions that

- (i) every irreducible component of genus 0 has at least three special points
- (ii) every irreducible component of genus 1 has at least one special point

guarantee that there are no continuous automorphism groups acting on $\overline{\mathcal{M}_{g,n}}$. Finite automorphism groups Aut are like gauge symmetries which are divided out. The resulting orbifold is the connected, irreducible, compact, non-singular *Deligne Mumford stack* of dimension $3g-3+n$, denoted also by $\overline{\mathcal{M}_{g,n}}$.

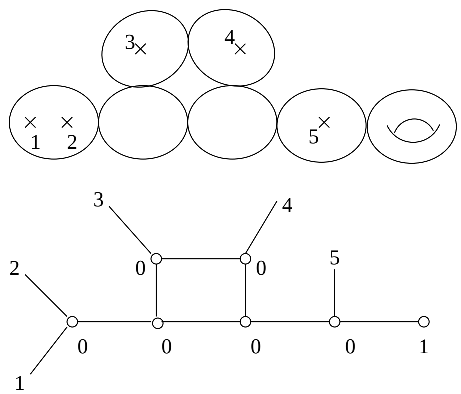

Fig. 11 This fi gure shows a stable degeneration of a genus 2 curve with 5 marked points in $\overline{\mathcal{M}_{2,5}}$ as actual confi guration above and as dual graph below.

The positive dimension of this space appears as an anomalous negative ghost number violation in the BRST quantization. In topological gravity it is compensated by insertion of descendant fields $\sigma_n(x)$ whose form degree is counted as positive ghost number. These descendant fields are constructed geometrically as the first Chern class of the complex line bundle $L_i = x_i^*(\omega)$ over $\overline{\mathcal{M}_{g,n}}$ in the universal curve $\mathcal{C}\overline{M_{g,n}}$, which is induced from the restriction of the holomorphic cotangent bundle $T^*\Sigma_g|_{x_i}$ of Σ_g to x_i. The universal curve is the fibration over $\overline{\mathcal{M}_{g,n}}$ whose fibers are the Riemann surfaces with n punctures described by the point $[\Sigma, x_1, \ldots, x_n] \in \overline{\mathcal{M}_{g,n}}$. $\omega = \mathcal{K}_{\mathcal{C}/\mathcal{M}}$ is the roughly the cotangent bundle along the fibres. More precisely since nodal singularities are allowed it is the corresponding *relative dualizing sheaf*. L_i are line bundles over $\overline{\mathcal{M}_{g,n}}$, see Fig. 12.

The first Chern class $\psi_i = c_1(L_i)$ might be represented by the $(1,1)$ curvature form (358)

$$\psi_i = -\frac{i}{\pi}\partial\bar{\partial}\log|\sigma_{x_i}|^2 \tag{121}$$

on $\mathcal{M}_{g,n}$, where σ_{x_i} is a meromorphic section of L_i. It can be wedged to define the general *descend operators* $\sigma_n(x_i) := \psi_i^n$ of form degree or ghost number $2n$. We can also consider the insertion of $\sigma_0(x) = \psi^0(x)$, the above mentioned *puncture operator*. What this means is that we change the moduli $\mathcal{M}_g$ to one $\mathcal{M}_{g,1}$ in which the diffeomorphism group in (42) is restricted to fix one point without doing anything else. The selection rule for a non vanishing correlator

$$\langle \sigma_{d_1} \ldots \sigma_{d_n} \rangle = \int_{\mathcal{M}_{g,r}} \psi_1^{d_1} \wedge \ldots \wedge \psi_n^{d_n} \tag{122}$$

is now given simply by counting form degrees of insertions against the dimension of $\mathcal{M}_{g,n}$, which yields the condition [202, 195]

$$\sum_{i=1}^{n}(d_i - 1) = 3g - 3 \ . \tag{123}$$

Two easy and universal properties of the correlators (122), called topological recursion relations [200], are the *puncture equation*, referred also to as the *string equation*

$$\langle \sigma_0 \sigma_{d_1} \dots \sigma_{d_n} \rangle = \sum_{d_i \neq 0} \langle \sigma_{d_1} \dots \sigma_{d_i - 1} \dots \sigma_{d_n} \rangle \tag{124}$$

and the *dilaton equation* [200]

$$\langle \sigma_1 \sigma_{d_1} \dots \sigma_{d_n} \rangle = (2g - 2 + n) \langle \sigma_{d_1} \dots \sigma_{d_n} \rangle \ . \tag{125}$$

Let us review the original arguments [200] that lead to (124, 125), which can be made mathematically rigorous [106]. In both equations a puncture is removed from the left relative to the right side and the nontrivial relation comes from loci in $\overline{\mathcal{M}}_{g,n+1}$, where this removed point x_0 is together with exactly one other x_j in a genus zero component S^2_j of the degenerate curve (the bold fibre in Fig. 12), so that its removal destabilizes $\overline{\mathcal{M}_{g,n}}$. We will discuss the generic case and leave the special $g = 0$, $n = 2$ and $g = 1$, $n = 1$ situations to the reader. The key point is that $L_i = x_i^*(\omega)$ over $\overline{\mathcal{M}}_{g,n+1}$ and $L_i' = x_i^*(\omega')$ over $\overline{\mathcal{M}}_{g,n}$, $i = 1, \dots, n$ are related in a non trivial way. If it would be the case that $L_i = \pi^*(L_i')$ then starting with the right hand side we could argue that the left hand side in (124, 125) vanishes due to (123).

These relevant issues occur at the divisors D_j in $\overline{\mathcal{M}}_{g,n+1}$ (in Fig. we show just D_1). The forgetful map $\pi : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ is a fibering map, whose fibers describes the position of the point x_0, which is essentially Σ. It lifts to the universal curve $\pi_{\mathcal{C}} : \mathcal{C}\overline{\mathcal{M}}_{g,n+1} \to \mathcal{C}\overline{\mathcal{M}}_{g,n}$ not as a fibering as $\pi_{\mathcal{C}}$ also contracts the unstable S^2_j. There is an isomorphism $\alpha : \overline{\mathcal{M}}_{g,n+1} \cong \mathcal{C}\overline{\mathcal{M}}_{g,n}$, but it is not compatible with the fibering $\pi : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$.

Now if s is a section of ω' then the evaluation $x_j^*(s)$ at x_j pulls back under π^* to a section $\pi^* x_j^*(s)$ of ω over $\overline{\mathcal{M}_{g,n+1}}$. A simple local model near the contracted S^2_j shows that $\pi^* x_j^*(s)$ vanished with order one at D_j. This implies $L_j = \pi^*(L_j) \otimes \mathcal{O}(D_j)$. With $\psi_j = c_1(L_j)$ and the properties about characteristic classes summarized in Sec. (9.3) one gets

$$\psi_j = \psi_j^* + [D_j] \ . \tag{126}$$

The algebraic identity

$$\psi_j^n = (\psi_j^*)^n + [D_j] \sum_{k=1}^{n-1} \psi_j^k (\psi_j^*)^{n-k-1} \tag{127}$$

simplifies to $\psi_j^n = (\psi_j^*)^n + [D_j](\psi_j^*)^{n-1}$ as $\psi_j = c_1(L_j)[D_j] = 0$, because L_j is trivial over D_j as the sphere S_2^j with its three special points is rigid.

So we can evaluate

$$\begin{aligned} \langle \sigma_0 \sigma_{d_1} \dots \sigma_{d_n} \rangle &= \int_{\overline{\mathcal{M}}_{g,n+1}} \mathbf{1} \wedge_{i=1}^n \psi_i^{d_i} = \sum_{j=1}^n \int_{\overline{\mathcal{M}}_{g,n+1}} [D_j] \wedge_{i=1}^n (\psi_i^*)^{d_i - \delta_{ij}} \\ &= \sum_{j=1}^n \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \wedge \dots \psi_j^{d_j - 1} \dots \wedge \psi_1^{d_1} = \sum_{j=1}^n \langle \sigma_{d_1} \dots \sigma_{d_j - 1} \dots \sigma_{d_n} \rangle \end{aligned} \tag{128}$$

Here we used $[D_i]\cdot[D_j]=0$ which follows from the definition and in the third equality we have integrated over the fiber of $\pi:\overline{\mathcal{M}}_{g,n+1}$ where $[D_j]$ represents a section with a simple zero. Very similarly one concludes that $L_0=\alpha^*(\omega')\otimes_{j=1}^n\mathcal{O}(D_j)$ is a degree $2g-g+n$ section of a line bundle over the fibre of π. We evaluate then again by integration over the fibre

$$\langle\sigma_1\sigma_{d_1}\ldots\sigma_{d_n}\rangle \;=\int_{\overline{\mathcal{M}}_{g,n+1}}\psi_0\wedge_{i=1}^n\psi_i^{d_i}=(2g-2+n)\langle\sigma_{d_1}\ldots\sigma_{d_n}\rangle \tag{129}$$

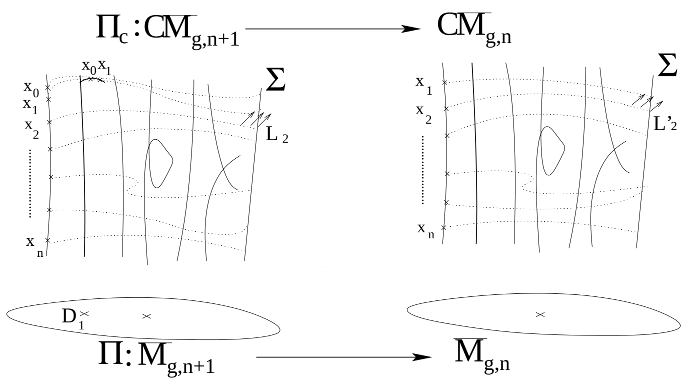

Fig. 12 Universal Curve $\mathcal{C}M_{g,n+1}$ and the forgetful map. The nodal and reducible fi bre are displayed, because there are such fi bres, but they plays no rˆole in the derivations of the *string and dilaton* equation. They would play a rˆole in recursion relations among different genera, which is hard from the algebraic point op view.

With the recursive relations (124,125) and the initial conditions that the moduli space of a three pointed sphere is a point $\langle\sigma_0\sigma_0\sigma_0\rangle=1$ and $\langle\sigma_1\rangle=\frac{1}{24}$ one can solve as an exercise all $g=0,1$ correlators. It seems natural to try next to consider maps which "forget" nodes to get a recursion among correlations with different genera. From the algebraic point of view taken above this turns out to be surprisingly difficult.

Let now $\{d_i\}$ the set of all nonnegative integers and define

$$F_g(t_0,t_1,\ldots)=\sum_{\{d_i\}}\langle\prod\tau_{d_i}\rangle_g\prod_{r>0}\frac{t_r^{n_r}}{n_r!}\ , \tag{130}$$

with $n_r=\mathrm{Card}(i:d_i=r)$ and

$$F=\sum_{g=0}^{\infty}\lambda^{2g-2}F_g\ , \tag{131}$$

the free energy of 2d topological gravity. Where we rescaled the operators $\tau_n=(2n+1)!!\sigma_n$ for latter convenience. [200] conjectured that the partition function $Z=e^F$ satisfies the *Virasoro constraints*

$$L_nZ=0,\qquad n\geq-1\qquad\qquad \text{with}\quad [L_n,L_m]=(n-m)L_{n+m} \tag{132}$$

with

$$\begin{aligned}
L_{-1}\; &= -\frac{1}{2}\frac{\partial}{\partial t_0}+\frac{1}{4}t_0^2+\sum_{i=1}^{\infty}\frac{2i+1}{2}t_i\frac{\partial}{\partial t_{i-1}},\\
L_0\; &= -\frac{1}{2}\frac{\partial}{\partial t_1}+\sum_{i=0}^{\infty}\frac{2i+1}{2}\frac{\partial}{\partial t_i}+\frac{1}{16},\\
L_n\; &= -\frac{1}{2}\frac{\partial}{\partial t_{n-1}}+\sum_{i=0}^{\infty}\frac{2i+1}{2}t_i\frac{\partial}{\partial t_{i+n}}+\frac{\lambda^2}{4}\sum_{i=0}^{n}\frac{\partial^2}{\partial t_{i-1}\partial t_{n-i}}\ ,
\end{aligned} \tag{133}$$

As an exercises one may check that (124,125) are equivalent to $L_{-1}Z = L_0 Z = 0$. It is well known [200] [59] that (132) is equivalent to the fact that Z is the τ function of the KDV hierachy and fulfills the dilaton equation.

6.4 Kontsevich model

All proofs of (132) are combinatorial. The first is by Kontsevich, who interprets a direct evaluation of the correlators as ribbon graphs of the shifted Airy function matrix model, which in turn can by viewed as the Akhiezer Baker function of the KdV hierarchy. This beautiful work [140] has been reviewed in many places e.g. [56, 54].

It employs ideas of open/closed string duality without using those terms. As mentioned in the introduction Kontsevich's hermitian matrix model is not related to 2d gravity by a double scaling limit. It is rather a direct combinatorial tool, whose ribbon graphs expansion calculate all correlators of 2d gravuty, i.e. the intersection numbers on the moduli space of n punctured genus g Riemann surfaces $\mathcal{M}_{g,n}$.

Important in associating combinatorial data to the cell decomposition of $\Sigma_{g,n}$ are Jenkins-Strebel quadratic differentials[22]. These are differentials $\phi = \phi(z)\mathrm{d}z^2$ on $\Sigma_{g,n}$ which define a flat non-degenerate metric $|\phi(z)||\mathrm{d}z|^2$ outside their discrete sets of zeros. A horizontal trajectory of ϕ is an curve in $\Sigma_{g,n}$ along which $\phi(z)$ is real and positive. Jenkins-Strebel quadractic differentials have the additional property that the union of nonclosed trajectories has measure zero. At a zero of order k of ϕ $k+2$ non-closed horizontal trajectories meet. Closed horizontal curves are concentric arround poles of ϕ. The following theorem makes this picture precise *Theorem* [182]: Let $\Sigma_{g,n}$ be a connected Riemann surface with n poles at the distinct points $x_1, \ldots, x_n$, $n > 2 - 2g$ and associated positive real numbers $p_1, \ldots, p_n$. Then there is a unique Jenkins-Strebel differential on $\Sigma_g \setminus \{x_1, \ldots, x_n\}$, whose maximal ring domains are n punctured disks surounding x_i with circumference k_i.

The non-closed orbits form a graph with valence $v \geq 3$ drawn on $\Sigma_{g,n}$. Thickening the edges inside $\Sigma_{g,n}$ one obtains a *ribbon graph* Γ_ϕ, which inherits the orientation of Σ_g. Vice versa one can reconstruct from the combinatorial data $\{p_i\}$ and the *oriented graph* Γ_ϕ the Riemann surface plus a Jenkin-Strebel differential. That is $\Sigma_{g,n}$ and ϕ are one to one to the *ribbon graph* Γ_ϕ with the total length of the edges making a closed loop fixed. The complement to Γ_ϕ in $\Sigma_{g,n}$ are all disks, i.e. Γ_ϕ defines a *cell decomposition* of $\Sigma_{g,n}$. These crucial facts are depicted in figure Fig. 13 and 14.

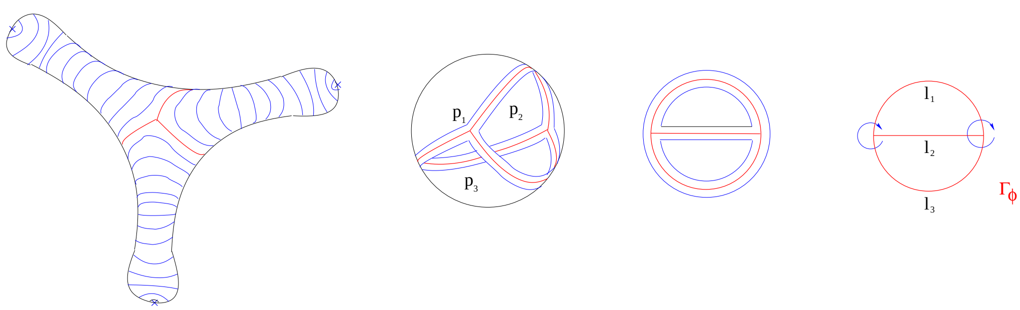

Fig. 13 From a planar graph to a genus 0 surface with 3 holes. The fat lines are the non-closed horizontal trajectories of a Jenkins-Strebel differential, with two first order zeros.

A metric on the ribbon graph is provided by associating to each edge a length l_i and consider the standard metric in $\mathbf{R}_+^{\#edges}$. With $\mathcal{M}_{g,n}^{comb}$ one denotes the *set of equivalence classes* of connected ribbon graphs with the above metric. From the metric of the graph one can reconstruct a metric in a conformal class on $\Sigma_{g,n}$ with a *unique complex structrure*. Therefore the map $\mathcal{M}_{g,n} \times \mathbf{R}_+^n \to \mathcal{M}_{g,n}^{comb}$, which is induced by associating to $\Sigma_{g,n}$ with a choice of $\{p_1, \ldots, p_n\}$ the critical graph of a Jenkins-Strebel differential with

[22] A subtle point in this association has been clarified in [217], where the reader finds also a review of Kontsevich's proof.

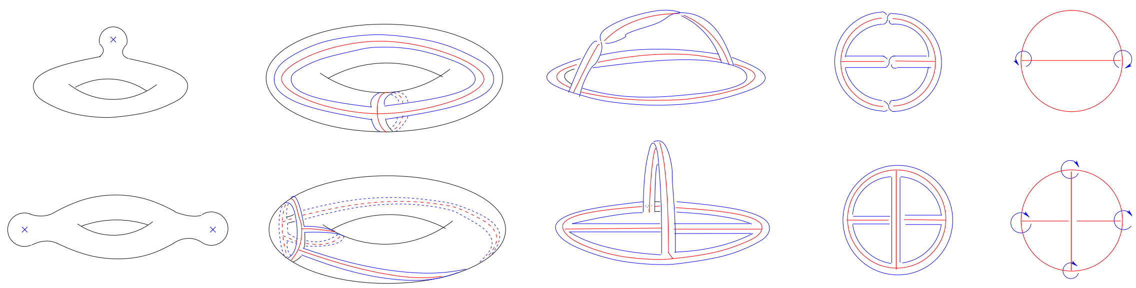

Fig. 14 From genus 1 surface with one and two holes to non-planar graphs.

edge length $\{l(e)\}$ is one to one. One can restrict to trivalent graphs as these correspond to the relevant top dimensional strata in $\mathcal{M}_{g,n}^{comb}$, then $\dim_{\mathbf{R}}(\mathcal{M}_{g,n}^{comb}) = 6g - 6 + 3n$.

Next one has to reconstruct the line bundles L_i used to define the operators $\psi_i = c_1(L_i)$ in Sec. 6.3 combinatorially. They are associated with the holes, which combinatorially become polygons bounding the ribbon graph. [140] denotes by $BU(1)^{comb}_{\leq N}$ the set of equivalence classes of series of the length $\{l_1, \dots, l_k\}$, $1 \leq k \leq N$ of edges in the polygons modulo cyclic permutations and denotes the direct limit of $BU(1)^{comb}_{\leq N}$ over all N by $BU(1)^{comb}$. Over $BU(1)^{comb}$ there is a contractable S^1 bundle $EU(1)^{comb}$, whose fibres are the polygons with edge lengths $\{l_1, \dots, l_k\}$. Now $\mathcal{M}_{g,n}^{comb}$ maps in an obvious way to $(BU(1)^{comb})^n$ by applying the construction to all boundaries. It is a key fact proven in [140] that these maps extend continuosly to a map $\Pi : \overline{\mathcal{M}}_{g,n} \times \mathbf{R}_+^n \mapsto (BU(1)^{comb})^n$, i.e. to the stable compactification of $\overline{\mathcal{M}}_{g,n}$ discussed in Sec. 6.3. On the i'th $BU(1)$ one can define the 2-form

$$\tilde{\omega}_i = \sum_{1 \leq m \leq n \leq k-1} \mathrm{d}\left(\frac{l_n}{p_i}\right) \wedge \mathrm{d}\left(\frac{l_m}{p_i}\right), \tag{134}$$

where p_i is the total length of the i'th polygon. These 2-forms pullback under Π to ω_i representing the class $c_1(L_i)$. Roughly speaking Π extends to a bundle map and the inverse image of the S^1 bundle over the i'th $EU(1)^{comb}$ is associated to the circle bundles in the complex line bundles L_i over $\overline{\mathcal{M}}_{g,n}$. Using $\mathcal{M}_{g,n} \times \mathbf{R}_+^n \sim \mathcal{M}_{g,n}^{comb}$ one has $\langle \tau_{d_1} \dots \tau_{d_n} \rangle = \int_{\pi^{-1}(\underline{p})} \prod_{i=1}^n \omega_i^{d_i}$, where $\pi : \mathcal{M}_{g,n}^{comb} \to \mathbf{R}_+^n$ is the projection on the length of all the holes. The integral is to be performed over the open strata in $\mathcal{M}_{g,n}^{comb}$ represented by the ribbon graphs Γ . To fix the signs, i.e. the orientation of these open strata, one evaluates the volume form $\mathrm{Vol} = \omega^n/d!$ on the complex $d = 3g - 3 + n$ dimensional fibre of the map π and uses $\langle \prod_i \tau_{d_i} \rangle > 0$. With $\omega = \sum_{i=1}^n p_i^2 \omega_i$ one obtains $\mathrm{vol}(\pi^{-1}(p_1, \dots, p_n)) = \int_{\pi^{-1}(\underline{p})} \mathrm{Vol} = \frac{1}{d!} \int_{\pi^{-1}(\underline{p})} \left(\sum_{i=1}^n p_i^2 \omega_i\right)^d = \sum_{|d_i|=d} \prod_{i=1}^n \frac{p_i^{2d_i}}{d_i!} \langle \tau_{d_1} \dots \tau_{d_n} \rangle$. Here the restriction $|d_i| = d$ on the sum comes from (123).

After a Laplace transformations $\int \mathrm{d}p_i e^{-\lambda_i p_i}$ of both sides and using the fact that ω has constant coefficients in the length $l(e)$ of the edges of Γ one gets the following main combinatorial identity

$$\sum_{|d_i|=d} \langle \tau_{d_1} \dots \tau_{d_n} \rangle_g \prod_{i=1}^n \frac{(2d_i - 1)!!}{\lambda_i^{2d_i+1}} = \sum_{\Gamma_{g,n}} \frac{2^{-\#v}}{|\Gamma_{g,n}|} \prod_{e \in \Gamma_{g,n}} \frac{2}{\lambda(e)} \ . \tag{135}$$

Here $\Gamma_{g,n}$ is summed over all trivalent graphs of the indicated topology, v, e are vertices and edges of the graph and $|\Gamma_{g,n}|$ is the order of the automorphism group of the graph. $\lambda(e)$ associates to a "ribbon" e in the graph $\lambda(e) = \lambda_{i(e)} + \lambda_{j(e)}$, the indices of the two "edges" of the ribbon.

The combinatorial expansion on the right hand side of (**??**) is the graph expansion the free energy of a hermitian $N \times N$ matrix model with the hermitian $N \times N$ matrix X as field and hermitian $N \times N$ matrix

Λ as source. The partition function is

$$Z(\Lambda) = C_\Lambda \int DX e^{\frac{i}{6}\mathrm{tr}(X^3)-\frac{1}{2}\mathrm{tr}(X^2\Lambda)} \tag{136}$$

where the normalization C_Λ is fixed so that the integral for the free theory gives $C_\Lambda \int DX e^{-\frac{1}{2}\mathrm{tr}(X^2\Lambda)=:\int \mathrm{d}\mu(\Lambda)=1}$. Now the main claim is that with the identification

$$t_i(\Lambda) = -(2i-1)!!\mathrm{tr}\Lambda^{-(2i+1)} \ . \tag{137}$$

on get the important identity

$$F(\underline{t(\Lambda)}) = \log(Z(\underline{t(\Lambda)})) = \sum_{\Gamma_{g,n}} \frac{2^{-\#v}}{|\Gamma_{g,n}|} \prod_{e\in\Gamma_{g,n}} \frac{2}{\lambda(e)} \ . \tag{138}$$

Here we identify the ribbon propagator $\int X_{ij}X_{kl} = \delta_{il}\delta_{jk} = \frac{2}{\Lambda_i+\Lambda_j} = \frac{2}{\lambda(e)}$. The proof of (138) is provided by the analysis of the disconnected Feymann graph expansion of the matrix model and established the equivalence (138) as asymptotic expansions. It is obvious that for finite N the t_i, symmetric functions in Λ_i, in (**??**) are become dependent for large i. In order to calculate a given intersection $\langle \prod_{i=1}^k \tau_{d_i}\rangle$ one has to go to high enough N to identify the right coefficient on the right handside. If the rank N of the matrix X is finite one probes a finite dimensional subspace in the infinite dimensional coupling space of 2d gravity, but the results in this subspace are exact and in particular independent of N.

More recently a second combinatorial proof has been given by Okounkov and Pandharipande [168]. Very recently a proof has been given by Mirzakhani [160], which establishes an interesting relation to the Weil-Peterson volume of the moduli space of hyperbolic Riemann surfaces with geodesic boundary conditions that awaits physical interpretation. It is surprising that the established solutions of the simplest model of topological string theory do not follow from the physical approach of closed string theory. However recently solutions of this system have been obtained using *open string* theory and open/closed duality[2] [82].

6.5 Physical approach to 2d gravity

There is a physical argument for recursion relations based on the contact term algebra of two dimensional gravity and sewing rules for string theory[195], which up to a normalization of $\langle\tau_0\tau_0\tau_0\rangle = 1$ reproduces all correlation functions and is equivalent to 132, see [56]. The recursion includes a reduction of the genus

$$\begin{aligned}\langle\tau_n \textstyle\prod_{k\in S}\tau_k\rangle_g &= \sum_{k\in S} P_k^{(n)} \langle \tau_{n+k-1} \prod_{k'\neq k}\tau_{k'}\rangle_g + \sum_{i+j=n-2} A_{ij}^{(n)}\langle\tau_i\tau_j\prod_{k\in S}\tau_k\rangle_{g-1} \\ &+ \sum_{h=1}^{g-1} \sum_{\substack{S=S_1\cup S_2 \\ i+j=n-2}} B_{ij}^{(n,h)} \langle\tau_i \prod_{k\in S_1}\tau_k\rangle_h \langle\tau_i\prod_{k\in S_2}\tau_k\rangle_{g-h}\end{aligned} \tag{139}$$

This recursion reads very naturally as if we could have reduced in addition to the unstable meeting of two points also the nodes and irreducible fibres in Fig. 12 and treat all boundaries of the moduli space $\overline{\mathcal{M}}_{g,n}$ at the same footing as in Fig. 15. [195] determine the $P_k^{(n)} = 2k+1$ and $A_{ij}^{(n)} = \frac{1}{2}$ and $B_{ij}^{(n,h)} = \frac{1}{2}$ using contact term manipulations. The puncture and the dilaton equation, which is implied in (139) can be established rigorously in this way. However for the determination of all A, B, P one needs the assumption of the constistency of surgery procedure at the level of the correlators, see Sec. 5.6, to restrict the contact term algebra. Therefore, even though (139) implies (132), the approach of [195] is not a quite a proof of (132).

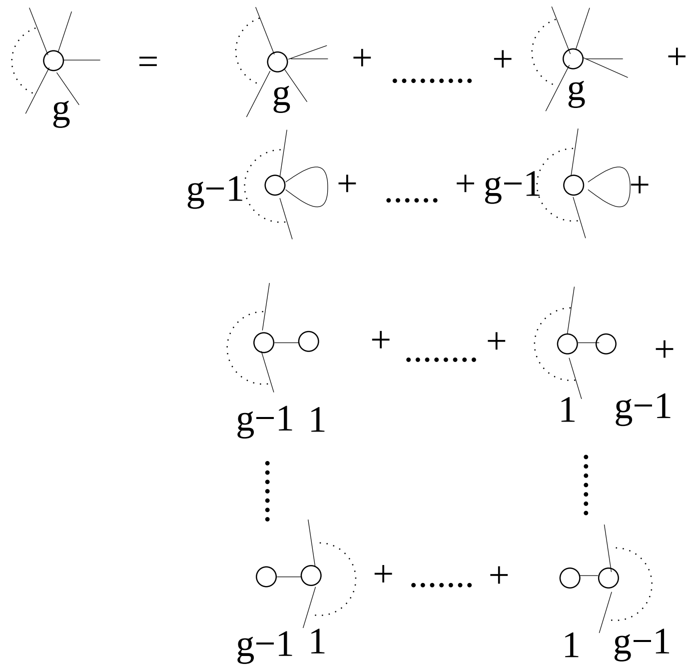

Fig. 15 Degenerations of a genus g surface corresponding to the codim one boundary in $\overline{\mathcal{M}}_{g,n}$ in the dual graph notations where closed lines are double points and open lines are operator insertions.

Let us sketch the argument [195] of the identification of the 2d field theory formalism with the geometrical approach. 2d gravity can be constructed as cohomological supersymmetric theory with two nilpotent operators Q representing the total BRST charge and $Q_- = Q_s - \bar{Q}_s$, where Q_s are the left and $\bar{Q}_s$ the right super charge. The decoupling of the WS metric is not complete $\{Q_s, \beta^k\} = \{Q, \beta^k\} = T^k$, so that Q and Q_s insertions in correlations act on the measure (308) and yield by (46) derivatives on $\mathcal{M}_{g,n}$. The decisive field is the 2d dilaton ϕ. Other fields have the following relation to ϕ

$$\begin{aligned} \psi - \bar{\psi} &= \tfrac{1}{2}\{Q_-, \phi\}, & \gamma_0 &= \tfrac{1}{2}\{Q, \{Q_-, \phi\}\} \\ (\omega, \bar{\omega}) &= \tfrac{1}{2}(\partial\phi, -\bar{\partial}\phi), & (\psi_0, \bar{\psi}_0) &= \tfrac{1}{2}(\partial\psi, -\bar{\partial}\psi), \qquad R = \mathrm{d}\omega = \partial\bar{\partial}\omega\ . \end{aligned} \tag{140}$$

The theory has a gauge fixing sector similar to the superstring and in particular anti-commutating (b, c) ghost and commuting (β, γ) ghosts with BRST symmetry $\delta_{brst}\omega = \phi_0 + \mathrm{d}c_0$, $\delta_{brst}c_0 = \gamma_0$, $\delta_{brst}\omega = \phi_0 + \mathrm{d}c_0$, $\delta_{brst}\psi_0 = \mathrm{d}\gamma_0$ and $\delta_{brst}\gamma_0 = 0$. The field equations imply $\gamma_0 = \frac{1}{2}(\partial\gamma + \gamma\partial\phi + c\partial\psi + c.c.)$. The main claim is that formally the ψ_i classes are

$$\psi_i \sim (\gamma_0 + \psi_0 + \mathrm{d}\omega)(x_i), \tag{141}$$

so that formally $\sigma_n \sim (\gamma_0 + \psi_0 + \mathrm{d}w)^n$. The point is that the insertion of $\langle(\gamma_0 + \psi_0 + \mathrm{d}\omega)\mathcal{O}\rangle_g$ produces by (140) and (46) a two-form on $\overline{\mathcal{M}}_g$ namely $\partial\bar{\partial}\langle\phi\mathcal{O}\rangle_g$, where the $\partial\bar{\partial}$ operators act on $\overline{\mathcal{M}}_{g,n}$ and $\mathcal{O}$ stands for cohomological states. Note that the $\partial, \bar{\partial}$ derivatives act in the direction of the complex moduli by (46) as well as in the direction of the fibre of the universal curve. There are operators $\mathcal{O}_i = e^{\pi(x_i)}$, so that $\langle\phi\mathcal{O}_i\rangle_g = \log|\sigma_{x_i}|^2$ hence by (121) we get the claimed relation. The puncture operator plays a special rôle in the field theory formalism and is given in the -1 picture [173] by $P(x) = c\bar{c}\delta(\gamma)\delta(\bar{\gamma})(x)$. In order to proof the puncture equation (124) one has to understand the contact term between P and σ_n, that is the integral

$$\begin{aligned}\int_{D_\epsilon} P|\sigma_n\rangle &= \int_{|x|<\epsilon} \mathrm{d}^2x P(x)|\sigma_n\rangle = \int_{|q|<\epsilon} \frac{\mathrm{d}^2 q}{|q|^2} G_0\bar{G}_0 q^{L_0}\bar{q}^{L_0} P(1)|\sigma_n\rangle \\ &= \int_{|q|<\epsilon} \frac{\mathrm{d}^2 q}{|q|^2} G_0\bar{G}_0 q^{L_0}\bar{q}^{L_0}\frac{1}{2} Q Q_-|\sigma_{n-1}\rangle = \int_{|q|<\epsilon} \mathrm{d}^2 q\partial_q\bar{\partial}_{\bar{q}}(q^{L_0}\bar{q}^{L_0}|\sigma_{n-1}\rangle) \\ &= \int_{|q|<\epsilon} \mathrm{d}^2 q\partial_q\bar{\partial}_{\bar{q}}(\log|q|^2)|\sigma_{n-1}\rangle = |\sigma_{n-1}\rangle\end{aligned} \tag{142}$$

where we replaced in the second equality the position dependence of the puncture operator by a neck of length $T = -\log|q|$, see figure 16. The insertion of the $G_0, \bar{G}_0$ comes from the integral over the superpartners of the modulus q. From the definition (141) and $\sigma_n = \psi^n$ as well as (140) follows the third equality. The $G_0, \bar{G}_0$ play the same rôle as the Q_+, Q_- in the derivation (93) namely to produce the derivatives $q\partial_q\bar{q}\bar{\partial}_{\bar{q}}$ from the anti commutator $\{Q, G_0\} = \{Q_-, G_0\} = L_0$. The logarithm occurs, because $[L_0, \phi(0)] = \phi_0 + 1$ with $\phi_0 = \oint \partial\phi$ and $\phi_0|\sigma_{n-1}\rangle = L_0|\sigma_{n-1}\rangle = 0$. Regular terms vanish under the integral. Hence one concludes that

$$P(x)|\sigma_n\rangle = \delta^{(2)}(x)|\sigma_{n-1}\rangle \tag{143}$$

from which (124) follows. The derivation of the dilaton equation is a very similar exercise. The rest of (139) is application of the sewing procedure of string perturbation theory with some consistency considerations restricting the contact algebra [195]. We will a make a similar construction in Sec. 8.14

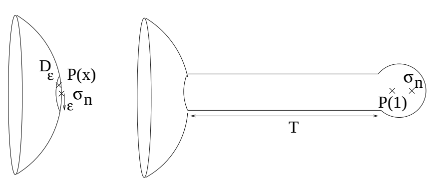

Fig. 16 Conformally equivalent definition of colliding points.

6.6 Integrals over the Hodge classes

Beside the ψ classes there are other important classes on $\mathcal{M}_{g,n}$. A smooth Riemann surface Σ_g has a g dimensional vector space of holomorphic differentials in $H^{1,0}(\Sigma_g) = H^0(\Sigma_g, K_{\Sigma_g})$. On a connected nodal curves there is an extension of this differentials. Namely on a curve of arithmetic genus g one has g meromorphic differentials ω, which are holomorphic outside the nodes, have at most a pole of order 1 at each node branch and the residua on the two node branches add up to zero. These vector space patch together to give a rank g vector bundle E on $\overline{\mathcal{M}_{g,n}}$, which is called the *Hodge bundle*[23]. In fact this construction applies likewise to $\overline{\mathcal{M}_{g,n}(M,\beta)}$, see below. The Chern classes of the Hodge bundle, sometimes referred to as λ_k classes, can be integrated over $\overline{\mathcal{M}_g}$. For $g \geq 2$ one gets [69] [68]

$$R_g = \int_{\overline{\mathcal{M}}_g} c_{g-1}^3(E) = \frac{|B_{2g}B_{2g-2}|}{2g(2g-2)(2g-2)!}\,. \tag{144}$$

Here B_g are Bernoulli numbers, e.g. $R_2 = \frac{1}{2880}, R_3 = \frac{1}{725760}, \ldots$. Using the Grothendieck-Riemann-Roch formula of Mumford for the Chern character of the Hodge bundle on $M_{g,n}$ the correlators involving

[23] A similar construction on the targetspace is discussed in Sec. 8.5.

$c_k(E)$ and ψ classes can be expressed as correlators involving only ψ classes and classes of boundary divisors from stable degenerations. The appearance of the Bernoulli numbers is from the expansion of the Todd class in the G-R-R formula. An recursive procedure in the genus for evaluating intersections with boundary classes has been developped in [69]. It ultimatelty reduces the above intersections to intersections of the ψ classes, which are fixed by the Virasoro constraints. C. Faber has written a Maple program, which calculates in this way recursively any given integral of λ_k and ψ classes over $\overline{\mathcal{M}}_{g,n}$.

6.7 The moduli space of maps

Let us now come to the original question of coupling the topological A-model to gravity. We want to construct a moduli space of maps $x : \Sigma_g \to M$, which send Σ_g into a class $\beta = [x(\Sigma)] \in H_2(M, \mathbf{Z})$, called $\mathcal{M}_g(M, \beta)$. The rough expectation is that the negative dimension of the moduli space (116,368) for $g > 1$ is offset by the dimension of the deformations space $\mathcal{M}_g$ of the Riemann surface (367). In other words we might hope to modify the complex structure j of Σ until it is compatible with the complex structure on M and a (j, J) holomorphic map satisfying the Cauchy-Riemann equation

$$\bar{\partial}_{j,J} x = \frac{1}{2}(\mathrm{d}x + J \circ \mathrm{d}x \circ j) = 0 \tag{145}$$

does exist. To see at least heuristically what the dimension of the moduli space of a stable compactification $\overline{\mathcal{M}}_{g,n}(M, \beta)$ is, consider the normal bundle exact sequence of an immersion of a non singular curve in M

$$0 \to T_\Sigma \to x^* T_M \to N_{\Sigma/M} \to 0\ . \tag{146}$$

The associated long exact sequence is

$$\begin{array}{cccc} & & 0 & \to H^0(\Sigma, T_\Sigma) & \to \\ H^0(\Sigma, x^* T_M) & \to & H^0(\Sigma, N_{\Sigma/M}) & \to H^1(\Sigma, T_\Sigma) & \to \\ H^1(\Sigma, x^* T_M) & \to & H^1(\Sigma, N_{\Sigma/M}) & \to 0 & . \end{array} \tag{147}$$

Let us interpret the terms as automorphism, deformations and obstructions for the maps x. As far as the domain curve is concerned we know that $H^1(\Sigma, T_\Sigma) - H^0(\Sigma, T_\Sigma) = \mathrm{Def}(\Sigma) - \mathrm{Aut}(\Sigma)$, and that the dimension of $\mathcal{M}_g$ is $3g - 3$. For fixed complex structure of the domain we can identify $H^0(\Sigma, x^* T_M)$ with the deformations and $H^1(\Sigma, x^* T_M)$ with the obstructions of the map x. The real objects of interest are $H^0(\Sigma, N_{\Sigma/M})$ and $H^1(\Sigma, N_{\Sigma/M})$, which are the deformations and obstructions of the map x without fixing the domain. In order to have a stable compactification $\overline{\mathcal{M}}_{g,n}(M, \beta)$ we must allow in general for marked points. In this case (147) becomes

$$\begin{array}{cccc} 0 & \to & \mathrm{Aut}(\Sigma, \underline{p}, x) & \to \mathrm{Aut}(\Sigma, \underline{p}) & \to \\ \mathrm{Def}(x) & \to & \mathrm{Def}(\Sigma, \underline{p}, x) & \to \mathrm{Def}(\Sigma, \underline{p}) & \to \\ \mathrm{Obs}(x) & \to & \mathrm{Obs}(\Sigma, \underline{p}, x) & \to 0 & . \end{array} \tag{148}$$

Now if a stable compactification $\overline{\mathcal{M}}_{g,n}(M, \beta)$ exist then $\mathrm{Aut}(\Sigma, \underline{p}, X) = 0$. Moreover at least in some relevant situations $\mathrm{Obs}(\Sigma, \underline{p}, X) = 0$ and since the alternating dimensions of long exact sequences is 0, we can calculate $\mathrm{Def}(\Sigma, \underline{p}, X)$, because we know $\mathrm{Def}(\Sigma, \underline{p}) - \mathrm{Aut}(\Sigma, \underline{p}) = 3g - 3 + n$ and $\mathrm{Def}(x) - \mathrm{Obs}(x) = h^0(x^*(TM)) - h^1(x^*(TM))$. The expected or virtual complex dimension of the moduli of stable maps is

$$\begin{aligned} \mathrm{vdim}_C \overline{\mathcal{M}}_{g,n}(M, \beta) &= h^0(x^*(TM)) - h^1(x^*(TM)) + \dim \mathrm{Def}(\Sigma, \underline{p}) - \dim \mathrm{Aut}(\Sigma, \underline{p}) \\ &= c_1(TM) \cdot \beta + (\dim_C M - 3)(1 - g) + n\ , \end{aligned} \tag{149}$$

where we calculated the first two terms contribution by (368) and the last two by (367) with addition of moduli for marked points.

This formula reflects the special rôle Calabi-Yau threefolds. By (149) the moduli space of the contributions to the zero point functions $\mathcal{F}^{(g)}(t)$ for all genera is zero dimensional, which reduces the problem of evaluating them to a *problem of counting points*, albeit a very complicated one. All topological theories will simplify in this way as the example in 3.1. That does not mean in general that all topological observable are integers, because discrete automorphism groups of the theory, which have to be identified in the path integral, weight some these points with $1/|Aut|$ factors. The remarkable fact about CY threefolds is that an infinite number of physically relevant objects can be reduced in this way. Further comments about the A-model coupling to gravity are exhibited in comparison with the B-model in Sec. 8.13.

One problem in this theory is that complex manifolds do not allow generic enough deformations so that the virtual dimension formula (149) is frequenly violated and the actual dimension of the moduli space is positive. There are two ways to overcome these class of problems:

Either we consider deformations under which the quantities under consideration are invariant. As mentioned below (114) a symplectic structure and a compatible almost complex structure are sufficient to define the A-model. Under this weaker conditions one can achieve the generic situation of a zero dimensional moduli space. Counting so called *pseudoholomorphic curves* is in this respect an easier approach to Gromow-Witten invariants.

Or we define a *virtual class* and defines formally the Gromow-Witten invariants as

$$r_\beta^g = \int_{\overline{\mathcal{M}}_{g,n}(M,\beta)} c_{g,n}^{vir}(M,\beta) \ . \tag{150}$$

In particular $c_{g,n}^{vir}(M,\beta)$ has to specify what class to integrate over all positive dimensional components of the moduli space that might occur. This problem of non-genericity due to obstructions is well known in intersection theory and the above method to overcome it called *excess intersection calculation* [80]. An easy illustration can be found in Sec. 6.16. In so called perfect obstruction theories the existence of $c_{g,n}^{vir}(M,\beta)$ is guaranteed. Below we follow an approach involving virtual localisation.

6.8 Idea of localisation

The successful setup of this point counting problem in the A-model is a very sophisticated problem, which needs several lectures in its own. Let us mention just some key ideas and some interesting issues with references to the literature. In the A-model we have a counting problems for each topological type of map $x:\Sigma_g\rightarrow M$ which are labeled by g and the class $\beta\in H_2(M,\mathbf{Z})$. The virtual dimension of the moduli space might be zero dimensional, but points have no hair. They are suitable characterized by starting with a bigger deformation space $\mathcal{M}$ and impose obstructions. In particular Kontsevich considered first maps into a toric variety M_T and imposed the restriction to maps in the Calabi-Yau variety on the moduli space $\overline{\mathcal{M}}_{g=0,0}(M_T,\beta)$ by integrating over the Chern class of an obstruction bundle [141]. The principal setup is as follows

$$\begin{array}{ccccccc} & & U_\beta=\pi_*\mathrm{ev}^*(V) & \stackrel{\pi_*}{\longleftarrow} & \mathrm{ev}^*(V) & \stackrel{\mathrm{ev}^*}{\longleftarrow} & V \\ & & \downarrow & & \downarrow & & \downarrow \\ \overline{\mathcal{M}}_{g,0}(M,\beta) & \stackrel{i}{\hookrightarrow} & \overline{\mathcal{M}}_{g,0}(M_T,\beta) & \stackrel{\pi}{\leftarrow} & \overline{\mathcal{M}}_{g,1}(M_T,\beta) & \stackrel{\mathrm{ev}}{\longrightarrow} & M_T\ . \end{array} \tag{151}$$

Here $\mathrm{ev}=\mathrm{ev}_1$ is the evaluation map $\mathrm{ev}:\overline{\mathcal{M}}_{g,n}(M_T,\beta)\rightarrow M_T$ defined by $\mathrm{ev}_i:(\Sigma,p_1\ldots,p_n,x)\mapsto x(p_i)$ and π is the forgetful map encountered in Sec. (6.3). The fibres of the bundle U_β over (σ,X) are $H^0(\Sigma,x^*(V))$ and one defines

$$r_\beta^g = \int_{\overline{\mathcal{M}}_{g,0}(M_T,\beta)} c^{vir}(U_\beta) \ . \tag{152}$$

In Kontsevichs principal example [141] $g=0$,$\beta=d$,$M_T=\mathbf{P}^4$ and $V=\mathcal{O}(5)$, i.e. the zero section s of V defines M the *quintic* in $\mathbf{P}^4$. We can use (365) to calculate $h^0(\Sigma_0,x^*(\mathcal{O}(5)))=5d+1$. Similarly

(149) gives $\mathrm{vdim}_C\overline{\mathcal{M}}_{0,0}(\mathbf{P}^4,d)=5d+1$ so that we can take indeed the top Chern class for $c^{vir}(U_d)=c_{5d+1}(U_d)$ to define a volume form on the moduli space. This counts zeros of the pull back and push forward $\tilde{s}$ of s which represents maps whose image is in s, the quitic threefold. Again the dimension count is optimistic and the general case requires the definition of a virtual fundamental class. However the key idea will apply, namely to push forward and pull back the torus action of the ambient space to moduli space of the maps $\mathcal{M}(g,0)(M_T,\beta)$ and to calculate (152) by techniques from equivariant cohomology on $\mathcal{M}(g,0)(M_T,\beta)$.

Let us give a heuristic picture how to use the induced torus on $\mathcal{M}$ to do the integral. For instance the question for the topological Euler number is point counting problem asking for the zero set of the generic section σ in the tangent bundle, a $\mathcal{C}^\infty$ vector field. We can use the Gauss-Bonnet theorem see Sec. 9.3 and write this as $\chi=\int_{\mathcal{M}}c_n=\int_{\mathcal{M}}R(g)\mathrm{d}V$. This is not a simplification, unless we have a good choice for g to perform the integral, which comes up if $\mathcal{M}$ admits symmetries. For instance on the sphere we can generate a vector field by rotating the sphere. This introduces a coordinate direction ϕ and we can choose the altitude θ as the second and pick the diagonal constant metric in these coordinates, which is flat everywhere but has δ curvature at the poles, which leads to the Euler number 2. The poles come of course from the two zeros of the vector field which generates an S^1 group action on S^2. This leads likewise to the conclusion that $\chi(S^2)=2$, see Fig. 17.

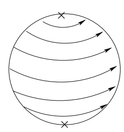

Fig. 17 Using the fi xpoints of the S^1 group action on S^2 to calculate its Eulernumber $\chi(S^2)=2$.

Points that contribute to the integrals can hence be singled out as fixpoints under group action of a group G[24]. The underlying principle is called *localization*. The key is to give general fixed sets additional structure, which describes the group action in their normal direction in a way that is useful to address global cohomological questions. The result which we need is the Atiyah and Bott localisation formula in *equivariant cohohomology* [13]. Learning about the group and the target from the action of the group is a highly developed subject [27, 53]. The principal construction is as follows. Let EG be a *contractible space* - unique up to Homotopy- on which G acts freely and the right and assume that G acts on the left on M. Then we consider the space $M_G=EG\times_G M$ whose points are equivalence classes $[f,m]$ under $(eg,m)\sim(f,gm)$. This space fibres over M/G. The fibres over $[Gm]$ are $BG_m=EG/G_m$, where $G_m=\{g\in G|gm=m\}$ is the stabilizer of m. One defines the *equivariant cohomology* $H^*_G(M)$ as the ordinary cohomology of M_G, i.e $H^*(M_G)$. For example if G acts freely on M, i.e. all G_m are trivial then since the fibre is contractible the cohomology of $H^*(M_G)$ is that of $H^*(M/G)$. In the other extreme that G acts trivial the cohomology is $H^*(M_G)=H^*(BG)\times H^*(M)$. Here we have to clearify what $H^*(BG)$ the cohomology of $BG=EG/G$ is. It depends only on the group G and can be understood a the equivariant cohomology of a point $H^*_G(\{pt\})$, where G can act only trivially. Since the ordinary cohomology of the point is trivial it is called cohomology class of pure weight (of the group action). However in equivariant cohomology the cohomology of point is a rich structure. An example of principle importance in the A-model are of course group actions of the algebraic torus $T=(\mathbf{C}^*)^d$, $\mathbf{C}^*=\mathbf{C}\setminus\{0\}$. The construction of EG for contineous groups requires a limit procedure, since there are no ordinary contractible spaces which allow for free actions of S^1 or $\mathbf{C}^*$. S^1 can be thougt of acting freely on the "contractible" space $EG=\lim_{n\to\infty}S^{2n+1}$, so that $BG=\lim_{n\to\infty}S^{2n+1}/S^1=\mathbf{P}^\infty$ and $H^*_{S^1}(\{pt\})=\mathbf{C}(\lambda)$ is the polynomial algebra in the variables $\lambda=c_1(H(\mathbf{P}^\infty))$, see Sec. 9.3. In the case of the torus action $T=(\mathbf{C}^*)^d$ one has as simple generalization

$$H^*_T(\{pt\})=H^*(BT)=H^*((\mathbf{P}^\infty)^d)=\mathbf{C}[\lambda_1,\dots,\lambda_d],\tag{153}$$

[24] Another way to single them out is a critical points of a Morse function.

the polynomial ring over $\mathbf{C}$ in d variables. $H_T^*(M)$ is a roughly speaking a cohomology theory with coefficients in polynomial algebra $\mathbf{C}[\lambda_1,\ldots,\lambda_d]$. The formal parameters λ can also be viewed as characters of the Lie group of T, i.e. maps $\lambda : T \to C^*$ that is $\lambda \in T^\vee \sim \oplus_{i=1}^s \mathbf{Z}$ and λ_i is a choice of a basis[**?**][141]. If M is non-singular then the T fixed sets $F \in M$ are also non-singular.

An important task is to relate classes $\phi \in H_T^*(M)$ to classes in the equivariant cohomology of the fixpoint loci $H^*(F) = H^*(F) \otimes H_T^*(\{pt\})$. In ordinary cohomology one has for maps $f : N \to M$ between compact orientable manifolds with $\mathrm{dim}M - \mathrm{dim}N = q$ a pushforward $f_* : H^*(N) \to H^{*+q}(M)$. If N is a fibering over M, i.e. $q < 0$, then f_* can be thought as integrating over the fibres. If f is the embedding $i : N \hookrightarrow M$ then i_* factors through the Thom isomorphism $\Phi_N : H^*(M, M-N) \sim H^{*-q}(N)$, i.e. with $H^{*-1}(M-N) \xrightarrow{\delta} H^*(M, M-N) \xrightarrow{j^*} H^*(M)$ one has $i_* = j^* \circ \Phi_N$. Moreover by the excision principle one identifies $H^*(M, M-N) \sim H_c^*(\nu)$, where ν is the normal bundle to N. The latter can be defined in any tubular neighborhood of N im M. The Thom class of the normal bundle is the Thom class $\Phi_N \cdot 1 \in H_c^q(\nu)$ and its restriction to N by the pullback of the inclusion map $i^* : H^*(M) \to H^*(N)$ is the euler class of ν

$$i^* i_* 1 = e(\nu) \ . \tag{154}$$

The consideration that lead to (154) goes through in equivariant cohomology. However a key difference and a main result in the latter case is that $i^* i_*$ is an isomomorphism up to torsion. As modules $H_T^*(M)$ and $H_T^*(F)$ are direct sums of a free part and a torsion part and much scrutiny in [**?**] is devoted to keep information of the torsion part. Similar as for $H^*(M, \mathbf{Z})$ where one can ignore the torsion part by passing to $H^*(M, \mathbf{Q})$ one can consider in the equivariant classes whose coefficients are rational functions in $\mathbf{Q}[\lambda_1,\ldots,\lambda_d]$. In this setting the equivariant euler class is invertible along F so that $(i_*)^{-1} = \frac{i^*}{e(\nu)}$. In this way one obtains for any equivariant class $\phi \in H^*(M_T)$

$$\phi = \sum_F \frac{i_* i^* \phi}{e(\nu_F)} \ . \tag{155}$$

The pushforward of the map $\pi^M : M \to \{pt\}$ given by integrating over the M and a similar map $\pi^F : F \to \{pt\}$ factors through so that $\pi_*^F i_* = \pi_*^M$. Applying that to both sides of (155) yields the integration formula of Atiyah and Bott

$$\int_M \phi = \sum_F \int_F \frac{i^* \phi}{e(\nu_F)} \ . \tag{156}$$

For example the euler class $e(TM) \in H^*(M_T)$ maps to $i^*(e(TM)) = e(\nu_F)$, the ratio in the integral is 1 and (156) calculates the euler number as the number of fixpoints $\chi(M) = \int_M e(TM) = \sum_F 1$.

6.9 Toric string backgrounds

Toric varieties of dim d are varieties M_T in which the d-dimensional algebraic torus $T = (\mathbf{C}^*)^d$ is embedded as dense open subset $T = (\mathbf{C}^*)^d \in M_T$ and T acts on the coordinates $x_l^{(i)}$ of T by multiplication $\lambda_i \in \mathbf{C}^* = \mathbf{C} - \{0\}$. The following is a convenient way to characterize the embedding of T in M_T. Consider

$$M_T = (\mathbf{C}^m - \mathrm{Z}(\{D_{i_1} \cdots D_{i_s}\}))/G \qquad d = m - r \tag{157}$$

where the $G \sim (\mathbf{C}^*)^r$ action on the homogeneous coordinates x_i, $i = 1,\ldots,m$ of $\mathbf{C}^m$ is specified by charge vectors $\vec{l}^{(k)}$

$$x_i \mapsto \mu_k^{l_i^{(k)}} x_i, \quad \text{with } l_i^{(k)} \in \mathbf{Z}, \quad \mu^{(k)} \in \mathbf{C}^*, \quad i = 1,\ldots,m, \quad k = 1,\ldots,r \ . \tag{158}$$

Let $(\mathbf{C}^*)^m$ act by multipliction on x_i then $T := (\mathbf{C}^*)^m/G$ defines the embedding of T into M_T and its action on x_i. We denote the pure phase rotation part of G and T by $G_R = U(1)^r$ and $T_R = U(1)^d$. $D_i := \{x_i = 0\}$ are divisors and the product $D_i \cdot D_i$ denotes intersection. $\mathrm{Z}(\{D_{i_1} \cdots D_{i_s}\})$ is the Stanley-Reisner ideal generated by this $\cdot$ product with D_i from certain sets of generators $\{D_{i_1} \cdots D_{i_s}\}$. The necessity to substract $\mathrm{Z}(\{D_{i_1} \cdots D_{i_s}\})$ is easily understood. In order to have a well defined quotient with strata of equal dimension we need the points in $\mathbf{C}^m - \mathrm{Z}(\{D_{i_1} \cdots D_{i_s}\})$ to have smooth r dimensional orbits under G and separable orbits under G_R. In particular that the data $l_i^{(k)}$ and $\{D_{i_1} \cdots D_{i_s}\}$ are not independent.

The definition (157) is quite obviously modelled as a generalization of the projective spaces $\mathbf{P}^d$ (326) for which $m = d+1$, $l^{(1)} = (1, \ldots, 1)$ is a m component vector and $Z(\{D_1 \cdots D_{d+1}\}) = \{D_1 \cdots D_{d+1}\}$. Note that the coordinates $x_l^{(i)}$ in (327) are invariant under $G = (\mathbf{C}^*)^1$ and T acts simply by multiplication on them. This generalizes in the following way. $\mathbf{C}[x_1 : \ldots : x_m]$ is the homogeneous coordinate ring of M_T and all local coordinates can be obtained as inhomogeneous coordinates by scaling r coordinates z_{i_k}, $k = 1, \ldots, r$ to 1. This requires the choice of a suitable combination of group generators in (158), once these are picked we identify $\mu^{(k)} \in \mathbf{C}^*$ with $z_{i_k} \neq 0$ in the scaling. By construction the inhomogeous local coordinates are invariant under G and the $l^{(k)}$ determine all coordinate transformations between the inhomogeneous coordinate patches.

To study M_T and its subspaces it is convenient to view the components of the $\vec{l}^{(k)}$ as coefficients of linear relations among m points $\{P\}$ in a lattice $N \sim \mathbf{Z}^d$ of rank d. I.e. the points are specified by integers $\nu_k^{(l)}$, $k = 1, \ldots d$, $l = 1, \ldots, m$. *Cones* σ_δ^i are positive linear subspaces of dimension $\delta = 0, \ldots, d$ in the real completion of the lattice $N_{\mathbf{R}} = N \otimes \mathbf{R} = \mathbf{R}^d$, whose edges are generated by points in $\{P\}$. Collection of these cones, which intersect each other at most at lower dimensional *faces* ($\sigma_\delta^i \cap \sigma_\delta^j = \sigma^k_{\tilde\delta < \delta}$), are called *fans* Σ and the properties of M_T have a nice geometrical, combinatorial and pictorial incarnation as properties of fans. In particular from the analysis of the fans one finds suitable choices of the $\vec{l}^{(k)}$ and $\{D_{i_1} \cdots D_{i_s}\}$ so that (157) defines a smooth toric variety. Apart from stating simple examples we will not go further in this subject, which is amply discussed in [167][81][105]. In Fig. 18 we show three fans of simple toric varieties.

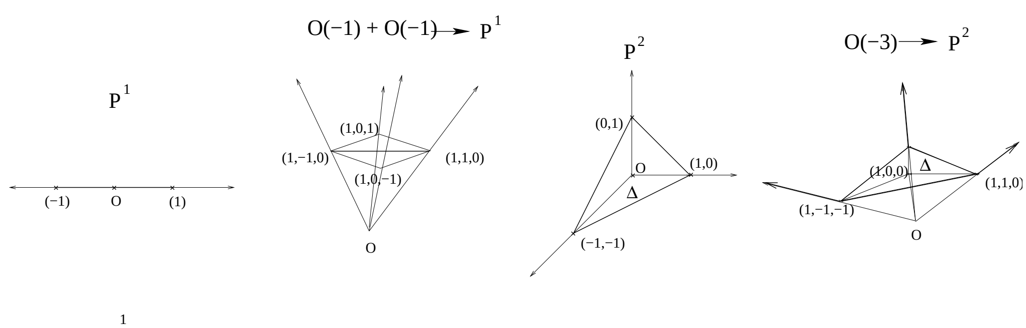

Fig. 18 Two fans for compact $\mathbf{P}^1$ with $l^{(1)} = (1,1)$ for compact $\mathbf{P}^2$ with $l^{(1)} = (1,1,1)$ and two for the non compact toric Calabi-Yau manifold $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbf{P}^1$ with $l^{(1)} = (1,1,-1,-1)$ and $\mathcal{O}(-3) \to \mathbf{P}^2$ with $l^{(1)} = (-3,1,1,1)$. Note that the fi rst fan is 1one dimensional, the third is two dimensional and the second and the fourth are three dimensional. This correspond to the complex dimensions of the M_T described by them.

One can also think about toric geometry as *symplectic quotient* construction. This is modelled as generalization of the definition of $\mathbf{P}^n = S^{2n+1}/S^1$, where one first restricts the moduli of x_i by $\sum_{i=1}^{n+1} |x_i|^2 = t$, a real S^{2n+1}, and divides then by the phase $U(1)$. The gauged linear $(2,2)$ supersymmetric σ model (GLSM) is a physical implementation for performing the quotient (157) in two steps [213]. One interpretes the coefficients of the vectors $\vec{l}^{(k)}$ as $U(1)$ charges of m chiral superfields ϕ_i with lowest scalar component x_i and the manifold M_T as the vacuum manifold parametrizes by the vacuum expectation values of the x_i denoted for brevity by the same symbol. The total gauge group is $U(1)^r$ and in the first step the absolute values of the x_i are restricted by the correponding D-terms constraint for the vev's

$$D^{(k)}(t) = \sum_{i=1}^{m} l_i^{(k)} |x_i|^2 = t^{(k)}, \quad k = 1, \dots, r \; . \tag{159}$$

The Fayet-Illioupoulos terms $t^{(k)} \in \mathbf{R}$ with $t^{(k)} > 0$ are identified with Kähler parameters of M_T. Gauge invariance requires as second step to divide by the G_R symmetry and defines $M_T = \cap_{i=1}^r (D^{(k)})^{-1}(t)/G_R$. The $t^{(k)} > 0$ constraints is necessary for smoothness of the gauge orbits[213]. E.g. for $\mathbf{P}^n$ this excludes precisely $D_1 \cdots D_{d+1}$. Similar as the non-linear $(2,2)$ σ model the above gauged linear σ has classically unbroken $U(1)_V$ and $U(1)_A$ symmetries. By a similar consideration of the transformation of the fermionic measure as in Sec. 9.4 one checks that the axial $U(1)_A$ parametrized by α develops an anomaly $\sim 2\alpha \sum_k b^{(k)} c_1(E^{(k)})$, where $c_1(E^{(k)})$ is the firts Chern class of the k'th $U(1)$ gauge bundle and $b^{(k)} = \sum_{i=1}^m l_i^{(k)}$. The theory has $\theta^{(k)}$ angles $k = 1, \dots, r$ which are shifted by the anomalous transformations by $\theta^{(k)} \to \theta^{(k)} + 2\alpha b^{(k)}$ and the $t^{(k)}$ become scale dependent by a one-loop contribution with $\mu \frac{\mathrm{d}}{\mathrm{d}\mu} r^{(k)} = b^{(k)}$. Hence the theory is $U(1)_A$ anomaly free and scale independent if $b^{(k)} = 0, \forall k$.

On the geometrical side it is easy to see the relevance of this condition for the existence of a trivial canonical class. To establish condition d.) in Sec. 9.8 we start in inhomogenous coordinates of a patch where the $(d,0)$-form is $\Omega = \mathrm{d}x_1^{(k)} \wedge \ldots \wedge \mathrm{d}x_d^{(k)}$. Coordinate transformations to other inhomogeneous coordinate patches are determined by the $l^{(k)}$ as explained in (158) cff. Now it is not hard check that the Jacobian $Jac\left(\frac{\partial x^{(k)}}{\partial x^{(i)}}\right)$ is a homogeneous function of degree $b^{(l)}$ in the variables $x^{(k)}$. It is therefore only possible to extend Ω trivially to all patches, iff $b^{(k)} = 0, \forall k$. As an exercise the reader may check by transforming between the two patches of $\mathbf{P}^1$ that the cotangent bundle $l\mathrm{d}x$ transforms as $\mathcal{O}(-2)$ and for $\mathcal{O}(-2) \to \mathbf{P}^1$, defined by $l^{(k)} = (-2,1,1)$, $\Omega = \mathrm{d}l \wedge \mathrm{d}x$ (l fibre x base direction) transforms trivially, see (400). To summarize one has the following

$$K(TM_T) \text{ is trivial} \Longleftrightarrow \beta^{(k)} = \sum_i l_i^{(k)} = 0, \; \forall k \Longleftrightarrow \begin{array}{c}\text{no } \mathrm{U(1)_A} \text{ anomaly \& no} \\ \text{scale dependence in GLSM}\end{array} \tag{160}$$

In Fig. 18 the second and the last fan represents Calabi-Yau manifolds with trivial canonical bundle. As a consequence of (160) all points generating these fans lie in a (hyper)plane in $N_{\mathbf{R}} = \mathbf{R}^3$. In contrast to the first fan $\Sigma_{\mathbf{P}^1}$ in $N_{\mathbf{R}} = \mathbf{R}$ and the third fan $\Sigma_{\mathbf{P}^2}$ in $N_{\mathbf{R}} = \mathbf{R}^2$, the fans for Calabi-Yau manifolds in $N_{\mathbf{R}} = \mathbf{R}^3$ do non cover this space $N_{\mathbf{R}}$. It can be shown in general that M_T is compact iff Σ covers $N_{\mathbf{R}}$ [167][81]. Hence toric varities with trivial canonical bundle can never be compact.

Each generator of a one dimensional cone i.e. all points $\nu^{(i)}$ other the O in Fig. 18 correspond to divisors. As further explained in [81][167][105] the divisors D_i are not independent, but fullfill the relations $R_k = \sum_{i=1}^m \nu_k^{(i)} D_i = 0$, $k = 1, \dots, d$. The ideal $Z(\{D_{i_1} \cdots D_{i_d}\})$ is generated by the intersection of those divisors, whose points do not lie on a common top dimensional cone. This information suffices to calculate the complete intersection ring, up to a normalisiation which is fixed by the Euler number, as $C[D_1, \dots, D_m]/Z[\{D_{i_1}, \dots, D_{i_d}, R_k\}]$. Independent divisor can be identified with the generators of the cohomology group $H^{n-1}(M_T, \mathbf{Z})$. In particular $h^{n-1}(M_T, \mathbf{Z}) = r$. The $l^{(k)}$ represent curves $\mathcal{C}^{(k)}$, which vanish on the k'th wall of the Kähler cone and $D_i \cdot \mathcal{C}^{(k)} = l_i^{(k)}$. In a nonsingular compact toric variety we can pick a basis $\mathcal{P}_1, \dots, \mathcal{P}_r$ of divisors $\mathcal{P}_k = \sum_{i=1}^m k_i D_i$ with $k_i \in \mathbf{Z}$ such that $\mathcal{P}_i \cdot \mathcal{C}^{(k)} = \delta_i^k$. The associated line bundles $L_i = [\mathcal{P}_i]$ generate the Picard-group of X. The class of any curve $\mathcal{C}$ is $\beta = [\mathcal{C} \cdot \mathcal{P}_1, \dots, \mathcal{C} \cdot \mathcal{P}_r]$. With suitable numeration of the points which we find $D_1 \cdot D_2$, $D_1 \cdot D_2$, $D_1 \cdot D_2 \cdot D_3$ and $D_1 \cdot D_2 \cdot D_3$ for the four cases in Fig. 18. $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbf{P}^1$ is the resolved conifold and the ambiguity in tringulating the correponding fan in Fig. 18 correponds two different possibilities to resolve the conifold by blowing up a $\mathbf{P}^1$.

Each toric variety comes with a natural symplectic structure which is given by the real 2-form in coordinates $x_k = |x_k| e^{i\theta_k}$

$$\omega = \frac{i}{2}\sum_{k=1}^{d} \mathrm{d}x_k \wedge \mathrm{d}\bar{x}_k = \frac{1}{2}\sum_{k=1}^{d} \mathrm{d}|x_k|^2 \wedge \mathrm{d}\theta_k = \sum_{k=1}^{d} \mathrm{d}u_k \wedge \mathrm{d}v_k \tag{161}$$

on $T \in M_T$. It extends via (157,158) over M_T. In the case of $\mathbf{P}^1 = S^2$ we have drawn in Fig. 19 the $S^1 \in \mathbf{C}^*$ action, which sweaps out the S^2. This yields a very useful interpretation of dual[25] toric diagrams as projections of M_T under the moment map, which forgets about the phase rotations $T^d_{\mathbf{R}} \in T \in M_T$ on the θ_k. The pictures show as linear subspace of $\mathbf{R}^d$ the base B of the $T^d_{\mathbf{R}}$ fibration parametrized by $|x_i|^2$ subject to (159). The information which cycles in $T^d_{\mathbf{R}}$ degenerate is easily reconstructable from B [152]. E.g. the two end points of the intervall in the $\mathbf{P}^1$ diagram are the loci where the S^1 degenerates. The relation to Fig. 17 and the usefulness of this point of view for localisation calculations should be obvious. E.g. in the $\mathbf{P}^2$ case B is the triangle Δ in Fig. 19. The direction of the edges of Δ hold the information, which cycle in a base $(a,b) \in H^1(T^2_{\mathbf{R}}, \mathbf{Z})$ degenerate. The correponding vanishing cycles are indicated in the Fig. 19. Inside of Δ the a and b cycle are non-degenerate, i.e. the generic fibre is $T^2_{\mathbf{R}}$, which degenerates over the 1d faces of Δ to S^1's and over the 0d faces, corners of Δ to $\{pt\}'s$. These are the fixpoints of T^2 and the remark after (156) yields $\chi(\mathbf{P}^2) = 3$. B in the last example in Fig. 19 is the open convex subspace of $\mathbf{R}^3$ bounded by the compact face Δ, and the non-compact faces F_1, F_2 and F_3, while in the second example it is the open convex subspace of $\mathbf{R}^3$ bounded by $F_1, \ldots, F_4$.

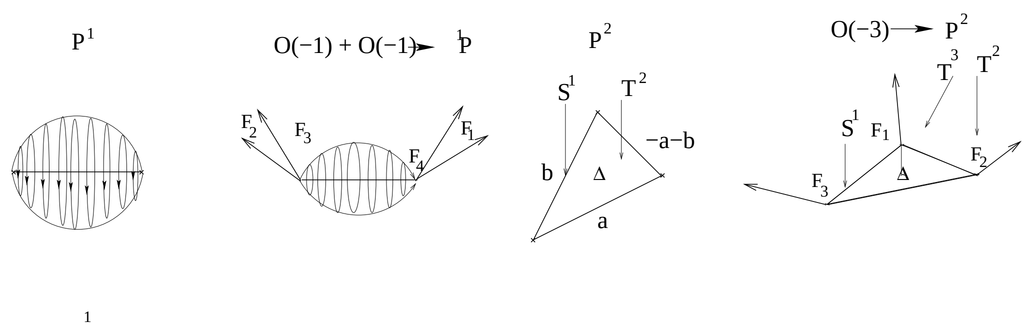

Fig. 19 Two fans for compact $\mathbf{P}^1$ with $l^{(1)} = (1,1)$ for compact $\mathbf{P}^2$ with $l^{(1)} = (1,1,1)$ and two for the non compact toric Calabi-Yau manifold $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbf{P}^1$ with $l^{(1)} = (1,1,-1,-1)$ and $\mathcal{O}(-3) \to \mathbf{P}^2$ with $l^{(1)} = (-3,1,1,1)$.

The conditions on $l^{(k)}$ in (160) reduce the dimension in which the points in the corresponding toric diagram are embedded by one, see Fig. 18. A similar reduction occurs for the toric diagrams representing the degenerations. A way to think about this is that instead of the generic $T^d_{\mathbf{R}}$ fibration over B one can consider a $\mathbf{R} \times T^{d-1}_{\mathbf{R}}$ fibration and one dimension less is necessary to describe the degenerations in $T^{d-1}_{\mathbf{R}}$ by directions in B. This is only possible if M_T is non-compact, which is the case for toric Calabi-Yau manifolds. To obtain this structure we establish it in a patch and show that is possible extend it over the non-compact toric CY manifolds. In a patch e.g. for $d = 3$ C^3 with coordinates x_1, x_2, x_3 we consider three Hamiltonians

$$r_{\alpha_1} = |x_1|^2 - |x_2|^2, \qquad r_{\alpha_2} = |x_3|^2 - |x_1|^2, \qquad r_{\mathrm{R}} = \mathrm{Im}(x_1 x_2 x_3)\ . \tag{162}$$

They parametrize the base and generate flows $\partial_\alpha x_k = \{r_\alpha, x_k\}_\omega$ etc, whose orbits define the fibre. The distinguished $\mathbf{R}$ orbit in the fibre is generated by r_{R}. E.g. for $d = 1$ the only toric CY is $\mathbf{C}$, which can be

[25] The reader will recognize that the pictures in Fig. 18 and 19 are dual in the sense that δ-dimensional subspaces are mapped to $d - \delta$ dimensional subspaces. For the detailed description of the pictures see exercise (164).

either viewed in polar coordinates $x = |x|e^{i\theta}$ as S^1 (θ) fibration over $\mathbf{R}$ ($|x|$) or in $x = u + iv$ coordinates as $\mathbf{R}$ (u) fibration over $\mathbf{R}$ (v). The latter fibration can be obtained as above by taking $r_{\rm R} = {\rm Im}(x) = v$ as base, while $\partial_R x = \{v, x\} = 1$ generates as orbit the real part u. In the general case r_R generates the real part of $x_1 \ldots x_d$, while $r_{\alpha_1}, r_{\alpha_2}$ generate independent phase rotations

$$\exp(\alpha_1 r_{\alpha_1} + \alpha_2 r_{\alpha_2}) : (x_1, x_2, x_3) \mapsto (e^{i(\alpha_1-\alpha_2)} x_1, e^{-i\alpha_1} x_2, e^{i\alpha_2} x_3) \tag{163}$$

designed not to affect the phase of $x_1 \ldots x_d$ and therefore the $\mathbf{R}$ fibration structure. To see that this fibration structure extends globally consider as above (160) inhomogeneous coordinates obtained by scaling with $l^{(k)}$. In each patch defined by $x_{i_k} = 1$, $k = 1, \ldots, r$ we can obtain the product $x_1^{(i)} \ldots x_d^{(i)}$ from the homogeneous coordinate product $x_1 \ldots x_m$ by scalings (158) with $\mu^{(k)} = x_{i_k}$. Precisely if $\beta^{(k)} = 0$ (160) $x_1 \ldots x_m$ is invariant under such scalings and defines gobally $r_R = {\rm Im}(x_1 \ldots, x_m)$ and therefore a global $\mathbf{R}$ fibration structure. The r_{α_i} can be defined in inhomogenous coordinates in a patch and the action can be extend by the usual coordinate transformations to other patches. Because the r_{α_i} act on the exponentials it is slightly more convenient to lift the multplicative relation between the coordinates to additive relations and described the r_{α_i} as follows. We pick $d-1$ independent generators r_{α_i} in the i'th patch defined by $x_{i_k} = 1$, $k = 1, \ldots, r$ and write them in homogeneous variables. Since $T_R = U(1)^m/G_R$ and the D_l in (159) generate G_R by the Poisson bracket and since moreover the Poisson bracket is linear, the r_{α_i} are defined only modulo addition of D_l. This ambiguity is of course fixed in any patch $x_{i_k} = 1$, $k = 1, \ldots, r$ by setting the coefficients of $|x_{i_k}|^2$, $k = 1, \ldots, r$ to zero. This ensures that r_{α_i} generates orbits within that patch. Note that the r$_{\alpha_i}$ do not change the phase of $x_1 \ldots x_m$ if the sum of the cofficients of the $|x_k|^2$ is zero. This sum does not change upon adding D_l, if all $\beta^{(l)} = 0$.

As an example we show in Fig. 20 the degeneration of T_R^2 for the $\mathcal{O}(-3) \to \mathbf{P}^2$ geometry, which is defined by $(-3, 1, 1, 1)$ acting on the coordinates (x_1, x_2, x_3, x_4). To cover the geometry we need three patches. Patch $x_4 \neq 0$, with coordinates $u_1 = x_1 x_4^3$, $u_2 = x_2/x_4$, $u_3 = x_3/x_4$, patch $x_3 \neq 0$ with coordinates $v_1 = x_1 x_3^3 = u_1 u_3^3$, $v_2 = x_4/x_3 = 1/u_3$, $v_3 = x_2/x_3 = u_2/u_3$, and patch $x_2 \neq 0$ with coordinates $w_1 = x_1 x_2^3 = u_1 u_2^3$, $w_2 = x_3/x_2 = u_3/u_2$, $w_3 = x_4/x_2 = 1/u_2$. We let (163) the action on the u_i in the first patch and collect the phase shifts on the variables u_i, $i = 1, \ldots, 3$ as $s_u = (\alpha_1 - \alpha_2, -\alpha_1, \alpha_2)$. With the same notation we have on the v patch $s_v = (2\alpha_2 + \alpha_1, -\alpha_2, -\alpha_1 - \alpha_2)$ and in the w patch $s_w = (-2\alpha_1 - \alpha_2, \alpha_2 + \alpha_1, +\alpha_1)$. The α_1, α_2 parametrize two independent cycles of the T_R^2 and can be fairly naturally be identified with a basis $\alpha_1/2\pi \sim (1, 0)$ and $\alpha_2/2\pi \sim (0, 1)$ of $H^1(T_R^2, \mathbf{Z})$. Note that $\alpha_i/2\pi$ also carries a natural integer structure, but the identification is of course up to $SL(2, \mathbf{Z})$. If $u_1 = u_3 = 0$, $(u_1 = u_2 = 0)$ or $[u_2 = u_3 = 0]$ the cycles $(0, 1)$, $((1, 0))$ or $[(1, 1)]$ degenerate. The former locii in the u-patch project on the $(r_{\alpha_1}, r_{\alpha_2})$-plane to the lines $r_{\alpha_1} = 0$, $(r_{\alpha_2} = 0)$ or $[r_{\alpha_1} + r_{\alpha_2} = 0]$. We define an association of $v_l = (m_1, m_2) \in H^1(T_R^2, \mathbf{Z})$ to lines in the $(r_{\alpha_1}, r_{\alpha_2})$-plane by $m_2 r_{\alpha_1} - m_1 r_{\alpha_2} = 0$. v_l is the cycle, which does not vanish along the line and its orientation is fixed by choice of the S^1 actions in s_u. The corresponding vanishing cycle fullfills $v \cdot v_l = 0$ and its orientation is fixed by $v \wedge v_l > 0$. In all patches we can read the v_l from the $s_{u,v,w}$. Equivalently we may transform the r_{α_i} to the patches $x_{4,3,2} = 1$ as explained above, namely by suitably adding D_l of (159). Then the projection to the $(r_{\alpha_1}, r_{\alpha_2})$-plane is performed as explained for the first patch above.

The lines in this figure correpond to stationary points in the Hamiltonian flows in the angular directions and also of the flow induced by r_R. The direction of r_R is perpendicular to the shown plane. The identification above is made so that the vectors v coincide with the direction of the vanishing cycle in $H^1(T_R^2, \mathbf{Z})$. The zero coefficient sum in (159,162) implies a zero force condition on each vertex i.e. $\sum_{i=1}^3 v_i = 0$. Smoothness correponds to $|v_i \wedge v_j| = 1$ for $v_i \neq v_j$ ending on a vertex. As an exercise one checks in Fig. 20 the vanishing cycles for the $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbf{P}^1$ examples. Further one may check that the generic T_R^d fibration can be obtained by choosing the Hamiltonians in a patch as

$$r_{\alpha_i} = |x_i|^2, \quad i = 1, \ldots, d, \tag{164}$$

and that the basis B and the torus degenerations in Fig. 19 are reconstructed very similar as above.

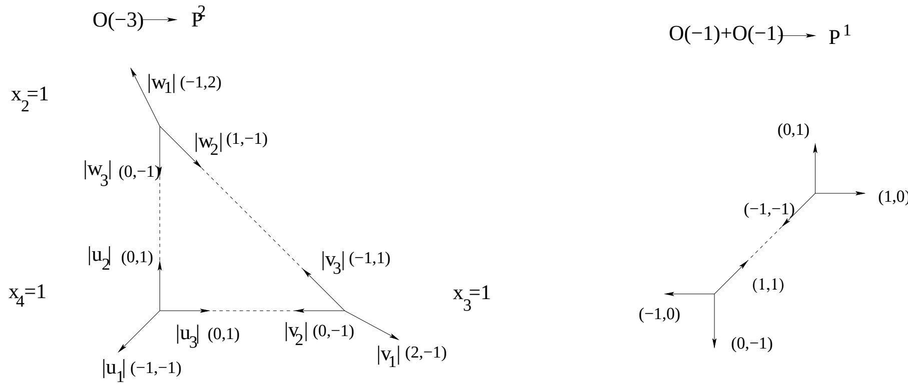

Fig. 20

6.10 Harvey-Lawson special Lagragian Branes

In the toric non-compact Calabi-Yau spaces one can define a very simple class of special Lagrangian branes. It is sufficient to discuss this for C^3 patch, where one the following three Lagragian cycles

$$\begin{array}{lllll} L_1: & r_{\alpha_1}=0, & r_{\alpha_2}=s_1, & r_R\geq 0, & \mathrm{Re}(x_1x_2x_3)=0\\ L_2: & r_{\alpha_1}=-s_2, & r_{\alpha_2}=0, & r_R\geq 0, & \mathrm{Re}(x_1x_2x_3)=0\\ L_3: & r_{\alpha_1}+r_{\alpha_2}=0, & r_{\alpha_1}=s_3, & r_R\geq 0, & \mathrm{Re}(x_1x_2x_3)=0 \end{array} \qquad (165)$$

The s_i are moduli. These submanifolds are Lagrangian, because the r_{α_i} constraints imply $\mathrm{d}|x_1|^2 = \mathrm{d}|x_2|^2 = \mathrm{d}|x_3|^2$ while $\mathrm{Re}(x_1x_2x_3)=0$ implies $\mathrm{d}(\theta_1+\theta_2+\theta_3)=0$, so that $\omega|_{L_i}=\mathrm{d}|x_3|^2\wedge\mathrm{d}(\theta_1+\theta_2+\theta_3)=0$. Similarly one shows with $\sum_{i=1}^3\theta_i=\pi/2$ that $\Omega|_{L_i}=\mathrm{d}r_R\wedge\mathrm{d}\alpha_1\wedge\alpha_2=\mathrm{vol}(L)$, i.e. L_i are special Lagrangian with the same calibration. The L_I are obviously T^2_R fibrations with fibres generated by r_{α_i} over r_R. The boundary conditions are so that one S^1, e.g. in the class $(0,1)$ for L_1 shrinks to zero and completes the halfline parametrized by r_R to $\mathbf{C}$, while the other in the class $(1,0)$ for L_1, has minimal radius $\sqrt{s_1}$. The topology of all L_i is therefore $S^1\times\mathbf{C}$. The geometry implies that there is natural holomorphic disk build as an S^1 fibration, the S^1 is in the $(1,0)$ class, over r_{α_2} bouding the topological non-trivial S^1 in L_1. The s_i are the volumes of these disks and in Fig. 21 all volumes are positive if $\mathrm{Re}(s_i)>0$. Similar as the Kähler parameters t get complexified by the integral of the B field $\int_C B$ the open string parameters s_i are complexified by the Wilson loop $\int_{S^1}A=\int_D B$. This complexification and holomorphicity renders singularities in the complex s space to codimension one. Similar as in the conifold flop transition the continuation to negative volumes including all multi coverings of the disk is not singular.

6.11 Localisation in the moduli space of maps

We spent some time now to explain the torus action in the target space M_T. The final goal is to use its implications for holomorphic maps from the world-sheet $x:\Sigma_g\to M_T$ into the target space. Recall that the key idea here is to pull back the torus action on the moduli space of stable maps, find its its fixpoints in $\overline{\mathcal{M}}_{g,n}(M_T,\beta)$ and use (156) to perform the integrals like (152) or more generally (150). First of all to define a fixpoint in $\overline{\mathcal{M}}_{g,n}(M_T,\beta)$ the geometric image $x(\Sigma_g)$ should not move under T^d. That is not to say that it pointwise fix. In particular a genus zero component of Σ_g can map to the S^2's that are the S^1 fibrations over the closed lines in Fig. 19 or 20, where the S^1 in the class specified by v_l acts on it, but it cannot map anywhere else. Marked points in Fig. 11 must map to the fixpoints of T^d otherwise the map would not be invariant under the torus action. Similarly the map of a higher genus component cannot multicover the $\mathbf{P}^1$ line with branch points. Such higher genus components must be contracted

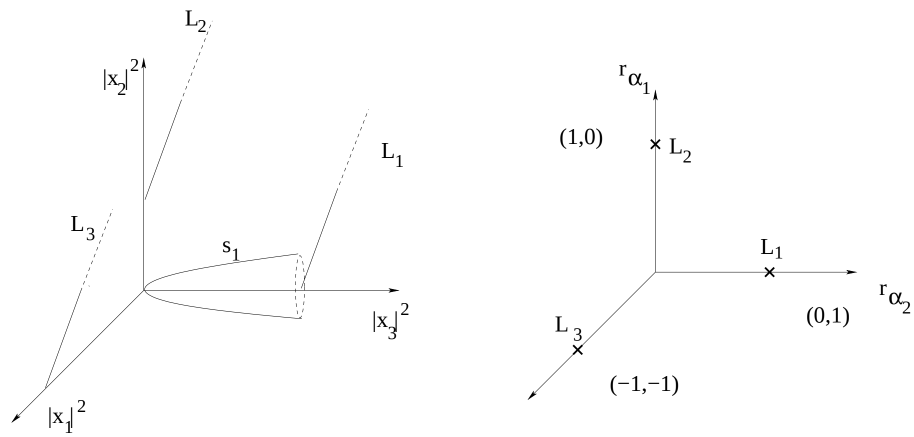

Fig. 21

to the fixpoints. The map of a genus zero component to a line is given in homogeneous coordiantes by $x(a_1 : a_2) = (0 : \ldots : 0 : a_1^{d_l} : 0 : \ldots : 0 : a_2^{d_l} : 0 : \ldots : 0)$. This is only part of the map which can carry degree d_l. The upshot is that the fixed points are labelled by "decorated graphs" Γ similar as in Fig. 11. The decoration indicates the genus g_v and target fixpoint of the contracted components (vertices) and the target line and the degree d_{l_i} of the uncontracted $g = 0$ components (lines). The total degree of the map is simply $d = \sum_i d_{l_i}$ and the total genus is $g = 1 + \chi(\Gamma) + \sum g_{v_i}$, where $\chi(\Gamma)$ is combinatorial Euler number of the graph. Fig. 22 shows the graphs for $g = 1$ and $d = 3$ maps. The i, j, k, l run over the fixpoints of the toric diagram of the traget space for which an embedding of the graph in the toric diagram is possible.

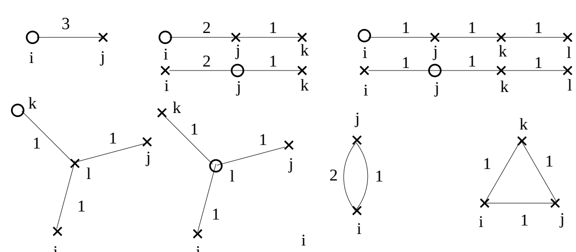

Fig. 22

After capturing the data of fixed maps in $\mathcal{M}_{g,n}(M, \beta)$ as graph Γ, we have to give equivariant expressions for $1/e(\nu_F) =: 1/e(\nu_\Gamma)$ and $i^*\phi =: i_\Gamma^*\phi$ in (**??**). The relevant model for $\mathcal{M}_{g,n}(M, \beta)$ is $\mathrm{Def}(\Sigma, \underline{p}, x)$ in (148) and we have to select normal directions in $\mathrm{Def}(\Sigma, \underline{p}, x)$, i.e. the ones which move the map out of the fixed configuration, this part will indicated by a $^{\mathrm{mov}_\Gamma}$ superscript. We can use the splitting of (148) to write

$$\frac{1}{e(\nu_\Gamma)} = \frac{1}{e(\mathrm{Def}(\Sigma, \underline{p}, x)^{\mathrm{mov}_\Gamma})} = \frac{e(\mathrm{Aut}(\Sigma, \underline{p})^{\mathrm{mov}_\Gamma})}{e(\mathrm{Def}(x)^{\mathrm{mov}_\Gamma})e(\mathrm{Def}(\Sigma, \underline{p}, x)^{\mathrm{mov}_\Gamma})} \tag{166}$$

One can provide expressions for the equivariant euler classes by constructing explicite sections of the bundles Aut, Def and Obs in local coordinates. The weights α_i of the torus action will be defined by (164) in a patch and extended by coordinate transformations over M_T. Let us consider as an example the map of a $\mathbf{P}^1$ component with degree d_l to a line (edge of the toric graph) $e \sim \mathbf{P}^1 \in M_T$. For this we want to compute the weights of the moving sections of $\mathrm{Def}(x)^{\mathrm{mov}_\Gamma} = H^0(\Sigma, x^* TM_T)$. We note the splitting of the tangent bundle in $TM_T|_e = T_e \oplus \mathcal{N}$ due to $0 \to T_l \to TM_T \to \mathcal{N} \to 0$. Since over $\mathbf{P}^1$ all complex vectors bundles split in complex line bundles we have $TM_T|_e = \mathcal{O}(2) \oplus_{i=1}^{d-1} \mathcal{O}(n_i)$. The first line bundle is the tangent bundle of $\mathbf{P}^1$ and the degrees of the line bundles in the normal directions can be straighforwardly calculated within toric geometry. Let $\beta = [d_1, \ldots, d_r]$ be the class of l given by by $D_{i_1} \cdots D_{i_{n-1}}$. Each D_i represents a normal direction $\mathcal{O}(n_i)$ with $n_i = \sum_{i=1}^r d_k (D_i \cdot \mathcal{C}^{(k)})$. We can always pick local toric coordinates such that z_e is a coordinate on e and $z_{j_1} \ldots z_{j_{n-1}}$ are normal coordinates. Locally the map $x : \mathbf{P}^1 \to e$ is given by $a^{d_e} = z_l$. For a $\deg(\mathrm{x}) = \mathrm{d_e}$ map we find the following basis of sections of $H^0(\Sigma, x^* TM_T)$: $a^i \frac{\partial}{\partial z_e}$, $i = 1, \ldots, 2d_l$ and $a^i \frac{\partial}{\partial z_{j_k}}$, $i = 1, \ldots, n_{j_k} d_l$, $k = 1, \ldots, n-1$. The toric coordinate transformations determine the weight α_l, α_{j_k} of the torus action on the tangential direction $z_e \mapsto \exp(i\alpha_l) z_e$ and the normal directions $z_{j_k} \mapsto \exp(i\alpha_{j_k}) z_{j_k}$ in terms of the basis $\alpha_1, \ldots, \alpha_d$ used in (164). The equivariant euler class of the moving part is the product of the torus weights of the sections, where the ones with trivial weights are omitted. A litte calculation yields $\frac{(-1)^{d_e} d^{2d_l}}{(d_e!)^2 \alpha_l^{2d_l}} \prod_{k=1}^{d-1} \prod_{i=1}^{n_{j_k}} \frac{1}{\frac{i}{d_e}\alpha_e - \alpha_{j_k}}$, as contribution to $\frac{1}{e(\nu_\Gamma)}$.

It is possible [141][94] to evaluate all expressions in (166) for a fixed map from the combinatorial data of the correponding graph in more or less closed form. E.g. the expression we evaluated above will appear then for every $\mathbf{P}^1$ component of Σ_g mapped to a line $e \in M_T$. To give the whole expression denote the vertices of the graph by v the genus of the irreducible component assiciated to this vertex $g(v)$. Note that v has to map to a fixpoint under T^d in M_T, which is a vertex of the toric diagrams of the types in Fig. 19,20. Let $e(v)$ be all edges of the graph ending on v and α_e the toric weights of the coordinate of the line e to which this edge is mapped. Note e must be a compact line in the toric diagram. We refer to the number of edges ending at v as the valence $\mathrm{val}(v)$. Further denote by $f(v)$ all lines (flags) that end on a vertex of the toric diagram and α_f its weight. In the cases of flags α_f is the weight the coordinate z_f that vanishes at the vertex. Flags and edges are very similar, only that flags can also stand for insertions of marked points on one vertex. Note that T^d-invariance implies that marked points can only be inserted at the vertices.

$$
\begin{aligned}
\frac{1}{e(\nu_\Gamma)} = \; & \prod_e \frac{(-1)^{d_e} d^{2d_l}}{(d_e!)^2 \alpha_l^{2d_l}} \prod_{k=1}^{d-1} \prod_{i=1}^{n_{j(e)_k}} \frac{1}{\frac{i}{d_e}\alpha_e - \alpha_{j(e)_k}} \prod_v \prod_{e(v)} (\alpha_e)^{\mathrm{val}(v)-1} \prod_{e,\mathrm{val}(v)=1} \frac{\alpha_e}{d_e} \\
& \times \begin{cases} \prod_v \left[\left(\sum_{f(v)} \frac{1}{\frac{\alpha_f}{d_f}} \right)^{\mathrm{val}(v)-3} \prod_{f(v)} \frac{1}{\frac{\alpha_f}{d_f}} \right], & \text{for g} = 0 \\ \prod_v \left[\prod_{e(v)} P_{g(v)}(\alpha_e, E^*) \right] \prod_v \left[\prod_{f(v)} \frac{1}{\frac{\alpha_f}{d_f} - \psi_f} \right], & \text{for g} > 0 \end{cases}
\end{aligned}
\qquad (167)
$$

Here $P_g(\lambda, E^*) = \sum_{r=0}^g \lambda^r c_{g-r}(E^*)$ and E is the Hodge bundle defined in Sec. **??**. Like E the field Ψ_f is descendant class on the moduli space $\mathcal{M}_{g(v),n}$ of the component of $\Sigma_{g,n}$, which is contracted a vertex

It remains to construct the obstruction bundle $\phi = c^{vir}(U_\beta)$. For compact M the virtual fundamental class $[\overline{\mathcal{M}}_{g,0}(M,\beta)]^{vir}$ does no embed into $[\overline{\mathcal{M}}_{g,0}(M_T,\beta)]^{vir}$ for $g > 0$ and it is therefore not clear[26] how to restrict in the moduli space of higher genus maps from those mapping to M_T to those mapping to M, see last section of[88].

The $g = 0$ localization for the quintic is discussed in [141], here we focus on the non-compact toric cases which can be solved to all genus by localization. We identify $V = K_B$, i.e. with the canonical bundle K_B over the compact base, e.g. $\mathcal{O}(-3)$ for the local $\mathcal{O}(-3) \to \mathbf{P}^2$ or $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ for the resolved conifold. Now we have to construct the class $\phi = c(U_\beta)$ and the moving part of its pushforward under i in

[26] I like to thank Tom Coates for a note regarding this and the reference [88].

equivariant cohomology. That was done in [135] and gives

$$i^*\phi = \prod_e \prod_{k=1}^{d-1} \prod_{l=1}^{d_e(-n_{j(e)_k})-1} -\frac{l\alpha_e}{d_e} - \alpha_{j(e)_k} \times \begin{cases} \prod_v \prod_{e'(v)} (\alpha_{e'})^{\mathrm{val}(v)-1} \,, & \text{for g} = 0 \\ \prod_v \prod_{e'(v)} (\alpha_{e'})^{\mathrm{val}(v)-1} P_{g(v)}(\alpha_{e'}, E^*) \,, & \text{for g} > 0 \,. \end{cases} \tag{168}$$

Here e' are the non-compact edges. Now (156) can be applied to give

$$\int_{\mathcal{M}_{g,n}(\beta,M_T)} c^{vir}(U_\beta) = \sum_\Gamma \frac{1}{|Aut(\Gamma)|} \int_{\mathcal{M}_\gamma} \frac{i_\Gamma^* \phi}{e(\nu_\Gamma)} \tag{169}$$

The integration in this formula is over suitable products of the classes ψ_f and $c_k(E^*)$ in the expansion of $\frac{i^*\phi}{e(\nu_F)}$ over the $3g(v)-3+n$ dimensional moduli space $\overline{\mathcal{M}}_{(v),n}$ of contracted components $\Sigma_{g(v),n}$ of the domain curve. This can be viewed as an excess intersection calculation. The result of these 2d-gravity integrals was described in Sec. 6.3. For a given g and class β all graphs have to be summed up to yield the contributions of this class. A considerable complication is that each graph comes with a particular automorphism factor $|Aut(\Gamma)|$, reflecting the discrete symmetries of the fixed map, by which one has to divide. It is easy to see that each edge gives a contribution d_e to $|Aut(\Gamma)|$, the rest of the symmetry factors can be obtained similarly like for a Feynmann graph expansion. The result will not depend on the value of the torus weghts α_i.

6.12 Localization of open string amplitudes

Heuristicly it is relatively easy to generalize these formulas to include open strings bounding the special Lagarangian branes discussed in Sec. 6.10. The boundaries of a Riemann-surface with h holes $\Sigma_{g,n,h}$ have to map to the non-trivial S^1 in L_i. Any disk component attaches to the rest of $\Sigma_{g,n,h-1}$ in a marked point and T^d invariance implies that this marked point is mapped to a fixpoint. Therefore each disk component i will map to a disk like the one shown in Fig. 21 with winding m_i around the $S^1 \in L$.

The relevant deformations $H^0(\Sigma, x^*T_{M_T})$ and obstructions $H^1(\Sigma, x * T_{M_T})$ are expressed by the weights [93]

$$\left.\frac{i_\Gamma^*\phi}{e(\nu_\Gamma)}\right|_{disk} = \left(\frac{1}{\prod_{k=1}^m \frac{k\alpha_f}{m}}\right)\left(\prod_{k=1}^{m-1}\left[\left(\frac{k\alpha_f}{m}\right) + \alpha_{N_i}\right]\right) \,. \tag{170}$$

The flag corresponds to the halfline in Fig. 21, which is the image of the disk under the moment map. The combinatoric of the open fixed graphs is rather abvious. We add to the figures in (22) flags, which are decorated by the winding m. Note that such a flag contributes $1/m$ to $|Aut(\Gamma)|$. Different then the closed string localization the open string localization is not based on a mathematical rigerous understanding of the open string moduli space. The evidence comes, among other considerations, from a comparison with the mirror calcalations in []. The result of the calculation has a residual dependence on one combination of torus weight. The choice of the normal bundel N_i can be absorbed in this weight depence, which is ralated to the framing ambiguity in Chern-Simons theory []. As an excercise one may that disk contribution with all windings m in Fig. 21 add up to the generating function

$$W = \frac{\prod_{j=1}^{m-1}\left(j + m\frac{\alpha_{N_i}}{\alpha_f}\right)}{m!m} e^{ms} \,. \tag{171}$$

This function is the superpotential for the $N=1$ theory which lives on $L \times M_{1,3}$.

6.13 Localization on $\mathcal{O}(-1)\oplus\mathcal{O}(-1)\to\mathbf{P}^1$ and $\mathcal{O}(-3)\to\mathbf{P}^2$.

The explicte formulas (169,167, 168,170) as well the solution of 2d gravity using the matrix model approach or the Virasoro constraints sublemented by the reduction of the integrals using the $c_k(E)$ classes as described below (144) give an in general combinatorial very tedious but complete solution for the closed and open string on non-compact toric Calabi-Yau spaces.

As a longer exercise the reader can check for example that the $g=0$ multi covering formula (183) comes out for $\mathcal{O}(-1)\oplus\mathcal{O}(-1)\to\mathbf{P}^1$. To do that one can make a choice for the values of the weights in which is apparent that most of the graphs vanish, see [68] [105]. Using special weights and closed expressions for certain classes of Hodge integral [68] prove the following all genus result. Let

$$\mathcal{F}(\lambda,t)=\sum_{g=1}^\infty \lambda^{2g-2}\mathcal{F}^{(g)}(t)=\sum_{g=0}^\infty\sum_{d=1}^\infty r^g_d\lambda^{2g-2}q^d\,, \tag{172}$$

where the r^g_d are the Gromow-Witten invariants defined in (150). $d\in\mathbf{Z}$ specifies the degree in $H_2(M,\mathbf{Z})$, which is generated by the $\mathbf{P}^1$ and $q:=\exp(2\pi t)$. The result of Faber and Pandharipande [68] gives all r^g_d by the formula

$$\mathcal{F}(\lambda,t)=\sum_{d=1}^\infty \frac{q^d}{d\left(\sin\frac{d\lambda}{2}\right)^2}\,. \tag{173}$$

In this special geometry we can understand all contributions as the multicovering of the $\mathbf{P}^1$, which is the only non-trivial holomorphic curve in this geometry, by maps of various degree and genus.

In general non-compact Calabi-Yau can support holomorphic curves in infinitly many classes β. E.g. for the closed string amplitudes on $\mathcal{O}(-3)\to\mathbf{P}^2$ [135] obtain

$$\begin{aligned}
F^{(0)} &= -\frac{t^3}{18}+3\,q-\frac{45\,q^2}{8}+\frac{244\,q^3}{9}-\frac{12333\,q^4}{64}+\frac{211878\,q^5}{125}\ldots\\
F^{(1)} &= -\frac{t}{12}+\frac{q}{4}-\frac{3\,q^2}{8}-\frac{23\,q^3}{3}+\frac{3437\,q^4}{16}-\frac{43107\,q^5}{10}\ldots\\
F^{(2)} &= \frac{\chi}{5720}+\frac{q}{80}+\frac{3\,q^3}{20}-\frac{514\,q^4}{5}+\frac{43497\,q^5}{8}\ldots\\
F^{(3)} &= -\frac{\chi}{145120}+\frac{q}{2016}+\frac{q^2}{336}+\frac{q^3}{56}+\frac{1480\,q^4}{63}-\frac{1385717\,q^5}{336}\ldots\\
F^{(4)} &= \frac{\chi}{87091200}+\frac{q}{57600}+\frac{q^2}{1920}+\frac{7\,q^3}{1600}-\frac{2491\,q^4}{900}+\frac{3865234\,q^5}{1920}\ldots\\
F^{(5)} &= -\frac{\chi}{2554675200}+\frac{q}{1774080}+\frac{q^2}{14080}+\frac{61\,q^3}{49280}+\frac{4471\,q^4}{22176}-\frac{65308319\,q^5}{98560}\ldots\,.
\end{aligned} \tag{174}$$

Due to the non-trivial holomorphic curves in all degrees it is hard to to give $\mathcal{F}(\lambda,t)$ in closed form, even though closed expressions for the $\mathcal{F}^{(g)}(t)$ can be given using mirror symmetry and the B-model [135]. The combinatoric of the A-model localisation calculation is involved. E.g. for the genus 5 degree 5 terms one has to sum over $\sim 10^4$ graphs.

6.14 BPS invariants for branes wrapping curves

Many fascinating topological and physical ideas enter the reintepretation of $\mathcal{F}^{(g)}(t)$ as BPS counting function[91]. The argument splits in a supergravity and a geometrical part

- The $N=2$ supergravity action contains terms $\sum_{g>0}\int_{M^4}\mathrm{d}^4x\mathcal{F}^{(g)}(t,\bar t)T_-^{2g-2}R_-\wedge R_-$, which couple the anti-selfdual part of the curvature R_+ with the anti-selfdual part of the graviphoton field strength T_-. The above terms are part of the component form of $\int_{M^4}\mathrm{d}^4x\mathrm{d}^4\theta\mathcal{F}^{(g)}(t,\bar t)(W^2)^{g-1}$, where $W^2=\epsilon_{ij}\epsilon_{kl}W^{ij}_{\mu\nu}W^{kl}_{\mu\nu}$ and $W^{ij}_{\mu\nu}=\epsilon^{ij}T_{\mu\nu}-R_{\mu\nu\eta\delta}\theta^i\sigma^{\eta\delta}\theta^j+\ldots$ is a chiral multiplet. The structure of $N=2$ supergravity in TypeII string on M implies that in the topological limit $\mathcal{F}^{(g)}(t)=\lim_{\bar t\to\bar t_0}\mathcal{F}^{(g)}(t,\bar t)$ is identified with the topological string free energy (209) [20][8]. It depends only on vector multiplets. This statements require like (52) a certain genericity assumptions.

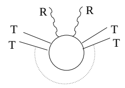

Fig. 23 BPS saturated one-loop graph contribution to $T^{2g-2}R^2$

Moreover supergravity puts the following restriction on this amplitude[8]. It is generated at one-loop and at one-loop only, the corresponding graph is shown in Fig. 23. The only particles which can contribute in the loop are BPS states. Their mass is determined by their charge. Once *mass* and *spin* of the BPS particle is known is contribution to $\mathcal{F}^{(g)}(t)$ can be evaluated by a Schwinger-loop calculation.

- In the geometrical consideration one has to identify the *mass* and *spin* of BPS particle with the geometrical properties of the embedded branes. The mass is easy and will be discussed below. The spin part is more complicated and is discussed in Sec. 6.16

Because the type II string coupling $g_s = \lambda$ is in a hyper multiplet and the above decoupling one expects that the strongly coupled M-theory- $\lambda \gg 1$ and the weakly coupling IIA description $\lambda \ll 1$ are equivalent points of view. The former description involves BPS states as coming from $M2$ branes the latter as coming from $D2 - D0$ bound states. In both cases the extended branes wrapping curves C in M in the class β. The mass is given straightforwardly as

$$m(\beta, k) = \beta \cdot t + 2\pi i k = \sum_{i=1}^{h^{1,1}} t_i \int_{C_\beta} \omega_i + 2\pi i k, \qquad \beta \in H^2(M, \mathbf{Z}), \qquad k \in \mathbf{Z} \;, \tag{175}$$

were the first term is the minimal volume of the curve on which the extended brane wraps. The second can be either viewed as the momentum k of $M2$ on the M theory circle or as the number k of $D0$ branes. The latter form in arbitrary number boundstates with the $D2$ brane.

Consider now an M-theory compactification on M to five dimensions. The space time BPS states fall into representations of the rotational group of the 5d Lorentz group $M = \mathrm{SO}(4) \simeq \mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$. As mentioned the low energy interpretation of the free energy $\mathcal{F}$ in 4d relates it to the 5d BPS spectrum through a Schwinger one loop calculation of the 4d $\int_{M^4} T_-^{2g-2} R_-^2$ effective terms[27]. Note that these 4d calculations are sensitive to the off shell quantum numbers, i.e. to $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$. Only BPS particles annihilated by the supercharges in the $(\mathbf{0}, \frac{\mathbf{1}}{\mathbf{2}})$ representation contribute to the loop. They couple to the anti-selfdual graviphoton field strength T and the anti-selfdual curvature R only via their left spin eigenvalues of their representation under M. The right representation content enters solely via its multiplicity and a sign $(-1)^{2j_R^3}$, in particular any contribution of long multiplets is projected out by these signs. To summarize, the dependence of $\mathcal{F}$ on the BPS spectrum is via a supersymmetric index

$$I(\alpha, \tau) = \mathrm{Tr}_{\mathcal{H}} (-1)^F e^{-\alpha j_L^3 - \tau H} \;, \tag{176}$$

where $F = 2j_L^3 + 2j_R^3$, and all spin information entering $\mathcal{F}$ is carried by $\left[(\frac{\mathbf{1}}{\mathbf{2}})_L + 2(\mathbf{0})_L\right]$ times the following combination

$$\sum_{j_L^3, j_R^3} (-1)^{2j_R^3} (2j_R^3 + 1) N^\beta_{j_R^3, J_L^3} [\mathbf{j_L}] = \sum_{g=0}^{\infty} n_\beta^{(g)} I_g \;. \tag{177}$$

[27] A similar one loop calculation corrects the effective gauge coupling $\frac{1}{g^2(G,p^2)}$ through threshold effects in heterotic strings [124].

The multiplicities of the BPS states $N^\beta_{j^3_R,J^3_L}$ enters only via the index like quantity $n^{(g)}_\beta$. Indeed the basis change of the left spin from $[\mathbf{j_L}]$ to

$$I_g = \left[\left(\mathbf{\frac{1}{2}}\right)_L + 2(\mathbf{0})_L\right]^{\otimes g} \tag{178}$$

relates the left spin to the genus g of C as explained in Sec. 6.16 and defines the integer Gopakumar-Vafa invariants n^g_β associated to a holomorphic curve C of genus g in the class β. The expansion of $\mathcal{F}$ in terms of these BPS state sums is now obtained by performing the Schwinger loop integral, which for given mass $m(\beta,k)$ and j^3_L quantum numbers is

$$\int_0^\infty \frac{\mathrm{d}\tau}{\tau} \frac{e^{-\tau m}}{\left(2\sin\frac{\tau\lambda}{2}\right)^2} \mathrm{Tr}(-)^F e^{-2\pi i J^3_L \lambda}\ . \tag{179}$$

In performing it for all $m(\beta,k)$ and g we note that I_g is a very convenient basis as $\mathrm{Tr}_{I_g}(-)^F e^{-2i\tau j^3_L\lambda} = \left(\sin\frac{\tau\lambda}{2}\right)^{2g}$ and that the sum over k gives a δ function, which makes the $\mathrm{d}\tau$ integration trivial, so that we get quite straightforwardly

$$\begin{aligned}\mathcal{F}(\lambda,t) &= \sum_{g=0}^\infty \lambda^{2g-2}\mathcal{F}^{(g)}(t)\\ &= \frac{c(t)}{\lambda^2} + l(t) + \sum_{g=0}^\infty \sum_{\beta\in H_2(M,\mathbf{Z})} \sum_{m=1}^\infty n^{(g)}_\beta \frac{1}{m}\left(2\sin\frac{m\lambda}{2}\right)^{2g-2} q^{\beta m}\\ &= \frac{c(t)}{\lambda^2} + l(t) + \sum_{g=0}^\infty \sum_{\beta\in H_2(M,\mathbf{Z})} \sum_{m=1}^\infty n^{(g)}_\beta (-1)^{g-1}\frac{[m]^{(2g-2)}}{m} q^{\beta m},\end{aligned} \tag{180}$$

with

$$q^\beta = e^{i\sum_{i=1}^{h^{1,1}} t_i \int_{C_\beta}\omega_i}, \qquad [x] := q_\lambda^{\frac{x}{2}} - q_\lambda^{-\frac{x}{2}}, \qquad q_\lambda = e^{i\lambda}.$$

The cubic term $c(t)$ in the Kähler parameters t_i is the classical part of the prepotential $\mathcal{F}^{(0)}$ given in (295) without the constant term, and $l(t) = \sum_{i=1}^h \frac{t_i}{24}\int_M \mathrm{ch}_2 J_i$ is the classical part[28] of $\mathcal{F}^{(1)}$. Using the expansion

$$\frac{1}{m}\frac{1}{\left(2\sin\frac{m\lambda}{2}\right)^2} = \sum_{g=0}\lambda^{2g-2}(-1)^{g+1}\frac{B_{2g}}{2g(2g-2)!}m^{2g-3} \tag{181}$$

and a $\zeta(x) = \sum_{m=1}^\infty \frac{1}{m^x}$ regularization of the sum over m with $\zeta(-n) = -\frac{B_{n+1}}{n+1}$, we see that for $g\geq 2$ the $\beta=0$ constant map terms from localization (144) [68]

$$\langle 1\rangle^M_{g,0} = (-1)^g\frac{\chi}{2}\int_{\mathcal{M}_g} c^3_{g-1} = (-1)^g\frac{\chi}{2}\frac{|B_{2g}B_{2g-2}|}{2g(2g-2)(2g-2)!} \tag{182}$$

are reproduced if we set $n^{(0)}_0 = -\frac{\chi}{2}$. This choice also reproduces the constant term proportional to $\zeta(3)$ in $\mathcal{F}^{(0)}$. In $\mathcal{F}^{(1)}$ there is a $\zeta(1)$ term which requires an additional regularization. More importantly expanding

[28] These terms do not follow entirely from the Schwinger-loop calculation and added here for completness.

(180) in λ and comparing with (209) predicts the *multicoving formulas* at all genus. Specialized to one Kähler class such that β is identified with the degree $d \in \mathbf{Z}$ we get

$$\begin{aligned}
\mathcal{F}^{(0)} &= \frac{D^3 t^3}{3!} + t\int_M c_2 \wedge \omega - i\frac{\chi}{2(2\pi)^3}\zeta(3) + \sum_{d=1}^{\infty} n_d^{(0)}\,\mathrm{Li}_3(q^d)\ ,\\
\mathcal{F}^{(1)} &= \frac{t\int c_2 J}{24} + \sum_{d=1}^{\infty}\left(\frac{1}{12}n_d^{(0)} + n_d^{(1)}\right)\mathrm{Li}_1(q^d)\ ,\\
\mathcal{F}^{(2)} &= \frac{\chi}{5760} + \sum_{d=1}^{\infty}\left(\frac{1}{240}n_d^{(0)} + n_d^{(2)}\right)\mathrm{Li}_{-1}(q^d),\\
\mathcal{F}^{(g)} &= \frac{(-1)^g\chi|B_{2g}B_{2g-2}|}{4g(2g-2)!(2g-2)} + \sum_{d=1}^{\infty}\left(\frac{|B_{2g}|n_d^0}{2g(2g-2)!} + \frac{2(-1)^g n_d^2}{(2g-2)!} \pm \ldots - \frac{g-2}{12}n_d^{g-1} + n_d^g\right)\mathrm{Li}_{3-2g}(q^d).
\end{aligned} \tag{183}$$

Using resummations like (181) one checks that the partition function $Z^{\rm hol} = \exp(\mathcal{F}^{\rm hol})$ has the following product form[29]

$$Z^{\rm hol}_{\rm GV}(M,\lambda,q) = \prod_\beta\left[\left(\prod_{r=1}^{\infty}(1-q_\lambda^r q^\beta)^{r n_\beta^{(0)}}\right)\prod_{g=1}^{\infty}\prod_{l=0}^{2g-2}(1-q_\lambda^{g-l-1}q^\beta)^{(-1)^{g+r}\binom{2g-2}{l}n_\beta^{(g)}}\right] \tag{184}$$

in terms of the invariants $n_\beta^{(g)}$. This product form resembles the Hilbert scheme of symmetric products written in terms of partition sums over free fermionic and bosonic fields with an integer $U(1)$ charge as well as the closely related product form for the elliptic genus of symmetric products. As it has already been pointed out in [90], it is also reminiscent of the Borcherds product form of automorphic forms of $O(2,n,\mathbf{Z})$, see [25] and [142] for a review. Here the idea is that integrality of the $n_\beta^{(g)}$ is related to the fact that they are Fourier coefficients of other (quasi)automorphic forms, see also [130].

6.15 BPS count, heteroric string and modular functions

As the above ideas originate to some extend from the duality of N=2 Type II to the heterotic string, some of the strongest predictictions for the n^g_β invariants on compact Calabi-Yau manifolds can be made if dual pairs of heterotic/type II compactifications are known, see Fig. 1. The relevant Calabi-Yau manifolds are K3 fibrations over $\mathbf{P}^1$[122] [133] and the heterotic weak coupling limit is translated to infinite volume limit of the base $\mathbf{P}^1$. The heterotic prediction relies on a perturbative WS one-loop calculation in the weak coupling limit and makes therefore only predictions for n^g_β if β is a class entirely in the $K3$ fibre. Information about other classes $\hat\beta$ is supressed, because of $q^{\hat\beta} \to 0$ in the weak coupling/infinite base limit. The one-loop (torus) amplitude is[9]

$$\mathcal{F}^g = \int_{\mathcal{F}} {\rm d}\tau\tau_2^{2g-3}\frac{1}{|\eta|^4}\sum_{even}\frac{i}{\pi}\partial_\tau\left(\frac{\theta\left[{a \atop b}\right](\tau)}{\eta(\tau)}\right)Z_g^{int}\begin{bmatrix}a\\b\end{bmatrix}, \qquad Z_g^{int}\begin{bmatrix}a\\b\end{bmatrix} = \langle : (\partial X)^{2g} : \rangle\ . \tag{185}$$

The integrand can be understood as an index on the heterotic WS theory very similar to (323)[108] and the integral over the fundamental region $\mathcal{F}$ of the torus can be calculated using the modular properties of the integrand in an ingeneous way[108][25] [154]. For the $K3$ fibrations without reducible fibres one finds in the holomorphic limit[138]

$$\mathcal{F}^{\rm hol}({\rm Fibre}_{\rm K_3},\lambda,{\rm q}) = \frac{\Theta({\rm q})}{\rm q}\left(\frac{1}{2\sin(\frac{\lambda}{2})}\right)^2\prod_{\rm n\geq 1}\frac{1}{(1-{\rm q}_\lambda{\rm q}^{\rm n})^2(1-{\rm q}^{\rm n})^{20}(1-{\rm q}_\lambda^{-1}{\rm q}^{\rm n})^2}\ . \tag{186}$$

[29] Here we dropped the $\exp(\frac{c(t)}{\lambda^2} + l(t))$ factor of the classical terms at genus $0, 1$.

Similar as in the case of elliptic Del-Pezzo surfaces S embedded in Calabi-Yau manifolds [116] the product factor can be interpreted as the Goettsches formula for the cohomology of the resolved Hilbert scheme of points on the surface S or the $K3$ respectively. The formula (186) can also be viewed as an extension of the analysis of [215] to a situation with less supersymmetry.

Example: The degree 12 hypersurface in the weighted projective space $WCP(1,1,2,2,6)$, see Sec. 9.10 is a K3 fibration, which is dual to the ST heterotic string discussed in [122]. In this case $\Theta(q)$ is [129] $\frac{\theta(q)}{\eta^{24}} = -\frac{2\sigma E_4 F_6}{\eta^{24}} = -\frac{2}{q} + 252 + 2496 q^{\frac{1}{4}} + 223752 q + \ldots$, where $\sigma(q) = \sum_{n\in\mathbf{Z}} q^{\frac{n^2}{4}}$ and $F_2(q) = \sum_{n\in\mathbf{Z}_{>0},\mathrm{odd}} \sigma_1(n) q^{\frac{n}{4}}$ generate the ring of modular forms for the congruence subgroup $\Gamma^0(4)$, and $F_6 = E_6 - 2F_2(\sigma^4 - 2F_2)(\sigma^4 - 16F_2)$. The embedding of the Picard lattice of the $K3$ into the Calabi-Yau M is specified by the replacement of $\lambda^{2g-2} q^l \to \frac{1}{(2\pi i)^{3-2g}} \sum_{n^2/4=l} \mathrm{Li}_{3-2g}(q^{\beta n})$ in (186), where β is the single class in the $K3$ fibre. Comparing with (180) one gets predictions in a closed form for n^g_β for all g and all β. Below are the first few listed

g	$\beta=1$	2	3	4	...
0	2496	223752	38637504	9100224984	...
1	0	-492	-1465984	-1042943520	...
2	0	-6	7488	50181180	...
3	0	0	0	-902328	...
4	0	0	0	1164	...
5	0	0	0	12	...
⋮	⋮	⋮	⋮	⋮	...

Many of these predictions from string duality have been checked in in [138] using the geometrical techniques described in the next section .

6.16 Geometric interpretation of the BPS numbers and their relation to Donaldson-Thomas invariants

As usual in theory of BPS solitons the degeneracy of the BPS states comes from the cohomology of the moduli space of the solitonic solutions, in this case of the brane solution. This moduli space is the vacuum manifold of the brane world volume theory, which is parametrized by the zero modes and the cohomological information is extracted by quantizing this zero mode sector as shortly discussed in Sec.4.1.

In the following we will discuss only single wrapped branes. For the M2 brane the eleven dimensional tangent space splits $0 \to N_8 \to T_{11} \to T_{M2} \to 0$. The normal space N_8 is decomposed into $N \times \mathcal{N}$, where $\mathcal{N}$ is the normal direction in the CY M and N are the spacial directions of $5d$ Minkowski space. The CY tangent space splits as well $0 \to \mathcal{N} \to T_M \to T_C \to 0$. The unbroken space-time symmetries $G_{N_8} = SO(4)_N \times U(2)_{\mathcal{N}}$ transversal to the brane become R-symmetries of the fields on the brane-world-volume. For holomorphic curves in n complex dimensional Kähler manifolds the generic structure group of normal bundle $SO(2(n-1))$ restricts because of property (ii) in Bergers list, Sec.9.9, to $U(n-1)_{\mathcal{N}}$. For Calabi-Yau manifolds it follows from the adjunction formula (402) and the vanishing of the first Chern-class that $c_1(\det(U(n-1))_{\mathcal{N}}) = c_1(T^*C)$, i.e. over C the $U(1)_{\mathcal{N}} \in U(n-1)_{\mathcal{N}}$ can be identified with the $U(1)_{\mathcal{N}}$ connection in the canonical bundle $K_C = T^*C$. This identification of the R-symmetry transformation of the normal bundle with the WS transformations on C leads to a natural twisting of the brane-world-volume theory [21].

Let us describe the transformation properties of theses fields on the brane under $G_T = SO(2,1)$ the Lorentzgoup on the brane and $G_{N_8} = SO(4)_N \times U(1)_{L,\mathcal{N}} \times SU(2)_{R,\mathcal{N}}$ the R-symmetry from the normal direction

- Before twisting the eight fermions $\psi \in [s, \mathbf{8}_s]$ transforms as spinor with helitity $s = \pm\frac{1}{2}$ under G_T and as spinor under $G_N = SU(2)_{L,\,N} \times SU(2)_{R,\,N} \times U(1)_{L,\mathcal{N}} \times SU(2)_{R,\mathcal{N}}$. The $U(1)_{L,\mathcal{N}}$ connection is identified with the connection in K_C. It changes the helicity of fields in $\sqrt{K}$ therefore by $0, \pm\frac{1}{2}$ depending on their $U(1)_{L,\mathcal{N}}$ charge

$$\begin{aligned} \psi \quad &\in \left[s, [(0,\tfrac{1}{2})_N \otimes (0,\tfrac{1}{2})_{\mathcal{N}}] \oplus [(\tfrac{1}{2},0)_N \otimes (\pm 1, 0)_{\mathcal{N}}]\right] \\ \psi_T \quad &\in \left[\pm\tfrac{1}{2}, (0,\tfrac{1}{2})_N \otimes (\tfrac{1}{2})_{R,\mathcal{N}}\right] \oplus \left[2(0), (\tfrac{1}{2},0)_N \otimes (0)_{R,\mathcal{N}}\right] \oplus \left[\pm 1, (\tfrac{1}{2},0)_N \otimes (0)_{R,\mathcal{N}}\right] \end{aligned} \tag{187}$$

here the $U(1)_{L,\mathcal{N}}$ charge is combined with the helicity in G_T to the first entry $h = \pm\frac{1}{2}, 0, \pm 1$ in the twisted representation ψ_T, which implies that the field is a section of K_C^h.

- For the eight bosons ϕ corresponding to the coordinates of the normal directions

$$\begin{aligned} \phi \quad &\in \left[0, [(\tfrac{1}{2},\tfrac{1}{2})_N \otimes (0,0)_{\mathcal{N}}] \oplus [(0,0)_N \otimes (\pm 1, \tfrac{1}{2})_{\mathcal{N}}]\right] \\ \phi_T \quad &\in \left[0, (\tfrac{1}{2},\tfrac{1}{2})_N \otimes (0)_{R\mathcal{N}}\right] \oplus \left[\pm\tfrac{1}{2}, (0,0)_N \otimes (\tfrac{1}{2})_{R\mathcal{N}}\right] \end{aligned} \tag{188}$$

Clearly the zero modes of the bosons transforming as $\left[\pm\frac{1}{2}, (0,0)_N \otimes (\frac{1}{2})_{R\mathcal{N}}\right]$ and the fermions transforming as $\left[\pm\frac{1}{2}, (0,\frac{1}{2})_N \otimes (\frac{1}{2})_{R\mathcal{N}}\right]$ correspond to deformations (and superdeformations) of C in the CY direction and parametrize the moduli space $\mathcal{M}_C$ of movements of C within M. Fermionic and bosonic zero modes form the field content of a supersymmetric σ model on $\mathcal{M}_C$ and after quantization one gets the cohomology of the moduli space of $\mathcal{M}_C$ weighted in addition with the R quantum number of the fermions modes from their $SU(2)_{N,R}$ transformation. The corresponding representations are identified with the Lefshetz decomposition of the cohomology of the Kähler manifold (351) $\mathcal{M}_C$. Other fermionic modes in ψ_T transform as 2 scalars the $(0,0)$ and $(1,1)$ form and g holomorphic and g antiholomorphic one forms on the genus g curve C if the latter does not degenerate. The corresponding zero modes are then forms on $(2g+2)$ dimensional torus, which form $SU(2)_{N,\,L}$ representations $[(\frac{1}{2},0)+2(0,0)]^{g+1}$, cff. (351). By the definition (177) only the multiplicity $(2j_R^3+1)$ and the sign $(-1)^{2j_R^3}$ of the cohomology of $\mathcal{M}_C$ are relevant for the determination of n_β^g. This alternating sum is just the Euler number $(-1)^m\chi(\mathcal{M}_C)$, with $m = \dim_C(\mathcal{M}_C)$. For classes β in M with non degenerate genus g curves we get therefore as coefficient of I_{g+1}

$$n_\beta^g = (-1)^m \chi(\mathcal{M}_C)\ . \tag{189}$$

An instructive example is a that of a ruled surface (RS) inside M. Familiar ruled surfaces are the Hirzebruch surfaces F_n fibrations of a $\mathbf{P}^1$ bundle over $\mathbf{P}^1$. More generally the base can be a higher genus surface Σ_g. We want to calculate the n_β^0 for the class of the fibre. The genus zero fibre curve $C = \mathbf{P}^1$ is smoothly embedded and zero is the maximal genus of a curve in the class. Due to the fibration structure of the RS the moduli space $\mathcal{M}_C = \Sigma_g$ is identified with the base. So (189) applies and gives $n_\beta^g = (-1)^1\chi(\Sigma_g) = 2g-2$. The embbeding of Σ_g is locally described by $\mathcal{O}(r)\otimes\mathcal{O}(s) \to \Sigma_g$ with $r+s = 2g-2$. Unless $g=0$ ($r=s=-1$) the curve Σ_g is not rigid in M and for $g>0$ the curve Σ_g can be deformed to $2(g-1)$ points in M, as in the Fig. 24.

The $SU(2)_{N,R}$ content before deformation is $R = 2g(0) - \left(\frac{1}{2}\right)$ with $\chi(R) = 2g-2 = -\chi(\Sigma_g)$ and after deformation $R' = (2g-2)(0)$ with $\chi(R') = 2g-2 = +\chi(2(g-1)\ pts)$. I.e. the total BPS numbers $N^g_{J_R^3,J_L^3}$ change by states with $\left[2(0) - \left(\frac{1}{2}\right)\right]$ right representation content, when the complex structure moduli space of M is deformed. So in contrast to the $n_\beta^{(g)}$, the $N^\beta_{j_R^3,j_L^3}$ are not invariant under the change of the complex structure. Notice that the successful microscopic interpretation of the 5d black hole entropy requires deformation invariance and relies on the index-like quantity n_β^g and not on $N^g_{J_R^3,J_L^3}$.

Example: Such ruled surfaces appear typically if one embedds the Calabi-Yau in a weighted projective space. E.g. the degree 14 hypersurface in $WCP^4(1,2,2,2,7)$, see Sec. 9.10, contains a ruled surface with a genus 15 curve as base[30]. The genus $g = 15$ curve is *semi* stable because the relevant complex

30 Such case have been investigated [137], [127], because there have interesting gauge symmetry enhancements, when the $\mathbf{P}^1$ shrinks.

deformation moduli are frozen as an artifact of the embedding. For other realization of the same family that is not necessarily the case.

Above is a good example to get a rough idea of some concepts of *virtual intersection theory*. The virtual dimension of the brane moduli space here is expected to be zero by (149) or here equivalently by (193). In this preferred situations the intersection problem is reduced to point counting, but the situtation might not be achievable as in the example above and the moduli space remains positive. In this particular case the *excess intersection calculation* amounts to integrate $c_1(T\Sigma_g)$ over Σ_g.

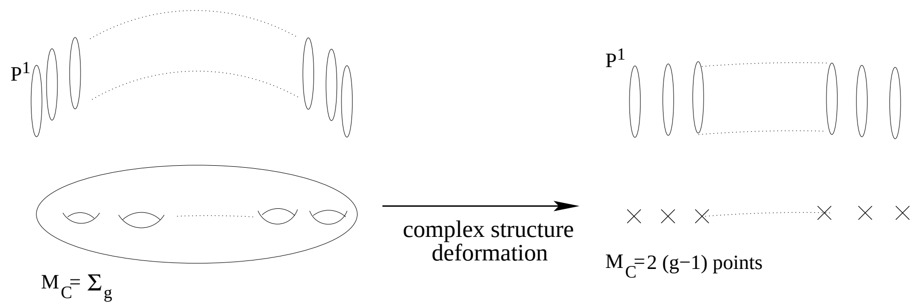

Fig. 24 The index n^g_β of the $D2$-$D0$ moduli space of the fi bre in a ruled surface is constant under complex defomations, while the $N^g_{j^3_L,J^3_R}$ jump.

In the type IIA picture one transversal direction parametrized previously by a scalar in $\left[0,(\frac{1}{2},\frac{1}{2})_N\otimes(0)_{R,\mathcal{N}}\right]$ is dualized on the 3d World-Volume to a $U(1)$ gauge field. The flat $U(1)$ connection has $2g$ zero modes on C exactly as the $\left[\pm 1,(\frac{1}{2},0)_N\otimes(0)_{R,\mathcal{N}}\right]$ fermions in ψ_T. Since these zero-modes parametrize the $2g$ dimensional torus $\mathrm{Jac}(C)$, called the Jacobian of C see [101] Chap 2.7, one gets a SQM on a space $\mathcal{M}$ with a fibration structure $\mathrm{Jac}(C)\to\mathcal{M}\to\mathcal{M}_\mathcal{C}$, see Fig. 25. The proposal [91] for the $\mathrm{SU}(2)_{N\ L}\times\mathrm{SU}(2)_{N\ R}$ action on $\mathcal{M}$ is that $H^*(\mathcal{M})=N^\beta_{j^3_R,j^3_L}[j^3_{R=base},j^3_{L=fibre}]$. Again one can conclude that the contribution n^g_β of smooth genus curves in the class β is the $(-1)^{2j^3{}_R}$ weighted sum of the right representations multiplying the non degenerate fibre contribution I_g in the representation decomposition. This is $(-1)^n\chi(\mathcal{M}_C)$. On the other extreme are the curves which are maximally degenerate. They have genus zero and come from genus g curves with g nodes. The Euler number of the fibres with δ nodes is $\chi(I_{g-\delta})=\delta_{g,\delta}$. Due to the fibration structure the Euler number of $\chi(\mathcal{M})$ is calculated as the Eulernumber of the locus in the base where the completly degenerate fibres sit times one. This is the $(-1)^{2j^3{}_R}$ weighted sum of the right representations on the cohomology of this locus and therefore

$$n^0_\beta=(-1)^{\dim_C(\mathcal{M})}\chi(\mathcal{M})\ . \tag{190}$$

In [128] a calculational scheme for the intermediate cases was given. E.g. if no *reducible fi bres* contribute one obtains

$$n^{g-\delta}_\beta=(-1)^{(\dim(\mathcal{M}_C)+\delta)}\sum_{p=0}^{\delta}b_{g-p,\delta-p}\chi(\mathcal{C}^{(p)}),\quad b_{g,k}:=\frac{2}{k!}\prod_{i=1}^{k-1}(2g-(k+2)+i),\quad b_{g,0}:=1\ . \tag{191}$$

Here $\mathcal{C}^{(p)}$ is the moduli space of the curve C with p points, e.g. $\mathcal{C}^{(0)}=\mathcal{M}_C$. In the case that C lies in a surface S in M, one can use similarly as in (186) formulas for the cohomology of Hilbert scheme to calculate $\chi(\mathcal{C}^{(p)})$, see [128] for examples.

As we saw above we obtain BPS states by wrapping D–branes on supersymmetric cycles in M. More generally we can wrap 6-branes on M itself, 4-branes on divisors and 2-branes on a curves $C\subset M$,

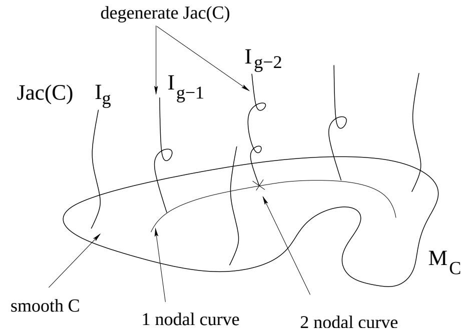

Fig. 25 Moduli space of $D2$-$D0$ brane bound states as a Jacobian fibration over the deformationspace $\mathcal{M}_C$.

possibly bound to some 0-branes. We leave out the 4-branes as we don't know an index yet carrying deformation invariant information. At the level of RR charges a configuration of the other branes can be cast into a short exact sequence of the form

$$0 \longrightarrow \mathcal{I} \longrightarrow \mathcal{O}_M \longrightarrow \mathcal{O}_Z \longrightarrow 0 \tag{192}$$

where $\mathcal{I}$ is the ideal sheaf describing this configuration and Z is the subscheme of M consisting of the curve C and the points at which the 0-branes are supported. Counting BPS states therefore leads to the study of the moduli space $I_k(M, \beta)$ of such ideal sheaves $\mathcal{I}$, which has two discrete invariants: the class $\beta = [Z] \in H_2(M, \mathbf{Z})$ and the number of 0-branes $k = \chi(\mathcal{O}_Z)$ plus an integral contribution form C. With the analogue of the Hirzebruch-Riemann-Roch theorem for sheaves, the Grothendieck-Riemann-Roch theorem[31] , one can calculate the virtual dimension of the deformations of ideal sheaves $\mathcal{I}$ inside a threefold M as [157]

$$\dim_{vir} = \dim \mathrm{Ext}^1_0(\mathcal{I}, \mathcal{I}) - \dim \mathrm{Ext}^2_0(\mathcal{I}, \mathcal{I}) = c_1 \cdot \beta \, . \tag{193}$$

This reflects again the special rôle of Calabi–Yau threefolds and one expects that the number of BPS states with these charges is obtained by counting points. As is in the case of Gromov–Witten invariants, these configurations can appear in families, and one has to work with the virtual fundamental class. However the situation is considered easier in many respects. For example there is no finite automorphism group acting on $I_k(M, \beta)$ so one expects directly integer BPS numbers as result. This number of points is called the Donaldson–Thomas invariant $\tilde{n}^{(k)}_\beta$ [64], [188].

Since both invariants, Gopakumar–Vafa and Donaldson–Thomas, keep track of the number of BPS states, they should be related. The relation is in fact a consequence of the S–duality in topological strings [165], and takes the following form. The factor in (184) coming from the constant maps gives the McMahon function $M(q_\lambda) = \prod_{n\geq 0} \frac{1}{(1-q_\lambda^n)^n}$ to the power $\frac{\chi}{2}$. This function appears also in Donaldson–Thomas theory [157], calculable on local toric Calabi–Yau spaces e.g. with the vertex [1]. However, in Donaldson–Thomas theory the power of the McMahon function is χ. Note also that if (180) holds then $\mathcal{F}$ or Z restricted to this class is always a finite degree rational function in q_λ symmetric in $q_\lambda \to \frac{1}{q_\lambda}$, since the genus is finite in a given class β. Thanks to this observation one can read from the comparison of the expansion of Z^{hol} in terms of Donaldson–Thomas invariants $\tilde{n}^{(m)}_\beta \in \mathbf{Z}$

$$Z^{\mathrm{hol}}_{\mathrm{DT}}(M, q_\lambda, q) = \sum_{\beta, k \in \mathbf{Z}} \tilde{n}^{(k)}_\beta q_\lambda^k q^\beta \tag{194}$$

[31] For Calabi-Yau 3 folds there is an even simpler agument that the differerence below vanishes. Serre duality applies the Ext groups and relates Ext^1_0 and Ext^2_0 on three folds with trivial canonocal bundle.

with the expansion in terms of Gopakumar–Vafa invariants [157]

$$Z^{\rm hol}_{\rm GV}(M,q_\lambda,q)(q_\lambda)^{\frac{\chi(M)}{2}} = Z^{\rm hol}_{\rm DT}(M,-q_\lambda,q) \tag{195}$$

the precise relation between $\tilde n_\beta^{(m)}$ and $n_\beta^{(g)}$. Eq. (180) and (184) then relate the two types of invariants to the Gromov–Witten invariants $r_\beta^{(g)} \in \mathbf{Q}$ as in (172).

7 Large N transitions and the topological vertex

The most effective method to solve the open and closed topological string on open toric Calabi-Yau manifolds imploys the connection of open topological string to Chern-Simons theory [199].

The procedure involves to steps. The first is to provide a building block of the open string amplitude on a $\mathbf{C}^3$ patch. Such a patch is defined by the trivalent vertices in the figures 20. The most general boundary conditions for the open string in this $\mathbf{C}^3$ geometry are stacks of arbitray numbers of D-branes wraping the three special Lagrangian submanifolds discussed in Sec. 6.10. The second step is a space-time surgery procedure for the amplitudes. It is based on the principles of localisation w.r.t. to the $\mathbf{C}^*$ actions of toric geometry. The half lines in Fig. 6.10 support only disks with arbitrary boundary conditions m_i on each of the S^1's of the brane configuration and the closed lines in Fig. 20 support only $\mathbf{P}^1$ components of the image curves. Since the latter can be glued by disks, it is suggestive that the sugery will procede by summing over all possible boundary conditions of the disks. More precisely the contribution to the degree of a map is fixed by the degree d_i of the $\mathbf{P}^1$ components of the image curves. Two disks with winding total M_i on the left and on the right glue to a degree $d_i = M_i$ component of the image curves. For this reason one has has to consider only finitely many boundary conditions if the degree of the map under consideration is fixed.

7.1 Chern-Simons Theory as Gauge Theory description of the Open String

One of the crucial insights used in the derivation of the vertex are the equivalence of the open topological string on T^*M_3 with Chern-Simons $U(N)$ gauge theory on the real three manifold M_3. In this geometry there is a canonical symplectic form $\omega = \sum_{i=1}^3 {\rm d}p_i \wedge {\rm d}q_i$ where q_i are coordinates on M_3 and q_i are coordinates of the cotangential bundel. The M_3 section is at $p_i = 0$ and obviously M_3 is Langrangian submanifold $\omega|_{M_3} = 0$. We can define an almost complex structure with coordinates $z_a = q_a + iq_a$. This is enough to define the A model. In general the complex structure is integrable and ω is Kähler. The form $\Omega = {\rm d}z_1 \wedge {\rm d}z_2 \wedge {\rm d}z_3$ is of type $(3,0)$ and no-where vanishing[32] and $\Omega|_{M_3} = vol(M_3)$ so that M_3 is special Lagrangian as well.

The special Lagrangian boundary conditions are the ones which respect the vector symmetry and the A twisting with Q_A as BRST operator is possible in this geometry. As we will see below the A type reduction of open string field theory in this geometry is not corrected by world-sheet instantons. It includes the coupling to worldsheet gravity and in absence of non-trivial maps this becomes similar as the B-model directly a problem of integrating over the open string moduli space $\mathcal{M}_{g,h}$. Like in the B-model (306) one can use the close similarity between the topological structures of the topological subsectors and the bosonic string provided by (61) in defining the measure on $\mathcal{M}_{g,h}$.

The key step is the reduction of the twisted open string field theory action on T^*M_3 to its zero mode sector. This action is defined by an integral over all open string field functionals Ψ with a BRST operator Q and $*$the folding star product

$$S_{OSFT} = \frac{1}{g_s}\int \frac{1}{2}\Psi * Q\Psi + \frac{1}{3}\Psi * \Psi * \Psi\ . \tag{196}$$

[32] With point d.) of Sec. 9.8 this also defines a local Calabi-Yau manifold.

In the T^*M_3 geometry (196) the zero mode sector is decribed by a Chern-Simons theory, whose gauge group is $U(N)$. In the reduction step the following identifications are made

$$\frac{1}{g_s} \to \frac{2\pi}{k+N}, \quad * \to \wedge, \quad Q \leftrightarrow Q_A \to \mathrm{d}, \quad \Psi \to A, \quad \int \to \int_{M_3}, \tag{197}$$

where A is a $U(N)$ gauge field one form corresponding to the trivial bundle over M_3, see [199] and [159] for a review of the reduction. Hence the actions reduces to the Chern-Simons action

$$S_{CS} = \frac{2\pi}{k+N} \int_{M_3} \frac{1}{2} A \wedge \mathrm{d}A + \frac{1}{3} A \wedge A \wedge A \,. \tag{198}$$

It is important to understand to what extend WS instanton corrections are captured by this action. Like in Sec. 6.1 the A model localization (114) implies that instantons are holomorphic maps of the WS to M. The Lagrangian condition is designed so that minimal surfaces bounding L are (j, J) holomorphic curves. We can integrate $\omega = \mathrm{d}\rho$ in T^*M_3 with $\rho = \sum_{i=1}^{3} p_i \mathrm{d}q_i$. The non BRST trivial part of the A-model action can now be written similar as in (112) as

$$\int_{\Sigma_{g,h}} \left(\partial_z x^i \partial_{\bar z} x^{\bar\jmath} g_{i\bar\jmath} - \partial_{\bar z} x^i \partial_z x^{\bar\jmath} g_{i\bar\jmath} \right) = \int_{\Sigma_{g,h}} x^*(\omega) = \int_{\partial\Sigma_{g,h}} x^*(\rho) = 0 \,, \tag{199}$$

where the last integral vanishes, because its integrand is pulled back from L where ρ vanishes. The right-hand side is positive unless x is a constant map (zero mode) and because of the boundary conditions it must map $\Sigma_{g,h}$ to L. The action (198) captures exactly these degenerate maps and is uncorrected in the T^*M geometry. However non-trivial open string instantons do exist, when ω is not a trivial class, i.e in particular in any compact CY and on more complicated non-compact examples where more SLAGS exist. In this case we have as usual a weight $t_i = \int_\Sigma x_i^*(\omega)$ for the bulk instanton action and gets an instanton corrected action

$$S_{corr} = \frac{2\pi}{k+N} \int_{M_3} \frac{1}{2} A \wedge \mathrm{d}A + \frac{1}{3} A \wedge A \wedge A \; + \sum_i^\infty \eta_i e^{-t_i} \mathrm{Tr} P \exp(\int_C A), \tag{200}$$

where $\eta_i = \pm$ is a determinant ratio.

A reduction for the B twisting can done on any Calabi-Yau space for the boundary of space filling D branes. In this case the identifications are

$$\Psi \to A, \quad Q \leftrightarrow Q_B \to \bar\partial, \quad * \to \wedge, \quad \int \to \int_M \Omega \wedge \tag{201}$$

lead to the holomorphic Chern-Simons actions of a field theory in six dimensions

$$S_{HCS} = \frac{1}{g_s} \int_M \Omega \wedge \left(\frac{1}{2} A \wedge \bar\partial A + \frac{1}{3} A \wedge A \wedge A \right). \tag{202}$$

Dimensional reduction of this action locally along the normal bundles to holomorphic curves in M lead tractable B-model open string calculations in non-compact Calabi-Yau manifolds [4][3] and the matrix model approach to the B-model [57][58].

7.2 Geometric transitions

We discussed in Sec. 7.1 a gauge theory description of the open topological string T^*M geometries. Geometric transitions link such open string geometries to a dual geometries for the closed topological string. More precisely the claim is that the large N gauge theory corresponds exactly to the closed topological string in the geometry after the transition. This is a topological version of t'Hoofts conjecture claiming a string description for large N QCD. In comparison with Maldacenas conjecture the topological closed string side corresponds to type IIB on ADS_5, while the topological gauge theory side corresponds to the $4d$ $N=4$ super Yang-Mills theory on the branes.

The simplest example of such a transition is the *conifold transition*. Consider the family of affine complex quadratic 3d hypersurfaces M_μ in $\mathbf{C}^4$

$$f(x,\mu)=y_1^2+y_2^2+y_3^2+y_4^2-\mu=0\,. \tag{203}$$

M_0 is a singular hypersurface, because $f(y,0)=0$ and $\frac{\mathrm{d}f(y,0)}{\mathrm{d}y_i}=0, \forall i$ have a common solution $\{y_i=0\}$, called a *nodal singularity* or *node*. One calls the point $\mu=0$ in the parameterspace where the node appears the *conifold point*. The name comes from the fact that for $\mu=0$ the solutions y of $f(y,0)$ can be rescaled $(f(y,0)=0)\to(f(\lambda y,0)=0)$ so M_0 forms a cone.

The node in M_0 can be smoothed in two ways. Either *deform* the hypersurface $M_0\to M_{\mu\neq 0}$. Then the node is *deformed* to an S^3 and the total smooth geometry is that of the cotangent bundle T^*S^3 of the three sphere. To see this, consider real $\mu>0$ and introduce real parameters (u_k,v_k) by $y_k=:u_k+iv_k$. Written as real equations (203) implies $\hat f=\sum_{k=1}^4 u_k^2=r^2$ with $r^2:=\mu+\sum_{k=1}^4 v_k^2>0$ and $\sum_{i=1}^4 u_k v_k=0$. From the first equation follows that u_k parametrize a compact S^3. We can chose a S^3 section of M_μ with radius $r^2=\mu$ for $v_i=0$. The second equation ensures that v_k parametrize the non-compact cotangent bundle of the sphere. To see this consider $\mathrm{d}\hat f=2\sum_{k=1}^4 u_k\mathrm{d}u_k=0$ and identify the cotangent direction $\mathrm{d}u_k$ with v_k. As a more detailed exercise one may cover T^*S^3 by patches and local coordinates $(\hat u,\hat v)$ and check that the $\hat v$ coordinates transform as cotangent bundle of S^3. As a further exercise one may show that as cone over $\lambda\in\mathbf{R}_0^+$ the base of M_0 is $S^3\times S^2$, see [36]. The reader should notice that the choice $\mu\in\mathbf{R}_0^+$ does not restrict the generality of the construction. A phase in $\mu=|\mu|e^{i\phi}$ can be absorbed by defining $y_k=:(u_k+iv_k)e^{\frac{i\phi}{2}}$ and modifies the choice of the complex structure in T^*S^3.

One can also *resolve* the node in M_0 by blowing up an $\mathbf{P}^1$. The idea of a blow up is to modify M_0 only over the singularity $S=\{y=0\}=\{w=0\}$. I.e. we search a smooth complex manifold $\hat M_0$ so that a biholomorphic map $\pi:(\hat M_0\setminus\pi^{-1}(S))\to(M_0\setminus S)$ exists. To find $\hat M_0$ we make a linear change to new complex coordinates $w_{1/2}=y_1\pm iy_2$ and $w_{3/4}=i(y_3\pm iy_4)$, so that M_μ is described by $w_1w_2-w_3w_4=\mu$. Now we define $\hat M_0$ by two equations

$$W\begin{pmatrix} w_5\\ w_6\end{pmatrix}=\begin{pmatrix} w_1 & w_4\\ w_3 & w_2\end{pmatrix}\begin{pmatrix} w_5\\ w_6\end{pmatrix}=0\,. \tag{204}$$

Here $(w_5:w_6)$ are homogeneous coordinates of $\mathbf{P}^1$, i.e. $(w_5,w_6)\sim(\rho w_5,\rho w_6)$ with $\rho\in\mathbf{C}^*$ and $(w_5,w_6)\neq(0,0)$. To see that (204) describes a smooth threefold we can view it e.g. as complete intersection defined by $f_1=w_1w_5+w_4w_6=0$ and $f_2=w_3w_5+w_2w_6=0$. A singularity of a complete intersection $f_1=0,\dots,f_r=0$ occurs if $\mathrm{rank}\left(\frac{\partial f_j}{\partial w_i}\right)<r$ for some point in $f_1=0,\dots,f_r=0$. This is not the case here, because $(w_5,w_6)\neq(0,0)$. Moreover as $(w_5,w_6)\neq(0,0)$ (204) enforces $\det W=w_1w_2-w_3w_4=0$ and every non-trivial solution to the latter equation fixes uniquely an equivalence class in $(w_5:w_6)$. This makes $\pi:(w_1,w_2,w_3,w_4,w_5:w_6)\mapsto(w_1,w_2,w_3,w_4)$ biholomorphic outside S and $\pi^{-1}(S)$. The sigular set S is the trivial solution $W\equiv 0$ in M_0. Over this point in M_0 the coordinate $(w_5:w_6)$ is unrestricted and parametrizes $\mathbf{P}^1$, so that $\pi^{-1}(S)=\mathbf{P}^1$. As an exercise choose coordinates for the two patches in $\mathbf{P}^1$, i.e. for $w_6\neq 0$, $(z=w_5/w_6, l_1=w_3, l_2=w_1)$ and for $w_5\neq 0$, $(\tilde z=w_6/w_5,\tilde l_1=w_2,\tilde l_2=w_4)$. (204) describes the transition functions for the non-compact l directions, which transform as the line bundle coordinates of $\hat M_0=\mathcal{O}(-1)\otimes\mathcal{O}(-1)\to\mathbf{P}^1$, see (400). Note that (204) identifies the $\mathbf{P}^1$ coordinate $z=w_4/w_1=w_2/w_3$ with the direction in which S is approached in $\mathbf{C}^4$. The geometry $M_{0,t}:=\hat M_0$ has also a parameter, namely the size $t=\int_{\mathbf{P}^1}\omega$ of the $\mathbf{P}^1$, which is not visible in (204). To make it visible we pass to the sympletic quotient construction by introducing variables x_i by $w_1=x_1x_3$, $w_2=x_2x_4$, $w_3=x_1x_2$ and $w_4=x_3x_4$, which fullfill the constraint

$w_1 w_2 - w_3 w_4 = 0$ identically. If $l^{(1)} = (1,-1,-1,1)$ acts by (158) on x_i then (157) defines $\hat{M}_0$. To see this identify the patches for $x_1 \neq 0$ given by $(z = x_4/x_1, l_1 = x_1 x_2, l_2 = x_1 x_3)$ and for $x_4 \neq 0$ by $(\tilde{z} = x_1/x_4, \tilde{l}_1 = x_2 x_4, \tilde{l}_2 = x_3 x_4)$ with the ones above. The constraint $|x_1|^2 + |x_4|^2 - |x_2|^2 - |x_3|^2 = t$ in the symplectic quotients contains with t the size of the $\mathbf{P}^1$. Note we could also identify $w_1 = x_1 x_2$, $w_2 = x_3 x_4$, $w_3 = x_2 x_4$ and $w_4 = x_1 x_3$ then $l^{(1)} = (-1,1,1,-1)$. The two identifications are related by a flop.

For the applications of the transition to toric non-compact Calabi-Yau manifolds it is importnant that the T^2 action defined in (163) is preserved during the transition. In the patch, where $x_4 \neq 0$ it acts as $(x_1, x_2, x_3, x_4) \mapsto (e^{-i(\alpha_1+\alpha_2)} x_1, e^{i\alpha_1} x_2, e^{-i\alpha_2} x_4, x_4)$. This translates to an action $w_{2/1} \mapsto e^{\pm i\alpha_1} w_{2/1}$ and $w_{4/3} \mapsto e^{\pm i\alpha_2} w_{4/3}$ on the w variables, which leaves the deformed conifold equation $w_1 w_2 - w_3 w_4 = \mu$ invariant. Hence it is possible to understand the T^*S^3 geometry also as an $T^2 \times \mathbf{R}$ fibration and reconstruct it from the degenerations of the $a = (1,0) \sim \alpha_1$ and the $b = (0,1) \sim \alpha_2$ cycle of T^2. The former vanishes at $w_1 = 0 = w_2$ and the latter at $w_3 = 0 = w_4$. Both loci in M_μ have the topology of a cylinder whose S^1 is the b and the a cycle respectively. Let us denote $w_3 w_4 = z$ and assume as before that $\mu \in \mathbf{R}$. One choses $\mathrm{Re}(z)$ and the two coordinates along the axis the cylinders as coordinates of the base $\mathbf{R}^3$ and α_1, α_2 and $\mathrm{Im}(z)$ as coordinates of the $T^2 \times \mathbf{R}$ fibre. The degeneration graphs cannot be drawn in two dimensions, because the value of $\mathrm{Re}(z)$ affects what cycle degenerates. E.g. for $z = 0$ and $z = -\mu$ the b- and the a- cycle degenerates. The line from $z = 0$ to $z = -\mu$ is drawn in the degeneration graphs as a dashed line. As it is shown in Fig. 26 over the upper half of the interval from $z = 0$ to $z = -\mu$ the topology of the fibration is that of a solid torus namely a S^1 in the class b fibered trivially over a disk D_a and in the lower half an solid torus build form an S^1 in the class a fibered trivially over D_b. As it is explained in [201] the way two solid tori can be glued topological to an S^3 is to glue the a and b cycles after an S transformation. This is the Heegaard glueing of S^3.

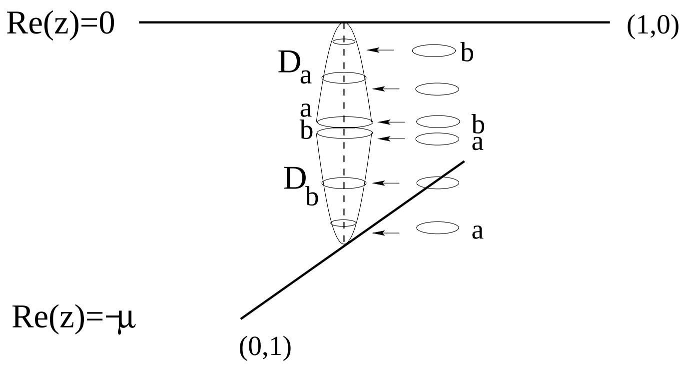

Fig. 26 The T^2 fi bration structure of the S^3 in the T^*S^3 geometry.

The transition can then neatly by depicted by the degeneration graphs in the $T^2 \times \mathbf{R}$ fibration. Closed lines in plane correspond to $\mathbf{P}^1 \sim S^2$ and dashed lines into the picture correpond to S^3. The diameter of both is visible as the length of the lines.

7.3 The closed string geometry for large N Chern-Simons theory on T^*S^3

This leads to the construction of the topological vertex [1] as reviewed in more detail in [159]. The topological vertex amplitude is the *building block* for calculation any closed or open string amplitude in any toric CY variety by

- Solving the general problem on a $\mathbf{C}^3$ patch for arbitrary conditions on three stacks of D-branes on Harvey-Lawson special Lagrangian cycles with topology $S^1 \times \mathbf{R}^2$[107] as in Fig. 29 This amplitude

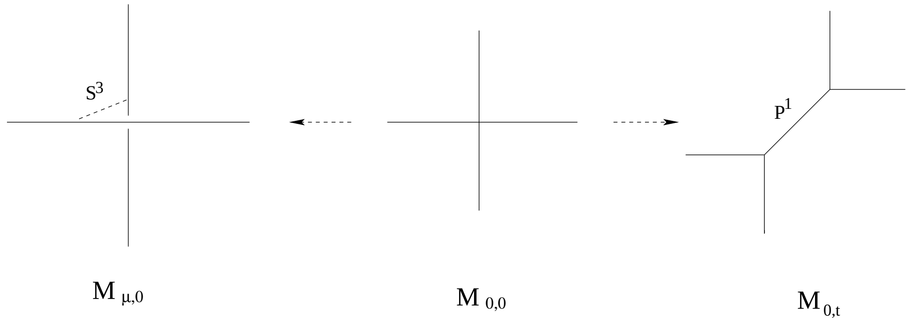

Fig. 27 The deneration loci of the T^2 fibrations in the conifold transitions. The precise nature of the fibration over the base on the left handside representing T^*S^3 is explained in Fig. 26. Here we show this figure from above. The fibration over the right handside representing $\mathcal{O}(-1)\oplus\mathcal{O}(-1)\to\mathbf{P}^1$ is explained in Fig. 20. The cross representing the singularity $M_{0,0}$ can also be seperated so that the middle line has slope -1, which is the flopped $\mathcal{O}(-1)\oplus\mathcal{O}(-1)\to\mathbf{P}^1$ geometry .

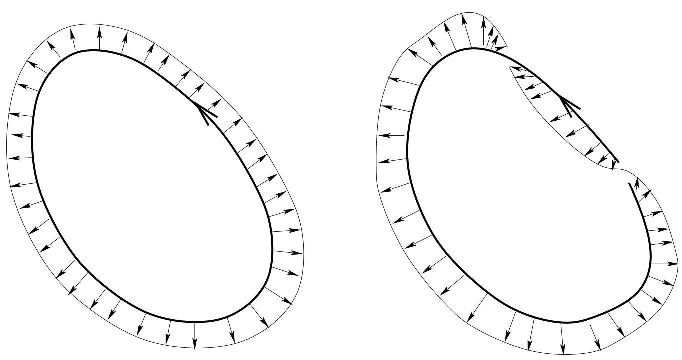

Fig. 28 The normal bundle to a link is not uniquely defined. In general one has an integral ambiguity. The choice made in the right picture leads to a self-linking number -1.

can be calculated in terms of the large N expansion of link invariants $W_{RR'}(q)$ of Chern-Simons theory on S^3 [1]. In a specific framing one has

$$C_{R_1R_2R_3}(q)=\sum_{R,Q_1,Q_2} N^{R_1}_{Q_1,R} N^{R_3^t}_{Q_3^t R} q^{\kappa_{R_2}/2+\kappa_{R_3}/2} \frac{W_{R_2^t Q_1}(q) W_{R_2 Q_3^t}(q)}{W_{R_2}(q)}, \tag{205}$$

where $N^{R_3}_{R_1R_2}$ are the usual tensor product coefficients and $\kappa_R=\sum_i l_i(l_i-2i+1)$ and l_i is the length of the row of the $i'th$ line in the Young-Tableaux of R. Note that $q=e^\lambda$ with λ the string coupling. i.e. C_{R_1,R_2,R_3} is exact in q and contains all genus information. All possible boundary conditions on the stack of N D-branes are encoded in R. Below we list the vertices with a total of up to 3 boxes at the outer legs

$$
\begin{aligned}
C_{\square\cdot\cdot} &= \frac{1}{q^{\frac{1}{2}}-q^{-\frac{1}{2}}},\\
C_{\square\square\cdot} &= \frac{q^2-q+1}{(q-1)^2}, \qquad C_{\square\!\square\cdot\cdot} = \frac{q^2}{(q+1)(q-1)^2}, \qquad C_{\substack{\square\\\square}\cdot\cdot} = \frac{q}{(q+1)(q-1)^2},\\
C_{\square\square\square} &= \frac{q^4-q^3+q^2-q+1}{q(q-1)^3}, \quad C_{\square\!\square\,\square\cdot} = \frac{q^{\frac{3}{2}}(q^3-q^2+1)}{(q+1)(q-1)^3},\\
C_{\substack{\square\\\square}\square\cdot} &= \frac{(q^3-q^2+1)}{q^{\frac{1}{2}}(q+1)(q-1)^3}, \qquad C_{\square\!\square\!\square\cdot\cdot} = \frac{q^{\frac{9}{2}}}{(q+1)(1+q+q^2)(q-1)^3},\\
C_{\substack{\square\!\square\\\square}\cdot\cdot} &= \frac{q^{\frac{5}{2}}}{(1+q+q^2)(q-1)^3}, \qquad C_{\substack{\square\\\square\\\square}\cdot\cdot} = \frac{q^{\frac{3}{2}}}{(q+1)(1+q+q^2)(q-1)^3}\,.
\end{aligned}
\tag{206}
$$

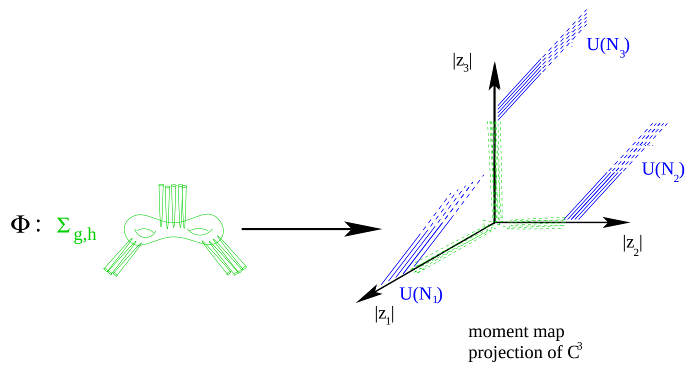

Fig. 29 Moment map projection of the vertex and an amplitude with genus 2 and boundary conditions specified by three representations R_i of $U(N_i)$ of three stack of D-branes wrapping Harvey-Lawson special Lagrangian cycles of topology $S^1 \times \mathbf{R}^2$.

- Providing gluing rules: If $\Gamma = \Gamma_1 \cup \Gamma_2$ and X_{Γ_i} are the associated toric varieties then

$$Z(X_\Gamma) = \sum_Q Z(X_{\Gamma_L})_Q (-1)^{l(Q)} e^{-l(Q)t} Z(X_{\Gamma_L})_{Q^t} \tag{207}$$

  with t is the Kähler parameter "size" of the connecting $\mathbf{P}^1$. The quantity $(-1)^{l(Q)}e^{-l(Q)t}$, with $l(Q)$ the number of boxes in the Young-Tableaux of the intermediate representation, can be viewed as *propagator*. Here again we ignore the data of the framing, which are essential to patch together arbitrary toric varieties.

For instance the Calabi-Yau geometry $/(-3) \to \mathbf{P}^2$ is covered by three patches, with the moment map projection as in Fig. 31 The partition function $Z_{\mathbf{P}^2}$ for closed strings is obtained by gluing three vertices with trivial representation $Q_i = \cdot$ on the outer legs by three propagators

$$Z_{\mathbf{P}^2} = \sum_{R_1,R_2,R_3} (-1)^{\sum_i l(R_i)} e^{-\sum_i l(R_i)t} q^{\sum_i \kappa_{R_i}} C_{\cdot R_2 R_3^t} C_{\cdot R_1 R_2^t} C_{\cdot R_3 R_1^t}\,. \tag{208}$$

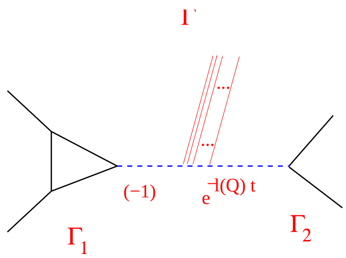

Fig. 30 Gluing of graphs along a connecting propagator

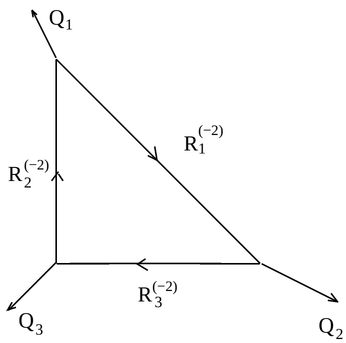

Fig. 31 The moment map projection that shows the degeneration of the torus action $(\mathbf{C}^2)^*$ on $\mathcal{O}(-3) \rightarrow \mathbf{P}^2$

All t represent the volume of the hyperplane $\mathbf{P}^1$, so that t is the single Kähler parameter of $\mathcal{O}(-3) \rightarrow \mathbf{P}^2$.

The calculation is easily performed and by taking the logarithm we get the generating function for the all genus contribution

$$\mathcal{F}(\lambda, t) = \sum_{\lambda=0}^{\infty} \lambda^{2g-2} \mathcal{F}^{(g)}(t) \ . \tag{209}$$

All $\mathcal{F}^{(g)}$ have an expansion $\mathcal{F}^{(g)} = \sum_\beta r_\beta^g q^\beta$, where the $r_\beta^g \in \mathbf{Q}$ are the Gromow-Witten invariants for the holomorphic map from Σ_g to a curve in the class $\beta \in H_2(M, \mathbf{Z})$ of the image curve in M.

8 The topological B-model

Since the axial $U(1)_A$, whose gauge connection is added to the spin connection to define the B-model, develops an anomaly of its current proportional to $\int_\Sigma \partial_\mu j_A^\mu \sim \int_\Sigma x^*(c_1(TM))$ the twisted B-model is only consistent for Kähler manifold with vanishing first Chern class, i.e. Calabi-Yau manifolds.

Our plan for the treatment of the B-model is as follows. In next two sections we will present the principal structure of the topological B-model and its coupling to gravity. We will then recall some facts about families of complex manifolds. The integrability of the complex structure deformations on Calabi-Yau manifolds will be presented in some detail following the proof of Tian, partly because it is one of the main classical results, but also because it leads directly to the formulation of Kodaira-Spencer theory of gravity. The behavior of the periods under infinitesimal deformations of the complex structure is the preparation for the derivation of the special Kähler geometry relation from geometry. After that we discuss two methods to obtain the Picard-Fuchs equations, which play a central role to actually solve the B-model. The quintic hypersurfaces is the main example, however we aim for a presentation, which paves the way for generalizations to the bulk of the known Calabi-Yau; complete intersections in weighted projective space.

8.1 The topological B without worldsheet gravity

The scalar BRST operator is in this case, see table 3,

$$Q_B = \bar{Q}_- + \bar{Q}_+ \,. \tag{210}$$

The scalar fields on the worldsheet are conveniently chosen as

$$\eta^{\bar{\imath}} := -(\psi_-^{\bar{\imath}} + \psi_+^{\bar{\imath}}), \qquad \theta_j := g_{j\bar{\imath}}(\psi_+^{\bar{\imath}} - \psi_-^{\bar{\imath}}) \,, \tag{211}$$

while the one form fields are

$$\rho_z^i := \psi_-^i \quad \text{of type } (1,0), \qquad\qquad \rho_{\bar{z}}^i := \psi_+^i \quad \text{of type } (0,1). \tag{212}$$

The supersymmetry transformation $\delta = \bar{\epsilon}\bar{Q}_+ + \bar{\epsilon}\bar{Q}_-$ is obtained by setting $\bar{\epsilon}_+ = -\bar{\epsilon}_- = \bar{\epsilon}$ and $\epsilon_\pm = 0$

$$\begin{aligned} \delta x_i &= 0, & \delta x^{\bar{\imath}} &= \bar{\epsilon}\eta^{\bar{\imath}} \\ \delta\theta_i &= 0, & \delta\eta^{\bar{\imath}} &= 0 \\ \delta\rho_\mu^i &= \pm i\bar{\epsilon}\partial_\mu x^i \,. \end{aligned} \tag{213}$$

The zero form observables $\mathcal{O}^{(0)}$ are now related to forms in $\Omega^{(0,p)}(M, \Lambda^q T^{0,1}M)$ with the identification of the scalar Grassmann fields on the worldsheet to forms and vectors on M $\eta^{\bar{\imath}} \leftrightarrow \mathrm{d}x^{\bar{\imath}}$ and $\theta_i \leftrightarrow \frac{\partial}{\partial x^i}$. I.e. to each form on M of type

$$W = \omega_{\bar{\imath}_1\ldots\bar{\imath}_p}^{j_1\ldots j_q} \mathrm{d}x^{\bar{\imath}_1} \wedge \ldots \wedge \mathrm{d}x^{\bar{\imath}_p} \frac{\partial}{\partial x^{j_1}} \wedge \ldots \wedge \frac{\partial}{\partial x^{j_q}} \tag{214}$$

we associate a 0-form operator on Σ

$$\mathcal{O}_W^{(0)} = \omega_{\bar{\imath}_1\ldots\bar{\imath}_p}^{j_1\ldots j_q} \eta^{\bar{\imath}_1} \ldots \eta^{\bar{\imath}_p} \theta_{j_1} \ldots \theta_{j_q} \,. \tag{215}$$

One checks that the Q_B operator is identified with the Dolbeault operator $\bar{\partial}$ which increases the anti holomorphic form degree

$$0 \xrightarrow{\bar{\partial}} \Omega^{00}(M, \Lambda^q T^{1,0}M) \xrightarrow{\bar{\partial}} \Omega^{01}(M, \Lambda^q T^{1,0}M) \xrightarrow{\bar{\partial}} \ldots \xrightarrow{\bar{\partial}} \Omega^{0d}(M, \Lambda^q T^{1,0}M) \xrightarrow{\bar{\partial}} 0 \,. \tag{216}$$

and one has with $\{Q_B, \mathcal{O}_W^{(1)}\} = -\mathcal{O}_{\bar{\partial}W}^{(0)}$ the identification

$$H_{Q_B}^* = \frac{\mathrm{Ker}\ Q_B}{\mathrm{Im}\ Q_B} = \bigoplus_{p,q=0}^{d} H^{0,p}(M, \Lambda^q T^{1,0}M) \,. \tag{217}$$

The selection rules from the R-symmetries are as before $\sum_i p_i = \sum_i q_i = d(1-g)$. It follows that for $g = 0$ we have again the possibility of a non-vanishing three point function $\langle \mathcal{O}_{A^{(i)}} \mathcal{O}_{A^{(j)}} \mathcal{O}_{A^{(k)}} \rangle$, if we consider three local operators $\mathcal{O}_{A^{(k)}}$ associated to

$$A^{(k)} = \omega_{\bar{j}}^{(k)\ i} \mathrm{d}x^{\bar{j}} \frac{\partial}{\partial x^i} \in H^1(M, T^{1,0}M) \,. \tag{218}$$

Eq. (213) shows that there is a fixpoint of the fermionic symmetry at the constant maps

$$\partial_\mu x^i = 0 \,. \tag{219}$$

We expect therefore that all contributions to the path integral are localized to constant maps. *This is the main simplification in the B-model.* For constant maps Σ_g is mapped to a point in M. These maps are of course much easier to control then the holomophic maps of the A-model and in particular they are not affected by the sizes, i.e. Kählerparameter of M. The B-model without worldsheet gravity is like a Kaluza-Klein reduction. By writing the action in the form

$$S = it \int_\sigma \{Q_B, V\} + tW \tag{220}$$

with

$$V = g_{i\bar{j}}(\rho_z \partial_{\bar{z}} x^{\bar{j}} + \rho^i_{\bar{z}} \partial_z x^{\bar{j}}) \tag{221}$$

and

$$W = \int_{\Sigma_g} (-\theta_i D\rho^i - \frac{i}{2} R_{i\bar{i}j\bar{j}} \rho^i \wedge \rho^j \eta^{\bar{i}} \theta_k g^{\bar{j}k}) \tag{222}$$

one can conclude the following. W does not depend on the complex structure of Σ, which decouples from the B-model at genus 0. The Kähler variations of W are Q_B exact and decouple likewise. It is also t independent as t can be absorbed in a field redefinition in W. For more details see [207]. In the off shell formulation of [146][147] one can simply write the complete action as Q commutator $S = \{Q_B, \tilde{V}\}$ which makes the above points more obvious.

Since the fixpoints of the fermionic maps of the B-model are constant maps, mapping all Σ to a point in the Calabi-Yau manifold M, their moduli space contains M and in the special case of the three punctured sphere, i.e. in the case of the three point function it is actually M, since these three points can be fixed on S^2 by an $SL(2, \mathbf{C})$ transformation and the sphere itself has no complex deformations. For this reason all we have to find is a canonical measure on M, which we integrate over M to get the three point function. Using Kaluza Klein reduction methods this measure has been found long ago [183]

$$C_{ijk}(z) = \langle \mathcal{O}^{(0)}_{A^i} \mathcal{O}^{(0)}_{A^j} \mathcal{O}^{(0)}_{A^k} \rangle = \int_M \Omega \wedge A^{(i)\ i_1}_{\bar{j}_1} A^{(j)\ i_2}_{\bar{j}_2} A^{(k)\ i_3}_{\bar{j}_3} \Omega_{i_1 i_2 i_3} \mathrm{d}x^{\bar{j}_1} \wedge \mathrm{d}x^{\bar{j}_2} \wedge \mathrm{d}x^{\bar{j}_3} \ . \tag{223}$$

Here $\Omega(z)$ is unique non-vanishing holomorphic $(3,0)$ form, which exists on every Calabi-Yau, see Sec. (9.8). Using the isomomorphism (231) $A \mapsto \hat{A}$ we can define a non-holomorphic two point function

$$N_{i\bar{j}} = \int_M \hat{A}^{(i)} \wedge \overline{\hat{A}}^{(\bar{j})} \ . \tag{224}$$

8.2 First order complex structure deformation

The expressions (223) and (224) depend as anticipated only on the complex structure of M and not on its Kähler structure. We saw in section 5.3 that deformations of the action by $\int_\Sigma \mathcal{O}^{(2)}_{A^k}$ with $A^{(k)} \in H^{(0,1)}_{\bar{\partial}}(M, TM)$ are first order complex structure deformations of M. Our aim is to explain in this section the local tangent space of the complex structure moduli space from a different point of view, put forward by Kodaira and Spencer [139] and to explain in the next section why the first order deformations on a Calabi-Yau manifold are unobstructed.

Consider a $2n$ real dimensional manifold and a covering of it by coordinate patches $\mathcal{U}_i$, $i = 1, \ldots, r$, which are homeomorphic to a neighborhood $U_i \in \mathbf{C}^n$ with coordinates $x^{(i)}_\alpha(p)$, $\alpha = 1, \ldots, n$. It is a complex manifold if the transition functions $f^{(jk)} : x^{(k)}(p) \to x^{(j)}(p)$, defined for $p \in \mathcal{U}_j \cap \mathcal{U}_k$, are biholomorphic. One attempts to define a family of complex manifolds M_z, by considering a family of transition functions $x^{(j)}_\alpha = f^{(jk)}_\alpha(x^{(k)}, z)$, which depend also holomorphically on the complex parameters

z. The difficulty is that some z dependence of $f_\alpha^{(ik)}(x^{(k)},z)$ corresponds just to different choices of local coordinates systems on the same complex manifold. In order to decide whether the $f^{(jk)}(x^{(k)},z)$ really induce changes of the complex structure [139] considers in every patch U_k an infinitesimal coordinate changes that is characterized by a holomorphic vector field $V^{(k)}(z) = \sum_{\alpha=1}^n \frac{\partial f_\alpha^{(k)}(x^{(k)},z)}{\partial z}\frac{\partial}{\partial x_\alpha^{(k)}}$. Next consider the composition of transition functions in $\mathcal{U}_i \cap \mathcal{U}_j \cap \mathcal{U}_k$. Per definition

$$f_\alpha^{(ik)}(x^{(k)},z) = f_\alpha^{(ij)}(f_1^{(jk)}(x^{(k)},z),\ldots,f_n^{(jk)}(x^{(k)},z),z) \tag{225}$$

holds. Differentiation w.r.t. to z gives

$$\frac{\partial f_\alpha^{(ik)}(x^{(k)},z)}{\partial z} = \frac{\partial f_\alpha^{(ij)}(x^{(j)},z)}{\partial z} + \sum_{\beta=1}^n \frac{\partial x_\alpha^{(i)}}{\partial x_\beta^{(j)}}\frac{\partial f_\beta^{(jk)}(x^{(k)},z)}{\partial z}\ . \tag{226}$$

Denote general vector fields by

$$A^{(jk)}(z) = \sum_{\alpha=1}^n \frac{\partial f_\alpha^{(jk)}(x^{(k)},z)}{\partial z}\frac{\partial}{\partial x_\alpha^{(j)}}, \qquad x^{(k)} = f^{(kj)}(x_j,z)\ . \tag{227}$$

Note that $A^{(kk)}(z) = 0$ since $f_\alpha^{(kk)} = x^{(k)}$ independently of z. Therefore eq. (226) written covariantly in terms of the vector fields (227) implies $A^{(kj)}(t) = -A^{(jk)}(t)$. For general i,j,k (226) is a Čech[33] 1-cocycle condition for the $A^{(ij)}$

$$A^{(ij)}(z) + A^{(ki)}(z) + A^{(jk)}(z) = 0\ . \tag{228}$$

The exact 1-cocycles come precisely from the infinitesimal coordinates changes setting $A^{(jk)}(z) = V^{(j)}(z) - V^{(k)}(z)$, while the true changes of complex structure correspond to 1-cocycles, which are not exact, i.e. elements of $H^1(M,A)$, where A are sheaves of vector fields $A = \mathcal{O}(TM)$. The Čech-Dolbeault theorem (334) with $F = \mathcal{O}(TM)$ implies that complex structure deformations are given by elements in $H^{0,1}(M,TM)$, which we also call A.

8.3 Unobstructedness of the complex deformation space

As explained in [139] the existence of a global complex structure deformation requires the vanishing of higher Čech cohomology groups for vector fields. Tian [189] and Todorov [192] have proven that these higher order conditions are automatically fulfilled on a Calabi-Yau space.

The elements $A(z) = A_{\bar\jmath}^{i}(x,z)\mathrm{d}x^{\bar\jmath}\frac{\partial}{\partial x^i}$ in $H^{(0,1)}(M,TM)$ in the complex moduli space can be used to deform the $\bar\partial$ operator to $\bar\partial_z = (\bar\partial + A(z))$ so that $\bar\partial_z f(x) = 0$, defines what a holomorphic function on M is w.r.t. the new complex structure. The requirement that $\bar\partial_z^2 = 0$ leads to

$$\bar\partial A(z) + \frac{1}{2}[A(z),A(z)] = 0, \tag{229}$$

where $[.,.]$ is the Lie bracket. For $\phi(x) = \phi^i(x)\partial_{x_i} \in \mathcal{L}^{0,p}(T)$, with $\phi^i = \frac{1}{p!}\phi(x)^i_{\bar\imath_1,\ldots,\bar\imath_p}\mathrm{d}x^{\bar\imath_1}\wedge\ldots\wedge\mathrm{d}x^{\bar\imath_p}$, and $\omega(x) \in \mathcal{L}^{0,q}(T)$ similarly defined one has

$$[\phi,\omega] = (\phi^i\wedge\partial_i\omega^j - (-1)^{pq}\omega^i\wedge\partial_i\phi^j)\partial_j, \tag{230}$$

giving above a $(0,2)$ form vector field from two $(0,1)$-form vector fields. Condition (229) is equivalent to the vanishing of the Nijenhuis tensor (329) [139].

[33] Čech cohomology made a prominent physical appearence in topological charge quantization in [6].

The main idea of the proof is that the existence of the holomorphic $(n,0)$ form induces an isomorphism

$$H^{(0,p)}(M,TM) \cong H^{n-1,p}(M)\ . \tag{231}$$

under which the condition (229) is converted into a cohomological question, which is solved by the $\partial\bar\partial$ lemma. This conversion of the deformation problem to a cohomological question, which is solved by an analog of the $\partial\bar\partial$ Lemma extends to deformations of G_2 metrics [121][112] as well as to the extended moduli space considered in [16].

Contraction with the homolomorphic $(n,0)$ form associates to $A = A^i_{\bar\jmath_1,\ldots,\bar\jmath_p}\mathrm{d}x^{\bar\jmath_1}\wedge\ldots\wedge\mathrm{d}x^{\bar\jmath_p}\frac{\partial}{\partial x^i} \in H^{(0,p)}(M,TM)$ an $\hat A \in H^{n-1,p}(M)$ as

$$\hat A = \frac{1}{(n-1)!}A^j_{\bar\jmath_1,\ldots,\bar\jmath_p}\Omega_{j,i_2,\ldots,i_n}\mathrm{d}x^{i_2}\wedge\ldots\wedge\mathrm{d}x^{i_n}\mathrm{d}x^{\bar\jmath_1}\wedge\ldots\wedge\mathrm{d}x^{\bar\jmath_p} \tag{232}$$

with the inverse

$$(\hat A)^\vee = \frac{1}{(n-1)!|\Omega|^2}\bar\Omega^{i,i_2,\ldots,i_n}\hat A_{i_2,\ldots i_n,\bar\jmath_1,\ldots,\bar\jmath_p}\mathrm{d}x^{\bar\jmath_1}\wedge\ldots\wedge\mathrm{d}x^{\bar\jmath_p}\frac{\partial}{\partial x^i} \tag{233}$$

where $|\Omega|^2$ is defined in (392). One checks that A is harmonic iff $\hat A$ is harmonic and the operation is invertible i.e. $A = (A^\wedge)^\vee$, which shows (231).

Since Ω is holomorphic the hat operation (232) commutes with $\bar\partial$ and we get

$$\bar\partial\hat A = \widehat{\bar\partial A} = -\frac{1}{2}\widehat{[A,A]} =: -\frac{1}{2}[\hat A,\hat A], \tag{234}$$

as equivalent to the condition (229).

The main technical instrument is the following Lemma (Tian-Todorov)

$$[\hat A,\hat B] := \widehat{[A,B]} = \partial(\widehat{A\wedge B}) - (D\cdot A)\wedge\hat B + \hat A\wedge(D\cdot B), \tag{235}$$

where $D\cdot A = (\partial_i A^i_{\bar\jmath_1\ldots\bar\jmath_p})x^{\bar\jmath_1}\wedge\ldots\wedge\mathrm{d}x^{\bar\jmath_p}$ is a contraction. The calculation is a straightforward exercise whose solution is made explicit in [189]. Eq. (235) becomes particularly useful, if one can choose "gauge" representatives for A and B so that $(D\cdot A) = (D\cdot B) = 0$. To control this "gauge" condition Tian considers a Taylor expansion $A(z) = A_1 z + A_2 z^2 + \ldots$ with A_i sections of $\Gamma(M,\Omega^{(0,1)}(TM))$ and starting data $\bar\partial_0 = \bar\partial$, i.e. $A(0) = 0$. To order z (229) states $\bar\partial A_1(x) = 0$ and we already argued that in order to get rid of complex coordinate transformations we should consider $A_1 \in H^{(0,1)}_{\bar\partial}(M,TM)$ only. One wants now to prove inductively that $\partial A_k + \frac{1}{2}\sum_{i=1}^{k-1}[A_i,A_{k-i}] = 0$ for $k>1$ which by (234) is equivalent to

$$\bar\partial\hat A_k = \frac{1}{2}\sum_{i=1}^{k-1}[\hat A_i,\hat A_{k-i}], \quad \text{for}\ \ k>1\ . \tag{236}$$

First step of induction: To first order in z one has simply as above $\hat A_1 \in H^{n-1,1}(M)$ and we pick the harmonic representative $\hat A_1$. In fact on compact Kähler manifolds it follows from (344,348) that every harmonic representative fulfills $\bar\partial A_1 = \bar\partial^* A_1 = 0$. Moreover with $\Delta_{\bar\partial} = \Delta_\partial$, see sect. 9.2 also $\partial\hat A_1 = 0$ holds. This implies $D\cdot A_1 = 0$ and by (235) $[\hat A_1,\hat A_1] = \partial(\widehat{A_1\wedge A_1})$ is ∂-exact. On the other hand for $\hat A_1 \in H^{n-1,1}(M)$ hence $\bar\partial A_1 = 0$ it is immediate from the definition of the bracket that $\bar\partial[\hat A_1,\hat A_1] = \bar\partial\partial(\widehat{A_1\wedge A_1}) = 0$. The ∂, $\bar\partial$ *Lemma* of Kähler geometry ([100], p 149) states that if a form $\eta\in\Omega^{p,q}$ is $\bar\partial$ closed and d-, ∂- or $\bar\partial-$ exact then it can be written as $\eta = \partial\bar\partial\psi$. Applied to the bracket we can write $[\hat A_1,\hat A_1] = \partial\bar\partial\psi_1$ for some $\psi_1\in\Omega^{1,1}$. Identifying $\hat A_2 = \frac{1}{2}\partial\psi_1$ we have constructed a solution to $\bar\partial\hat A_2 + \frac{1}{2}[\hat A_1,\hat A_1] = 0$.

General induction: If for some N one has solved for $\hat{A}_i$ with $\partial\hat{A}_i = 0$ and $\bar{\partial}\hat{A}_i + \frac{1}{2}\sum_{j=1}^{i-1}[\hat{A}_j, \hat{A}_{i-j}] = 0$, $i = 1, \ldots, N$, then

$$\sum_{j=1}^{N}[\hat{A}_j, \hat{A}_{N+1-j}] = \partial \sum_{j=1}^{N}(A_j \wedge A_{N+1-j})^{\wedge} \tag{237}$$

and one also checks that

$$\begin{aligned}\bar{\partial}\left(\sum_{j=1}^{N}[\hat{A}_j, \hat{A}_{N+1-j}]\right) &= \bar{\partial}\partial\left(\sum_{j=1}^{N}[A_j, A_{N+1-j}]\right)^{\wedge}\\ &= \frac{1}{2}\partial\left(\sum_{j=1}^{N}\sum_{k=1}^{j-1}[[A_k, A_{j-k}], A_{N+1-j}] - [A_j, [A_k, A_{N+1-j-k}]]\right)^{\wedge} = 0\,.\end{aligned}$$

Here we used first (235), then the fact that $\bar{\partial}$ and $\wedge$ commutes, (237) for A_k with $k \leq N$ and the Jacobi identity for (230). By the $\partial, \bar{\partial}$ Lemma one can set $\hat{A}_{N+1} = \frac{1}{2}\partial\psi_N$ and since $\partial\hat{A}_{N+1} = 0$ the induction proceeds. Moreover one has arguments that the series converges in $H^{n-1,1}(M)$ [189].

Hence there exist always a family of Calabi-Yau manifolds with varying complex structure parameters, whose complex dimension is $h^{(0,1)}(M, TM)$. Tians and Todorovs result is very important also with respect to the world sheet theory, where is very not-trivial to establish that a deformation of type (47) is exactly marginal and does lead to family of $N = 2$ SCFTs.

Mirror statement On a Calabi-Yau threefold one has the above mentioned isomorphism between $H^{(0,1)}(M, TM)$ and $H^{2,1}(M)$, which is induced by the unique $(3,0)$ form Ω. Thanks to the above isomorphism the A-model and B-model physical operators are associated to $H^{p,q}$ and we mirror symmetry can be interpreted as the following identification of these spaces $H^{p,q}(M) \leftrightarrow H^{d-p,q}(W)$. Here M and W are mirror manifolds. As a corollary one has $\chi(M) = -\chi(W)$ if d is odd.

8.4 Kodaira-Spencer gravity as space-time action for the B-model

There are three space time actions known, which reproduce as classical equations of motion the unobstructedness of complex structures on the Calabi-Yau. Kodaira-Spencer gravity [20], Hitchins three-form action [112] and Hitchins general threeform action [113]. The first[20] and the last [172][**?**] reproduce the B-model also at one loop. But even Einsteins gravity poses no problem up to one loop [191]. While it is not clear how the suggested spacetime descriptions make sense as full quantum theory, the worldsheet B-model approach makes remarkable predictions at higher loops.

Kodaira-Spencer theory of gravity is theory on M which couples exclusively to the complex moduli of M. Its tree level result reproduces the B-model without the coupling to worldsheet gravity, i.e. its genus zero sector[20]. It is a space time gravity theory in the sense that is does couple to the Calabi-Yau metric as far as complex structure dependence is concerned. It reproduces the (229) in the form $\bar{\partial}A(z) + \frac{1}{2}\partial(A(z)\widehat{\wedge}A(z)) = 0$ as its equation of motion and its Feynman graph expansion corresponds to the iterative solution to that equation exactly in the form as given above. In fact by the $\partial, \bar{\partial}$-Lemma we have shown e.g. in the second induction step that one has an ψ_1 with $\partial\bar{\partial}\psi_1 = [\widehat{A_1, A_1}]$, hence $\hat{A}_2 = \frac{1}{2}\partial\psi_1$. By (235) the first statement means also $\bar{\partial}\psi = (A_1\widehat{\wedge}A_1)$. Combining the two facts one gets a solution for $\hat{A}_2$ in the form

$$\hat{A}_2 = -\frac{1}{2\bar{\partial}}\partial(A_1\widehat{\wedge}A_1) = \mathcal{P}(A_1\widehat{\wedge}A_1)\,. \tag{238}$$

We have used a "gauge" $\partial\hat{A}_k = 0$ and it is easy to see that the recursive solution comes with the freedom $\hat{A}_k + \bar{\partial}\lambda$, which one can fix be requiring $\bar{\partial}^* A_k = 0$. We can then define the "propagator" as $\mathcal{P} =$

$-\frac{1}{2\bar\partial}\partial = -\bar\partial^*\frac{1}{2\Delta_{\bar\partial}}\partial$. With this "propagator" one can recursively write the solutions to $\hat A_k$. E.g. $\hat A_3 = 2\mathcal{P}(A_1 \wedge (\mathcal{P}(\widehat{A_1 \wedge A_1}))^\vee)^\wedge$. It follows from the construction of A_k that only $\hat A_1$ fulfills the Laplace equation, while A_k for $k > 1$ correspond to "massive modes."

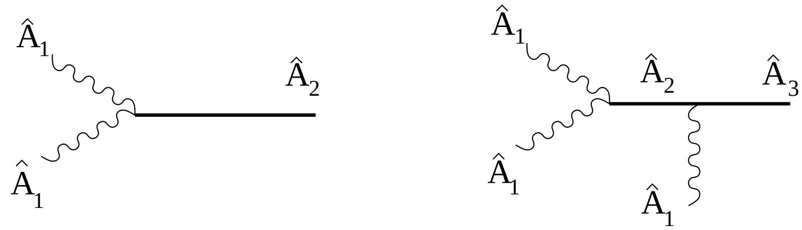

Fig. 32 Perturbative solution of the Kodaira-Spence equation in Tians form $\bar\partial A(z) + \frac{1}{2}\partial(\widehat{A(z)\wedge A(z)}) = 0$ by Feynmann graphs with massless fi elds (weavy lines) and massive fi elds (solid lines).

It is not hard to see [20], that the Kodaira-Spencer action

$$\lambda^2 S(\hat A_1, \hat A_m, z_0) = \int_M \frac{1}{2}\hat A_m \mathcal{P}\hat A_m + \frac{1}{6}((A_1 + A_m)\wedge(A_1 + A_m))^\wedge \wedge (A_1 + A_m)^\wedge \qquad (239)$$

has $\bar\partial(\hat A_1 + \hat A_m) + \frac{1}{2}\partial((A_1 + A_m)\wedge(A_1 + A_m))^\wedge = 0$ as e.o.m. and reproduces the Feynman graph expansion above. Here we have defined as A_m the massive part of $A(z)$ and z_0 is background value of the complex structure. It has further be shown that (239) is the reduction of closed string field theory to the topological modes and it has been argued that its path integral defines the generating function for all correlators of the topological B-model coupled to worldsheet gravity. However the action has not been made sense of as quantum theory. So its solution is indirect by means of the holomorphic anomaly equation of the topological B-model. Nevertheless the divergent factors in the graph expansion of (239) lead to an analysis of the leading behavior at the boundaries of the complex moduli space of the Calabi-Yau space once the ones of the three point couplings are known. For one modulus t one gets $F_g \sim \frac{[\partial_t^3 C_{ttt}]^{2g-2}}{[\partial_t C_{ttt}]}$. This result is useful to fix the holomorphic ambiguity.

8.5 The periods and infinitesimal deformations of the complex structure

The integral (223) can expressed in terms of holomorphic functions on the complex moduli space parametrized by z, which are integrals of the holomorphic $(3,0)$-fom over a fixed topological basis of three cycles of M

$$X^k(z) = \int_{A_k}\Omega(z), \qquad F_k(z) = \int_{B^k}\Omega(z), \qquad k = 0,\ldots,h_{2,1}\ . \qquad (240)$$

These are called *period integrals* of periods for short. Here we have chosen an integral symplectic basis of A and B cycles of the integral homology $H_3(M,\mathbf{Z})$ such that $A_k \cap B^l = \delta_k^l$, while $A^i \cap A^j = B_i \cap B_j = 0$. The choice of such a basis in $H_3(M,\mathbf{Z})$ and its dual basis (α_i, β^j) in the integral cohomology $H^3(M,\mathbf{Z})$ with

$$\int_M \alpha_k \wedge \beta^l = \int_{A_l}\alpha_k = -\int_M \beta^l \wedge \alpha_k = -\int_{B_k}\beta^l = \delta_k^l \qquad (241)$$

is unique of to an $\mathrm{Sp}(h^3,\mathbf{Z})$ transformation. The two dual symplectic bases (A^k, B_k) and (α_i, β^j) are topologically and do in particular not depend on the complex structure. What we call $(n,0)$ form $\Omega(z)$ does depend on the complex structure. This dependence is captured by the period integrals, w.r.t to the fixed basis (α_i, β^j)

$$\Omega(z) = X^k(z)\alpha_k - F_k(z)\beta^k\ . \qquad (242)$$

The symplectic group over $\mathbf{C}$ is defined by

$$M^\dagger \Sigma M = \Sigma, \qquad M \in Sp(h^3, \mathbf{C}) \qquad \text{with} \quad \Sigma = \begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix} . \tag{243}$$

Ω is a symplectic invariance and we have a natural action on the period vector

$$\Pi := \begin{pmatrix} X^k \\ F_k \end{pmatrix} \qquad \text{by} \qquad \tilde{\Pi} = M\Pi . \tag{244}$$

The X^k are homogeneous projective coordinates of the complex structure moduli space and one can choose locally inhomogeneous coordinates

$$t^k = \frac{X^k}{X^0} \qquad k = 1, \ldots, h := h^{2,1} \tag{245}$$

as the complex structure parameters[99, 189]. This can be viewed as local Torelli theorem for Calabi-Yau manifolds. A global Torelli is proven for $K3$ (and Enriques surfaces) [14], but seems not to hold on general Calabi-Yau manifolds.

In virtue of (245) the F_k must be expressible as functions of t. The precise relation comes from the infinitesimal calculus describing changes of the $(n, 0)$-form Ω in $H^n(M)$ under changes of the complex structure. The decomposition of $H^n(M)$ into (p, q) type $H^n(M) = \oplus_{p+q=n} H^{p,q}(M)$ varies over the complex moduli space parametrized by t. We are concerned with $n = 3$. One wants to describe the varying of $H^{p,q}(M_t)$ as a bundle $\mathcal{H}^{p,q}$ over the moduli domain $D(M)$ of M , called the *Hodge bundle*. However the spaces $H^{p,q}$ do not fiber holomorphically over $D(M)$. One defines therefore first a Hodge filtration $\mathbf{F}^*(M) = \{\mathbf{F}^p(M)\}_{p=0}^n$ by $\mathbf{F}^p(M) = \bigoplus_{a\geq p} H^{a,k-a}(M)$, with $H^n(M, \mathbf{C}) = \mathbf{F}^p(M) \oplus \overline{\mathbf{F}^{k-p+1}(M)}$. Obviously $H^{p,q}(M)$ is recovered as $H^{p,q}(M) = \mathbf{F}^p(M) \cap \overline{\mathbf{F}^q(M)}$ and one has an isomorphism $H^{p,q}(M) = \mathbf{F}^p(M)/\mathbf{F}^{p+1}(M)$. The $\mathbf{F}^p(M_t)$ form holomorphic bundles $\mathcal{F}^p$ over $D(M)$ and the holomorphic Hodge bundle $\mathcal{H}^{p,q}$ can be defined as $\mathcal{H}^{p,q} = \mathcal{F}^p/\mathcal{F}^{p+1}$, see [101] for a precise definition of $D(M)$. There is a bilinear form on $H^n(M, \mathbf{Z})/\text{torsion}$

$$Q(\phi, \psi) = (-1)^{n(n-1)/2} \int_M \phi \wedge \psi \tag{246}$$

with the following properties

$$Q(H^{pq}, H^{p'q'}) \quad = 0, \qquad \text{unless } \ p' = n - p \ \text{ and } \ q' = n - q \tag{247}$$

$$S(\psi, \psi) \equiv i^{p-q} Q(\psi, \bar{\psi}) \quad > 0, \qquad \text{unless } \ \psi = 0 \text{ in } H^{p,q} . \tag{248}$$

In mathematical terms Q is called a *polarization* on the *Hodgestructure* $H^n(M, \mathbf{Z})/\text{torsion}$ and (247) and (248) are the first and second *Riemann bilinear relations*, see [100, 101]. In particular $\mathcal{H}^{3,0}$ defines a line subbundle $\mathcal{L}$ in $H^3(M)$ and $\Omega(z)$ defines a section of it. Since is is expandable in the fixed integer frame (α_k, β^l) by the periods (242) it has a flat connection that is called *Gauss-Manin connection*. The Picard-Fuchs equations that the periods fulfill, which we derived latter, can be viewed as one manifestation of the flatness of the Gauss-Manin connection. Despite the fact that the connection is flat the period vector Π (244) will have a monodromy $G \in \mathrm{Sp}(h^3, \mathbf{Z})$, if transported around loops Γ_{z_0} encircling singular points z_i in the complex moduli space. To understand the possibility of a monodromy remember that the moduli space is *not simply connected*. Singular or orbifold loci of M are cut out. As exemplified at the end of Sec. 9.7 not simply connected manifolds can have non trivial holonomy of flat connections[34]. The monodromy group is generated by transport around all loops γ_i in $H^1(\mathcal{M})$

$$\Pi(z) = M_{\gamma_{z_i}} \Pi(z), \qquad M_{\gamma_{z_i}} \in Sp(h^3, \mathbf{Z}) \ , \tag{249}$$

[34] This monodromy is called a "Wilson line" in physics.

where one has relations, e.g. in the situation depicted in figure 33 one has $M_{\gamma_\infty}^{-1} = M_{\gamma_0} M_{\gamma_1}$. The homotopy group of $\mathcal{M}$ and the symplectic monodromies around the loops determine the period vector as solution to a Riemann-Hilbert problem.

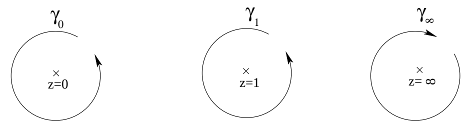

Fig. 33 Moduli space of a one complex parameter Calabi-Yau manifold compactified to $\mathbf{P}^1$ with three singular points. In general singularities are divisors in $\mathcal{M}$.

By taking a derivative w.r.t. the complex structure coordinates z^k the $(3,0)$ form changes as follows

$$\frac{\partial\Omega}{\partial z^k} = c_k(z,\bar{z})\Omega + \hat{A}^{(k)}(z), \tag{250}$$

where $\hat{A}^{(k)}(z) \in H^{2,1}$ is a basis and $c_k(z,\bar{z})$ depends on the complex moduli as made explicite after (258). This can be seen as follows. Let as in section (8.2) $f^\mu(x,z)$ define a family of holomorphic coordinates on M, which vary with the complex structure parameter z, so that $x^\mu = f^\mu(x,z_0)$. Via $f^\mu(x,z)$ the $(3,0)$-form $\Omega = \frac{1}{3!}h(f)\epsilon_{\mu\nu\rho}\mathrm{d}f^\mu \mathrm{d}f^\nu \mathrm{d}f^\rho$ depends on the complex structure z and by derivation we get

$$\frac{\partial\Omega}{\partial z^k} = \frac{1}{3!}\frac{\partial h}{\partial z^k}\epsilon_{\mu\nu\rho}\mathrm{d}f^\mu \mathrm{d}f^\nu \mathrm{d}f^\rho + \frac{1}{2!}h\epsilon_{\mu\nu\rho}\mathrm{d}f^\mu \mathrm{d}f^\nu \frac{\partial(\mathrm{d}f^\rho)}{\partial z^k}\ . \tag{251}$$

To analyze $\frac{\partial(\mathrm{d}f^\rho)}{\partial z^k}$ requires an infinitesimal calculus in the neighborhood of the reference complex structure z_0. It is easy to convince oneself that the $(0,1)$ part $\left.\frac{\partial(\mathrm{d}f^\rho)}{\partial z^k}\right|_{(0,1)} = A^{(k)\ \rho}_{\bar{\jmath}}\mathrm{d}z^{\bar{\jmath}}$, where $A^{(k)} \in H^{(0,1)}(M, T^{1,0}M)$ is the object we encountered in Sec. 8.2. The isomorphism (231) implies then (250). Upon taking further derivatives we get

$$\begin{aligned} \frac{\partial}{\partial X^i}\Omega \in &\quad \mathbf{F}^2 = H^{3,0}\oplus H^{2,1} \\ \frac{\partial^2}{\partial X^i \partial X^j}\Omega \in &\quad \mathbf{F}^1 = H^{3,0}\oplus H^{2,1}\oplus H^{1,2} \\ \frac{\partial^3}{\partial X^i \partial X^j \partial X^k}\Omega \in &\quad \mathbf{F}^0 = H^{3,0}\oplus H^{2,1}\oplus H^{1,2}\oplus H^{0,3}. \end{aligned} \tag{252}$$

8.6 Special Kähler geometry

Let us discuss the consequences of the first property (247), which follows from simple consideration of type. If we insert (242) in $\int_M \Omega \wedge \frac{\partial}{\partial X^k}\Omega = 0$, a consequence of (252) and (247), we can conclude that $F_k = \frac{1}{2}\frac{\partial}{\partial X_k}\sum_i X^i F_i$. That implies that the F_i are indeed not independent but determined as derivatives of the single function[35]

$$F = \frac{1}{2}\sum_{i=0}^{h} X^i F_i \tag{253}$$

called the prepotential. Note that F is not a symplectic invariant. It follows further from the first transversality that F is homogeneous of degree 2 in X^a, i.e. $\sum_{a=0}^h X^a \frac{\partial}{\partial X_a}F = 2F$. The implication of the second

[35] Note that on even complex dimensional Calabi-Yau manifolds there will be no relative sign in (241) basis nor in (242) and $\int_M \Omega\wedge\Omega = 2X^a F_a = 0$ gives already an algebraic relation between the periods. Using further transversalities one find an intriguing mix between algebraic and differential relations between the periods in the even case.

line in (252) $\int_M \Omega \wedge \frac{\partial^2}{\partial X_i \partial X_j}\Omega = 0$ follows already from the degree two homogeneity of F and contains no new information. The last line of (252) shows that $\int_M \Omega \wedge \frac{\partial^3}{\partial_{X^a}\partial_{X^b}\partial_{X^c}}\Omega$ is nonzero and we calculate

$$C_{abc}(t) = \int_M \Omega \wedge \frac{\partial^3}{\partial_{X^a}\partial_{X^b}\partial_{X^c}}\Omega = \frac{\partial^3}{\partial_{X^a}\partial_{X^b}\partial_{X^c}} F = (X^0)^2 \frac{\partial^3}{\partial_{t_a}\partial_{t_b}\partial_{t_c}}\ \mathcal{F}^{(0)}(t) \ , \tag{254}$$

where a, b, c runs form 1 to h^{21}. To derive this we used (240,241,242) and the homogeneity of degree two of F to pass to the inhomogeneous variables t. Each of the three derivatives w.r.t. to the complex structure parameters $\frac{\partial}{\partial X^k}$ has to hit one $\mathrm{d}f^\kappa$ in $\Omega = \frac{1}{3!}h(f)\epsilon_{\mu\nu\rho}\mathrm{d}f^\mu \mathrm{d}f^\nu \mathrm{d}f^\rho$ to produce the $(0,3)$ part. It is clear by (251) that the eq. (254) is up a normalization equivalent to (223). It turns out that mirror symmetry identifies $C_{ijk}(t) = \frac{\partial^3}{\partial_{t_a}\partial_{t_b}\partial_{t_c}}\ \mathcal{F}^{(0)}(t)$ at a special point in the moduli space with (117). The right hand side of (254) is not covariant. It is valid only in the coordinate system defined by the periods X^a or the inhomogeneous coordinates t^a. The period expression is however valid in any parametrization of the complex structure. If we make a coordinate transformations of the latter $X^a \to z^a(X)$ we need no covariant derivatives on the right hand side to compensate for the derivatives of $\frac{\partial z^a}{\partial X^b}$ by z, because by (252) only terms contribute, for which all derivatives by z act on $\Omega(z)$. In any complex structure coordinates we can therefore express the triple couplings in terms of the period integrals as

$$C_{ijk} = \int \Omega \wedge \partial_i \partial_j \partial_k \Omega = \sum_{l=0}^{h} (X^l \partial_i \partial_j \partial_k F_l - F_l \partial_i \partial_j \partial_k X^l) \tag{255}$$

and C_{ijk} transforms like $\mathrm{Sym}^3 T\mathcal{M} \otimes \mathcal{L}^{-2}$ under Kähler- and general coordinate transformations in the complex moduli space $\mathcal{M}$. Note that C_{ijk} is by (244 an symplectic invariant, if the derivative is w.r.t. to invariant complex structure parameters, such as the z in Sec. 8.7. The triple coupling are the *Yukawa couplings* of the moduli fields in the effective action of heterotic string compactifications, see e.g. [92, 173].

Let us come to the two point function (224) and is relation to (248). As we have discussed the $(3,0)$ form $\Omega(z)$ lies a complex line bundle $\mathcal{H}^{3,0}$. This bundle is called the vacuum bundle $\mathcal{L}$ in physics. Is has a natural gauge transformation $\Omega \mapsto e^{f(z)}\Omega$ where $f(z)$ is holomorphic, which leads to another nowhere vanishing $(3,0)$ form. We have by (248) a positive *hermitian norm* $S(\Omega,\Omega) = ||\Omega||^2 := i\int_M \Omega \wedge \bar{\Omega}$, which is is related to the norm (392) by a volume factor $||\Omega||^2 = iV|\Omega|^2$. We define a now potential

$$K = -\log i \int_M \Omega \wedge \bar{\Omega} \ , \tag{256}$$

which will turn out to be Kähler potential of the moduli space metric. Clearly the gauge transformation become Kähler transformations $K \mapsto K - f - \bar{f}$ and e^K is a section of real line bundle. We can define a candidate Kähler metric on the moduli space

$$G_{a\bar{b}} = \partial_a \bar{\partial}_{\bar{b}} K \ . \tag{257}$$

Note by (358) that the Kähler form to this metric is the *curvature form* $\mathcal{R}$ of the hermitian metric $S(\Omega,\Omega)$ on $\mathcal{L}$. Using (250) we can relate this metric to (224)

$$G_{a\bar{b}} = -\frac{\int_M \hat{A}^{(a)} \wedge \hat{A}^{(\bar{b})}}{\int_M \Omega \wedge \bar{\Omega}} \ . \tag{258}$$

These couplings (224) are the kinetic terms of the moduli fields [92, 173]. We determine $c_k(z,\bar{z}) = -\partial_{z_k} K$.

Let us compare that metric $G_{a\bar{b}}$ with the standard way one defines a metric on the space of metrics on M. The metric on the Calabi-Yau moduli space factorizes at least locally in the Kähler- and the complex structure deformations space, see Sec. 5.3 and [34, 33] for further background,

$$2G_{a\bar{b}}\delta z^a \delta z^{\bar{b}} = \frac{1}{2V}\int_M g^{m\bar{n}} g^{k\bar{l}} \delta g_{mk} \delta g_{\bar{n}\bar{l}} \det(g_{ab})^{\frac{1}{2}} \mathrm{d}x^6 \ , \tag{259}$$

where we just took the complex structure deformations into account. The metric (259) is called the Weil-Peterson metric of the complex moduli space. In Sec. 5.3 we have already identified pure deformations of the metric with elements in $H^1(M, TM)$, the precise relation is $\delta g^{(a)}_{\bar{m}\bar{n}} = \frac{\partial g_{\bar{n}\bar{m}}}{\partial z^a}\delta z^a = -2A^{(a)\ i}_{\bar{n}} g_{i\bar{n}}\delta z^a$. Using (233) in (259) we note the remarkable fact that the two metrics (257) and (259) coincide. This was first proven in [189] and implies the *local Torelli theorem* as well as the fact that the holomorphic sectional curvature of the Weil-Peterson metric is negative and bounded away form zero [189].

From (256) and (242) follows a simple $\mathrm{Sp}(h^3, \mathbf{Z})$ invariant formula for its Kähler potential in terms of the periods

$$K = -\log i\left(\bar{X}^{\bar{a}}\frac{\partial F}{\partial X^a} - X^a\frac{\partial \bar{F}}{\partial \bar{X}^{\bar{a}}}\right) = -\log i\Pi^\dagger\Sigma\Pi \ . \tag{260}$$

This statement in terms of the inhomogeneous coordinates $t_i = X^i/X^0$, $i = 1, \ldots, h^{2,1}$ reads

$$e^{-K(t,\bar{t})} = i|X^0|^2(t^i - \bar{t}^{\bar{\imath}})(\partial_i\mathcal{F}^{(0)} - \bar{\partial}_{\bar{\imath}}\overline{\mathcal{F}}^{(0)}) - 2(\mathcal{F}^{(0)} - \overline{\mathcal{F}}^{(0)}) \ . \tag{261}$$

As it obvious the $C_{ijk}(t) \in \mathrm{Sym}^3 T^*\mathcal{M} \otimes \mathcal{L}^2$ as well as the real Kähler potential $K(t\bar{t})$ derive from the holomorphic section $\mathcal{F}^{(0)}(t) \in \mathcal{L}^2$ over the complex moduli space $\mathcal{M}$. This justifies the name prepotential for $\mathcal{F}^{(0)}$ and the structure defined by (257),(261) and (254) supplemented with the requirement that the Chern class represented by the curvature two form $\mathcal{R}$ of the vacuum line bundle $\mathcal{L}$ defines an even integral class[36] on $\mathcal{M}$ is known as *special K¨ahler geometry*.

The integrability condition for the existence of $\mathcal{F}^{(0)}$, given $G_{i\bar{\jmath}} = \partial_i\bar{\partial}_{\bar{\jmath}}K(t,\bar{t})$ and C_{ijk}, is

$$R^l_{i\bar{k}j} = -\bar{\partial}_{\bar{k}}\Gamma^l_{ij} = [D_i, \partial_{\bar{k}}]^l_j = G_{i\bar{k}}\delta^l_j + G_{j\bar{k}}\delta^l_i - C_{ijm}\bar{C}^{ml}_{\bar{k}} \tag{262}$$

The upshot of *special K¨ahler geometry* is that the relevant quantities are fixed by the section $\mathcal{F}$ of the holomorphic line bundle $\mathcal{L}^2$ over the compactified moduli space. As it is well known in complex geometry such sections are fixed by a finite set of data, basically a Riemann-Hilbert problem to find sections of the Hodge-bundle, which observe certain monodromies. This fact underlies our ability to solve the two derivative effective action of $N = 2$ gauge theories exactly.

This structure we have discussed here mainly form the geometrical point of view has been independently discovered in the vector multiplet moduli space of $N = 2$ supergravity theories in four dimensions [51, 52, 49]. The connection to string compactifications has been made in [33, 185] and a more mathematical view is offered in [73]. In making contact with the supergravity literature note that [51, 52, 49] uses for the homogeneous sections

$$L^I = e^{\frac{K}{2}}X^I, \qquad M_I = \kappa e^{\frac{K}{2}}F_I, \tag{263}$$

over $\mathcal{M}$, which are not holomorphic $\partial_{\bar{k}}X^I = \bar{\partial}_{\bar{k}}F_I = 0$, but covariantly holomorphic with respect to the Kähler connection $D_{\bar{k}} = (\partial_{\bar{k}} - \frac{1}{2}K_{\bar{k}})$, i.e. $D_{\bar{k}}L^I = D_{\bar{k}}M_I = 0$, with the effect that $i(\bar{L}^I M_I\kappa^{-1} - L^I\bar{M}_I\bar{\kappa}^{-1}) = 1$. In particular the earlier literature on $N = 2$ black holes [71, 186] uses $\kappa = 2i$, because the gravitino variations have been worked out in this conventions [52]. In the inhomogeneous coordinates $t^I = \frac{L^I}{L^0} = \frac{X^I}{X^0}$ the Kähler factor cancels.

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[36] A K¨ahler manifold (261) whose K¨ahler form is the curvature two-form $\mathcal{R}$ of line bundle $\mathcal{L}$ representing a class in $H^2(\mathcal{M}, \mathbf{Z})$ is called K¨ahler-Hodge in the mathematical literature. As is was pointed out in [48] the fermions already in $N = 1$ susy require that $[\mathcal{R}]$ is an even integral class.

8.7 Picard-Fuchs equation from the symmetries of the ambient space

Let us now discuss an explicit simple example of such a mirror symmetry computation. The principle example is the quintic in the projective space $\mathbf{P}^4$, which is discussed in great detail in the paper [37]. It is defined as the zero locus of a homogeneous polynomial of degree 5 in x_i, e.g.

$$P=\sum_{i=1}^{5} a_i x_i^5 + a_0 \prod_{i=1}^{5} x_i = \sum_{i=1}^{5} x_i^5 - z^{-\frac{1}{5}} \prod_{i=1}^{5} x_i = 0 \tag{264}$$

The z appears here as one of the 101 possible complex structure deformations of the full family of quintics. A deformation is generate by perturbing $P_0 = \sum_{i=1}^5 x_i^5$ with a parameter multiplying a monomial of degree 5. We count (5) x_i^5, (20) $x_i^4 x_j$, (20) $x_i^3 x_j^2$, (30) $x_i^2 x_j^2 x_k$, (30) $x_i x_j x_k^3$, (20) $x_i x_j x_k x_l^2$, (1) $\prod_{i=1}^5$, with $i,j,k,l=1,\ldots 5$ hence 126 monomials. Not all of those lead to independent complex structure deformations, because the complex linear transformations of the coordinates x_i of $\mathbf{P}^4$ leads to completely equivalent forms of the constraint. The group of those has dimension 5^2-1. Finally there is one relation by $P=0$ leading to 101. The symmetric deformation in (264) is chosen with hindsight, because we can see it as the unique complex structure deformation on the mirror manifold of the quintic W. The mirror is constructed as $\mathbf{Z}_5^3$ orbifold of the original quintic M. The orbifold is generated by phase rotations on the homogeneous coordinates $\mathbf{P}^4$

$$x_i \to \exp(2\pi i g_i^{(\alpha)}/5) x_i, \quad \alpha=1,2,3, \quad i=1,\ldots,5\ , \tag{265}$$

with $g^{(1)}=(1,4,0,0,0)$, $g^{(2)}=(1,0,4,0,0)$ and $g^{(3)}=(1,0,0,4,0)$. It leaves precisely the perturbing monomial $\prod_{i=1}^5 x_i$ invariant. This one deformation parameter z can be identified with the one Kähler deformation t of the original quintic M which has Hodge numbers $h^{1,1}=1$ and $h^{2,1}=101$. The one element in $H^{1,1}(M)$ comes from the restriction of the unique Kähler form of $\mathbf{P}^5$ to the hyper surface. The 101 elements of $H^1(M,TM)$ we counted above and explained their relation to $H^{2,1}(M)$ before.

The holomorphic $(3,0)$ form can written explicit in every patch U_l of $\mathbf{P}^4$ as a residuum expression[98]

$$\Omega(z)=\int_\gamma \frac{a_0\mu}{P}\ , \tag{266}$$

where the contour surrounds the single pole at $P=0$ inside $\mathbf{P}^4$ and the measure is

$$\mu=\sum_{k=1}^{5}(-1)^k w_k x_k \mathrm{d}x_1\wedge\ldots\wedge\widehat{\mathrm{d}x_k}\wedge\ldots\wedge \mathrm{d}x_5\ . \tag{267}$$

In each coordinate patch U_l, $x_l=1$ and $\mathrm{d}x_l=0$ so the sum (267) collapses to a single term. The w_k makes (267) applicable to hypersurfaces in weighted projective space $WCP[w_1,\ldots,w_5]$, which are generalizations of $\mathbf{P}^4$, see (398). An important consistency condition for Ω is its invariance under the $\mathbf{C}^*$ action $x_i\to\lambda x_i$. Let us consider the parametrization of the complex structure by the parameters a_i, $i=0,\ldots,5$ in $P=\sum_{i=1}^5 a_i x_i^5 + a_0\prod_{i=1}^5 x_i$. Theses are redundant parameters and can be "gauged" by the $G_{\mathbf{P}^4}=PGL(N,\mathbf{C})\times\mathbf{C}^*$ transformation on the homogeneous parameters $(x_1:\ldots:x_5)$ of $\mathbf{P}^4$ to one parameter. Let us summarize the "gauge invariances" of $\Omega(\underline{a})$, which are obvious from (266) and (267).

- It is invariant under the change $a_i\to\rho a_i$ with $\rho\in\mathbf{C}^*$. Defining the logarithmic derivative $\theta_i = a_i\frac{\partial}{\partial a_i}$, this homogeneity of degree 0 is expressed as

$$\sum_{i=0}^{5}\theta_i\Omega(\underline{a})=0\ . \tag{268}$$

- It is invariant under the $\mathbf{C}^*$ actions $(a_i, a_j) \to (\rho^{-5} a_i, \rho^5 a_j)$, $i, j = 1, \ldots, 5$ with $\rho \in \mathbf{C}^*$. These are compensated on P by $G_{\mathbf{P}^4}$ transformations $(x_i, x_j) \to (\rho x_i, \rho^{-1} x_j)$, which leave the form μ invariant. As differential relations one has

$$(\theta_i - \theta_5)\Omega(\underline{a}) = 0, \;\; i = 1, \ldots, 5 \, . \tag{269}$$

These two equations mean that $\Omega(\underline{a}) = \Omega(z)$ does depend only on the combination $z = -\frac{a_1 a_2 a_3 a_4 a_5}{a_0^5}$, where we chose the sign for latter convenience. Instead of fixing the gauge immediately we first notice the obvious differential relations

$$\left(\frac{\partial}{\partial a_0}\right)^5 \frac{\Omega(\underline{a})}{a_0} = \left(\prod_{i=1}^{n} \frac{\partial}{\partial a_i}\right) \frac{\Omega(\underline{a})}{a_0} \, . \tag{270}$$

With $\theta_i = a_i \frac{\partial}{\partial a_i}$, $\theta = z\frac{\mathrm{d}}{\mathrm{d}z}$, the commutator $[\theta_i, a_i^x] = x a_i$ and $\theta_0 = -5\theta$ as well as $\theta_i = \theta$ for $i = 1, \ldots, 5$ we rewrite

$$\begin{aligned} \left(\frac{\theta_0}{a_0}\right)^5 \frac{\Omega(\underline{a})}{a_0} &= \frac{1}{a_1 a_2 a_3 a_4 a_5} \left(\prod_{i=1}^{5} \theta_i\right) \frac{\Omega(\underline{a})}{a_0} \\ \frac{a_1 a_2 a_3 a_4 a_5}{a_0^5} \left(\prod_{k=1} (\theta_0 - k)\right) \Omega(\underline{a}) &= \left(\prod_{i=1}^{5} \theta_i\right) \Omega(\underline{a}) \\ z \prod_{k=1} (5\theta + k)\Omega(z) &= \theta^5 \Omega(z) \end{aligned} \tag{271}$$

The last line means that the factorizing differential operator $\mathcal{D} = \theta\mathcal{L} = \theta[\theta^4 - z\prod_{i=1}^4 (\theta + i)]$ annihilates $\Omega(z)$ and it also annihilates the periods

$$\Pi_i(z) = \int_{\Gamma_i} \Omega(z) \tag{272}$$

with $\Gamma_i \in H^3(W)$. One checks that $\mathcal{L}\Omega(z)$ is already exact, i.e. $\int_{\Gamma_i} \mathcal{L}\Omega(z) = 0$ so that the periods $\Pi_i(z) = \int_{\Gamma_i} \Omega(z)$, which correspond to the four independent cycles $\Gamma_i \in H_3(W)$ are determined by the four solutions of differential equation

$$[\theta^4 - 5z \prod_{i=1}^{4} (\theta + i)] \, \Pi(z) = 0 \, . \tag{273}$$

Note that the mirror has $h^{2,1} = 1$ and hence 4 elements in the middle cohomology $H^3(M, \mathbf{Z}) = H^{3,0} \oplus H^{21} \oplus H^{12} \oplus H^{03}$. The four period integrals over the dual four homology 3-cycles, which are invariant under the $\mathbf{Z}_5^3$ group correspond to four independent solutions of eq (273). The 3-cycles are in a fixed topological basis of $H^3(M, \mathbf{Z})$. This basis is independent of the complex structure. The trick in the derivation of the differential equation was to fix the gauge symmetry at the very end (last line of (271)). This results in a considerable simplification in the derivation of the period equations compared with the Griffith reduction method discussed below. The method is adjusted to derive the systems of Picard-Fuchs operators of multi parameter Calabi-Yau hypersurfaces and complete intersections in toric ambient spaces, which have the corresponding $\mathbf{C}^*$ actions, see [115][138]. It will give in general as above differential operators allowing for too many solutions, which need to be reduced to lower order differential operators. In the simplest case this is accomplished by factorization. As one example of this type consider the hypersurface of degree 12 in $\mathbf{P}(1, 1, 2, 2, 6)$, which has $h^{1,1}(M) = 2$ and $h^{2,1}(M) = 128$. We mod M out by an $\mathbf{Z}_{12} \times \mathbf{Z}_6 \times \mathbf{Z}_6$ acting as

$$x_i \to \exp(2\pi i g_i^{(\alpha)}/12) x_i, \quad \alpha = 1, 2, 3, \quad i = 1, \ldots, 5 \, , \tag{274}$$

with $g^{(1)} = (1, 11, 0, 0, 0)$, $g^{(2)} = (2, 0, 10, 0, 0)$ and $g^{(3)} = (2, 0, 0, 10, 0)$. The invariant constraint, which we interpret as mirror admits two complex structure deformations $h^{2,1}(W) = 2$

$$P = a_1 x_1^{12} + a_2 x_2^{12} + a_3 x_3^6 + a_4 x_4^6 + a_5 x_5^2 + a_0 \prod_{i=1}^{5} x_i + a_6 (x_1 x_2)^6 \tag{275}$$

It is convenient to express the multiplicative relation between the monomials in (275) in vectors[37]

$$l^{(1)} = (-6; 0, 0, 1, 1, 3, 1) \qquad l^{(2)} = (0; 1, 1, 0, 0, 0, -2) \tag{276}$$

such that equations corresponding to (270) are now written as

$$\prod_{l_i^{(b)}<0} \left(\frac{\partial}{\partial a_i}\right)^{-l_i^{(b)}} \frac{\Omega(\underline{a})}{a_0} = \prod_{l_i^{(b)}<0} \left(\frac{\partial}{\partial a_i}\right)^{l_i^{(b)}} \frac{\Omega(\underline{a})}{a_0} \qquad b = 1, 2\,. \tag{277}$$

Similar symmetry considerations as above lead to the conclusion that $\Pi(\underline{z})$ depends only on

$$z_b = (-1)^{l_0^{(b)}} \prod_i a_i^{l_i^{(b)}}, \qquad b = 1, 2 \tag{278}$$

and the reduction of (277) leads after factorization to the differential operators $\theta_i = z_i \frac{\mathrm{d}}{\mathrm{d}z_i}$

$$\begin{aligned} \mathcal{D}_1 &= \theta_1^2(\theta_1^2 - 2\theta_2) - \textstyle\prod_{i=0}^{2}(6\theta_1 - (2i+1)) z_1 \\ \mathcal{D}_2 &= \theta_2^2 - \textstyle\prod_{i=1}^{2}(2\theta_2 - \theta_1 - i) z_2\,. \end{aligned} \tag{279}$$

We will discuss the solution to (273,279) below.

Let us first perform the integral over the small circle γ say in the patch U_k, i.e. $x_k = 1$ to bring the expression of the $(n, 0)$ form to one which is familiar from the study of Riemann surfaces. In order to do reduce one integration over $\mathrm{d}x_i$ to the residuum integration $\int \frac{\mathrm{d}p}{p} = 2\pi i$ we perform a coordinate transformation from $(x_1 \ldots \widehat{x_k} \ldots x_5)$ to $(x_1 \ldots \widehat{x_k} \ldots \widehat{x_i} \ldots x_5, P)$ under which the measure $\mathrm{d}x_1 \wedge \ldots \widehat{\mathrm{d}x_k} \ldots \wedge \mathrm{d}x_4$ goes to $\left(\frac{\partial P}{\partial x_i}\right)^{-1} \mathrm{d}x_1 \wedge \ldots \widehat{\mathrm{d}x_k} \ldots \widehat{\mathrm{d}x_i} \ldots \wedge \mathrm{d}x_5 \wedge \mathrm{d}P$. Because of transversality $\mathrm{d}P = 0$ has no common solution with $P = 0$ and we can always pick an k and i so that $\left(\frac{\partial P}{\partial x_i}\right) \neq 0$ for $P = 0$. Therefore the integrand will have a single pole at $\frac{1}{P}$ and integration leads to

$$\Omega(z) = \frac{a_0 w_k x_k \mathrm{d}x_1 \wedge \ldots \widehat{\mathrm{d}x_k} \ldots \widehat{\mathrm{d}x_i} \ldots \wedge \mathrm{d}x_5}{\frac{\partial P}{\partial x_i}}\,. \tag{280}$$

This form of the $(n, 0)$ form is analogous to the well known $(1, 0)$ form $\Omega \sim \frac{\mathrm{d}x}{y}$ in the case of an elliptic curve realized as cubic in $\mathbf{P}^2$ with the inhomogeneous equation in the $z = 1$ patch given in the Weierstrass form $y^2 = 4x^3 - g_2 x - g_3$. It can be verified that it is nowhere vanishing [98].

8.8 Picard-Fuchs equation from the Dwork-Griffith reduction method

From the formal definition of the period $\Pi(z) = \int_{\Gamma_i} \Omega(z)$, with Ω given in (266) we can alternatively derive a fourth order differential equation for the period in terms of the moduli z by the Dwork-Griffiths reduction method. We mention this methods, because in general the symmetries of the ambient space are

[37] They will identifi ed with the generators of the Mori cone in Sec. 8.9.

not sufficient to find the full set of Picard-Fuchs equations. The key observation for this algorithm comes as follows. Consider on the ambient space $\mathbf{P}^{m-1}(w_1,\ldots,w_m)$ the $(m-2)$-form

$$\Phi = \frac{a_0}{p^r}\sum_{i<j}(-1)^{i+j}(w_jx_jA_i - w_ix_iA_j)\mathrm{d}x_1\wedge\ldots\wedge\widehat{\mathrm{d}x_i}\wedge\ldots\wedge\widehat{\mathrm{d}x_j}\wedge\ldots\wedge\mathrm{d}x_n\ .$$

Here $A_i(x)$ are homogeneous of degree d_i in x, i.e. $\sum_{k=1}^m x_kw_k\frac{\partial}{\partial_k}A_i = d_iA_i$. We further assume that $c_1(M)=0\leftrightarrow\sum_{i=1}^m w_i = d$, where d is the homogeneous degree of P, $\sum_{k=1}^m x_kw_k\frac{\partial}{\partial_k}P = Pd$. With this assumptions the total derivative of Φ simplifies

$$\begin{aligned}\mathrm{d}\Phi = &\sum_{k=1}^m\left(\frac{a_0 r}{P^{r+1}}A_k\partial_kP - \frac{a_0}{P^r}\partial_kA_k\right)\mu\\ &+\frac{a_0}{P^r}\sum_{j=1}^m(d(1-r)-w_i+d_i)A_i(-1)^j\mathrm{d}x_1\wedge\ldots\wedge\widehat{\mathrm{d}x_j}\wedge\ldots\wedge\mathrm{d}x_n\ .\end{aligned}$$

If we choose now the A_j so that $A_j=0$ for $j\neq k$ and $d_k = d(r-1)+w_k$ for $f(x):=A_k(x)$ the second term vanishes. In other words if $\frac{\partial}{\partial x_k}\left(\frac{f(x)a_0}{p^r}\mu\right)$ is homogeneous of degree 0 w.r.t. the coordinate weights w_i then

$$\frac{a_0rf\partial_kP}{P^{r+1}}\mu = \frac{a_0\partial_kf}{P^r}\mu \tag{281}$$

holds under the integration sign.

Let us mention in passing that for Calabi-Yau manifolds defined by a transversal complete intersections of s polynomials, i.e. as the zero set $P_1=\ldots=P_s=0$ in a weighted projective space the analog of (266) is

$$\Omega = \int_{\gamma_a}\ldots\int_{\gamma_s}\prod_{k=1}^s\frac{a_0^{(k)}}{P_s}\mu, \tag{282}$$

where γ_i are circles around the $P_i=0$ and similar as before $\frac{\partial}{\partial x_k}\left(f(x)\prod_{k=1}^s\frac{a_0^{(k)}}{P_k}\mu\right)$ is exact iff it is of total degree zero. This leads to the partial integration rule[98]

$$\sum_{k\neq j}\frac{n_k}{n_j-1}\frac{P_j}{P_k}\frac{f\partial_iP_k}{\prod_{l=1}^sP_l^{n_l}}\mu\ = \frac{1}{n_j-1}\frac{P_j\partial_i}{\prod_{l=1}^sP_l^{n_l}}\mu - \frac{f\partial_iP_j}{\prod_{l=1}^sP_l^{n_l}}\mu, \tag{283}$$

where we omitted the factor $\prod_{k=1}^s a_0^{(k)}$, which is however of relevance for a scaling argument as in (271).

The idea is to take up to four derivatives of the period $\Pi(z)$ w.r.t. the complex structure moduli z, and rewrite the emerging expression by the repeated use of the partial integration rules (281) or (283) w.r.t. x_i into expressions, which have lower powers of P in the denominator and lower homogeneous degree polynomials in x in the numerators. Eventually all emergent terms can be manipulated into the form of moduli dependent functions times lower derivatives of $\Pi(z)$ w.r.t. to the moduli z. The relation derived in this way is one Picard-Fuchs operator. For the quintic one starts with four derivatives of $\Pi(z)$ and the emerging relation is of course the same 4th order generalized hypergeometric differential equation as in (273). In the multi moduli examples one has to consider various derivatives of $\Pi(z)$ w.r.t. to different combinations z as starting point and the calculation becomes quite tedious. Nevertheless one can give criteria when the left ideal of differential relations is sufficient to determine $\Pi(z)$ and systematize the calculations somewhat using a Groebner basis for the ring of monomials in the x [114, 115].

8.9 Explicite periods and monodromies

A solution to (273) will correspond a priori to an arbitrary linear combination of period integrals. To understand the physical duality symmetries and the mirror map of the model it is important to find a basis of solutions which corresponds to an integral basis of $H^3(M,\mathbf{Z})$. This can be achieved by requiring that the monodromy group is realized by a subgroup of $\mathrm{Sp}(4,\mathbf{Z})$. In rescaled variables $z \to \tilde{z} = 5^5 z$ (273) has regular singular points at $\tilde{z} = 0, 1, \infty$. I.e. the moduli space is $\mathbf{P} \setminus \{0, 1, \infty\}$ and we drop the tilde from the z. At $z_0 = 0$ the *indical equation*, i.e. the condition on α in solving (273) with a local power series ansatz $\Psi(z) = (z - z_0)^\alpha \sum_{n=0} a_n (z - z_0)^n$, is $\alpha^4 = 0$. This degeneracy of solutions implies that beside the unique power series solutions one has three logarithmic solutions. Because of the logarithms the mondromy around this point has in a suitable basis an upper triangular from with a maximal shift symmetries. Near $z_0 = 1$ the indicial equation has solutions $\{0, 1, 1, 2\}$ and near $z_0 = 1/z = 0$ one has solutions $\{1/5, 2/5, 3/5, 4/5\}$ for α. The latter implies that one has an order 5 monodromy around $z = \infty$. The order two degeneration of the solutions at $z_0 = 1$ indicate three power series and one logarithmic solution. The monodromies around these special points are easily worked out. We refer to the basis (296), which is the canonical large radius basis of the mirror. For the quintic further input data needed in (296) are $\int c_2 \omega = 50$ and $A_{11} = \frac{11}{2}$. In this basis and referring to the rescale variable z the monodromies are

$$M_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 5 & -3 & 1 & -1 \\ -8 & -5 & 0 & 1 \end{pmatrix}, \quad M_1 = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad M_\infty^{-1} = \begin{pmatrix} -4 & 3 & -1 & 1 \\ 1 & 1 & 0 & 0 \\ 5 & -3 & 1 & -1 \\ 8 & -5 & 0 & 1 \end{pmatrix}. \tag{284}$$

In this parametrization $z = \infty$ is the *Gepner point*, $z = 1$ is the *conifold point* and $z = 0$ correponds to *large volume* on the mirror. Our notation is that monodromies which go counter clock wised are positive, see Fig. 33. One has of course the relation $M_\infty^{-1} = M_1 M_0$. Remarkable is the monodromy M_0 around $z = 0$. This is the point of maximal unipotency. A monodomy is called quasi-unipotent of index at most k if here is some N so that

$$(T^N - \mathbf{1})^{k+1} = 0 \tag{285}$$

As it has been shown [148] if the period map is semi stable the monodromy is unipotent. This means $N = 1$. Moreover [179] shows that the maximal k that occurs as monodromy of periods is $k = \dim_C(M)$. M_0 saturates this bound and is of the maximal unipotency 3. This means in particular that a solution with cubic logarithm appears at this point. As was argued in [37] discovering (292) is that this structure is needed to map to the large radius expansion of the mirror manifold given by (**??**). A corollary to the mirror conjecture is then that all Calabi-Yau manifolds have at least one point of maximal unipotent monodromy [161].

The monodromies in original paper [37] have been worked out in variable $\psi = z^{-\frac{1}{5}}$. This yields in the above basis

$$m_\infty = \begin{pmatrix} -19 & 32 & -16 & 4 \\ 5 & -7 & 4 & -1 \\ 25 & -40 & 21 & -5 \\ -40 & 64 & -32 & 9 \end{pmatrix}, \quad m_1 = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 0 & 1 & 0 \\ -1 & 1 & -1 & 0 \\ -5 & 8 & -4 & 1 \\ 3 & 5 & 3 & 1 \end{pmatrix}, \tag{286}$$

with $m_{\alpha^k} = A^{-k} m_1 A^k$. In the unfolded moduli space there are five copies of the conifold and encircling all five yields $m_\infty = M_0^5$, see Fig. 34.

There are general theorems that guarantee that the analysis of solutions and mondromy performed by Candelas et al. for the quintic[37] extends to any family of Calabi-Yau manifolds over its complex moduli space $\mathcal{M}$. Let us summarize some of the relevant general results

- 1.) As we know from Tian-Todorov $\mathcal{M}$ is $h_{n,1}$ dimensional and unobstructed, see Sec. 8.3.
- 2.) Viehweg shows that $\mathcal{M}$ is a quasi-projective scheme, see [196] for review.
- 3.) It is not known in generality what singular fibres can occur in the family. However all singularities appear at most in complex codimension one in $\mathcal{M}$. The correponding loci S, the discriminant components of the Picard-Fuchs system, in $\mathcal{M}$ can have themselves singularities and tangencies. Application of general theorems about desingularizations of Hironaki [110] guarentees that the latter can always be resolved so that $\hat{S}$ are specified by smooth divisors with *normal crossing*, i.e. with no tangencies.
- 4.) The theorem of W. Schmidt[179] puts restrictions on the singularities of the periods at the boundary of $\mathcal{M}$. In particular no period can be degenerate worse then with $\log(z)^{\dim_C}$ at the components of $\hat{S}$.

In practice 2.) guarantees that there is a compactification of $\mathcal{M}$ while 3.) and 4.) guarantee that a local solution of all periods can be obtained everywehre in $\mathcal{M}$ solving an ansatz with infinite power series and logarithms of finite power. Monodromies for more parameter families have been investigated in [38][123] [39]. The case (275) shows some features of the resolution of S annd has been reviewed in same detail in [134].

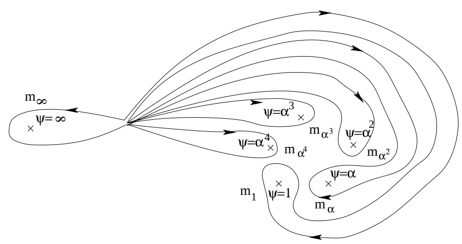

Fig. 34 Quintic monodromies in the unfolded Ψ modulispace

8.10 Integrality of the mirror map

While the integrality of instanton expansion of the $\mathcal{F}^{(g)}$ has found, at least physically, a completly satisfactory explanation as counting of BPS states, see Sec. 6.14, the integral expansion of all known mirror maps at the point of maximal unipotent monodromy remains mysterious.

We exponentiate (292) invert it and expand $z(q)$ in $q = e^t$. Call $j_q = \frac{1}{z(q)}$ in analogy with the normalized $j_e(q)$ $Sl(2,\mathbf{Z})$ invariant function of the elliptic curve. Both expansions have positive integral coefficients

$$\begin{aligned} j_e &= \tfrac{1}{q} + 744 + 196884\,q + 21493760\,q^2 + 864299970\,q^3 + 20245856256\,q^4 + \ldots \\ j_q &= \tfrac{1}{q} + 770 + 421375\,q + 274007500\,q^2 + 236982309375\,q^3 + 251719793608904\,q^4 + \ldots \end{aligned} \tag{287}$$

The integrality should be related to monodromy group $\Gamma \in \mathrm{Sp}(4,\mathbf{Z})$ generated by M_0 and M_1, but it is unknown what the integer coefficients are counting. For the example of degree 12 in $\mathbf{P}(1,1,2,2,6)$ we get

d	rational	elliptic
1	2 875	0
2	609 250	0
3	317 206 375	609 250
4	242 467 530 000	3 721 431 625
5	229 305 888 887 625	12 129 909 700 200
6	248 249 742 118 022 000	31 147 299 733 286 500
7	295 091 050 570 845 659 250	71 578 406 022 880 761 750
8	375 632 160 937 476 603 550 000	154 990 541 752 961 568 418 125

Table 4 BPS degeneracies $n^{(g)}_{\beta=d}$ associated to rational and elliptic curves on the Quintic in $\mathbf{P}^4$

for each of the two functions $j_1 = \frac{1}{z_1}$ and $j_2 = \frac{1}{z_2}$ an integral two parameter expansion

$$\begin{aligned} j_1 = \; & \tfrac{1}{q_1} + 744 + 196884\, q_1 + 21493760\, q_1^2 + 864299970\, q_1^3 + \dots \\ & q_2 \left(-\tfrac{1}{q_1} + 480 + 1403748\, q_1 + 1203172608\, q_1^2 + \dots\right) \\ & \vdots \\ j_2 = \; & \tfrac{1}{q_2} + 2 + q_2 + q_1 \left(\tfrac{1}{q_2} 240 - 240 - 240\, q + 240\, q^2\right) + \\ & q_1^2 \left(\tfrac{1}{q_2} 70920 - 57600 - 26640\, q_2 - 57600\, q_2^2 + 70920\, q_2^3\right) \dots \end{aligned} \tag{288}$$

The occurance of the j-function[38] at $j_e = j_1|_{q_2=0}$ has been related to string duality between type II on to the heterotic string on $K3 \times T2$ [122, 123], see [134] for a review. These primitive observations may point towards number theoretic applications of topological string theory. Intriguing observations for Calabi-Yau manifolds over finite fields have been made in [40]

8.11 Solutions to the Picard-Fuchs equations for all complete intersection in toric ambient spaces

For a Calabi-Yau in an general toric ambient space one can determine the generators of the Mori cone of M. These are vectors, which represent curves $C^{(a)}$, $a = 1, \dots, h_{11}$ in the Calabi-Yau space M that are dual to the Kählercone

$$l^{(a)} = (l^{(a)}_{0,1}, \dots, l^{(a)}_{0,r}; l^{(a)}_1, \dots, l^{(a)}_n), \quad \text{for } a = 1, \dots, h_{1,1}(M) = h_{2,1}(W). \tag{289}$$

Their first entries $l^{(a)}_{0,1}, \dots, l^{(a)}_{0,r}$ are the (multi)degree(s) of the algebraic constraints $P_1 = 0, \dots, P_r = 0$ defining the Calabi-Yau manifold w.r.t to the dual divisors of the $C^{(a)}$. The second set of entries $l^{(a)}_1, \dots, l^{(a)}_n$ are the intersections of the curve $C^{(a)}$ with the toric divisors of the ambient space. These curves and the intersection numbers can be determined purely combinatorial from the toric description of the ambient space, see [115] for details. E.g. for the quintic one has $l^{(1)} = (-5; 1, 1, 1, 1, 1)$.

With these data and and the classical intersections numbers $\kappa_{abc} = D_a \cap D_b \cap D_c$, which is also determined combinatorial (it is $\kappa_{111} = 5$ for the quintic), one can write down a local expansion of the periods convergent near the large complex structure point, which is characterized by its maximal unipotent monodromy. We review in the following just the essentials and refer to [115] for further details. A particular set of local coordinates z_a on the complex structure moduli space on W is defined by

$$z_b = (-1)^{\sum_a l^{(b)}_{0,a}} \prod_{i=1}^{n} a_i^{l_i^{(b)}} \qquad b = 1, \dots, h^{21}(W) \tag{290}$$

in terms of a_i, the coefficients in the polynomial constraints of the complete intersection in the torus variables (264). A point of *maximal unipotent monodromy* is then always at $z_b = 0$. Let $\varpi_{a_1,\ldots,a_s}$ be obtained by the Frobenius method[38] from the coefficients of the holomorphic function $\varpi(\vec{z},\vec{\rho})$ defined as

$$\begin{aligned} \varpi(z_1,\ldots,z_h,\rho_1,\ldots,\rho_h) &= \sum_{\{n_a\}} c(n_1\ldots n_h,\rho_1\ldots\rho_h)\prod_{a=1}^h z_a^{n_a+\rho_a} \\ c(n_1,\ldots,n_h,\rho_1,\ldots,\rho_h) &= \frac{\prod_{m=1}^r \Gamma(1-\sum_{a=1}^h \hat{l}_m^{(a)}(n_a+\rho_a))}{\prod_{i=1}^n \Gamma(1+\sum_{a=1}^h l_i^{(a)}(n_a+\rho_a))} \\ \varpi_{a_1,\ldots,a_s}(z_1,\ldots,z_h) &= \left(\tfrac{1}{2\pi i}\right)^s \partial_{\rho_{a_1}}\ldots\partial_{\rho_{a_s}}\varpi(z_1,\ldots,z_h,\rho_1,\ldots,\rho_h)|_{\{\rho_a=0\}}\,. \end{aligned} \tag{291}$$

Define also $\sigma_{a_1,\ldots,a_s} = (\varpi_{a_1,\ldots,a_s}(z_1,\ldots,z_h)|_{\log(z_a)=0})/\varpi(z_1,\ldots,z_h,\rho_1,\ldots,\rho_h)|_{\{\rho_a=0\}}$. At the large complex structure point the mirror map defines natural flat coordinates on the Kähler moduli space of the original manifold M

$$t^a = \frac{X^a}{X^0} = \frac{1}{2\pi i}(\log(z_a)+\sigma_a), \qquad a=1,\ldots,h\,, \tag{292}$$

where $X^0 = \varpi(z_1,\ldots,z_h,\rho_1,\ldots,\rho_h)|_{\rho=0}$ is the unique holomorphic period at $z_a = 0$ and $X^a = \varpi_a$ are the logarithmic periods. Double and triple logarithmic solutions are given by [115]

$$w_a^{(2)} = \frac{1}{2}\sum_{b,c=1}^h \kappa_{abc}\varpi_{bc}(z_1,\ldots,z_h), \quad a=1,\ldots,h. \tag{293}$$

$$w^{(3)} = \frac{1}{6}\sum_{a,b,c=1}^h \kappa_{abc}\varpi_{abc}(z_1,\ldots,z_h)\,, \tag{294}$$

where κ_{abc} are the classical intersection numbers $\kappa_{abc} = D_a\cap D_b\cap D_c$.

The prepotentials $F^{(0)}(X^I)$ in homogeneous or $\mathcal{F}^{(0)}(t^a)$ in inhomogeneous coordinates can now be written as

$$\begin{aligned} F^{(0)} &= -\frac{\kappa_{abc}X^aX^bX^c}{3!X^0} + A_{ab}\frac{X^aX^b}{2} + c_aX^aX^0 - i\chi\frac{\zeta(3)}{2(2\pi)^3}(X^0)^2 + (X^0)^2 f(q) \\ &= (X^0)^2\mathcal{F}^{(0)} = (X^0)^2\left[-\frac{\kappa_{abc}t^at^bt^c}{3!} + A_{ab}\frac{t^at^b}{2} + c_at^a - i\chi\frac{\zeta(3)}{2(2\pi)^3} + f(q)\right] \end{aligned} \tag{295}$$

where $q_a = \exp(2\pi i t^a)$, $c_a = \frac{1}{24}\int_X \mathrm{ch}_2 J_a$ and χ is the Euler number of X. The real coefficients A_{ab} are not completely fixed. They are unphysical in the sense that $K(t,\bar{t})$ and $C_{abc}(q)$ do not depend on them. A key technical problem[39] in the calculation is to invert the exponentiated mirror map (292) to obtain $z_i(\underline{t})$. An integral symplectic basis for the periods is given by

$$\Pi = X^0\begin{pmatrix} 1 \\ t^a \\ 2\mathcal{F}^{(0)} - t^a\partial_{t^a}\mathcal{F}^{(0)} \\ \partial_{t^a}\mathcal{F}^{(0)} \end{pmatrix} = X^0\begin{pmatrix} 1 \\ t^a \\ \frac{\kappa_{abc}t^at^bt^c}{3!} + c_at^a - i\chi\frac{\zeta(3)}{(2\pi)^3} + 2f(q) - t^a\partial_{t^a}f(q) \\ -\frac{\kappa_{abc}t^bt^c}{2} + A_{ab}t^b + c_a + \partial_{t^a}f(q) \end{pmatrix} \tag{296}$$

[38] The holomorphic period $\varpi(z_1,\ldots,z_h)$ can also be directly integrated using a residuum expression for the holomorphic $(3,0)$ form [115].

[39] We wrote an improved code for that [144].

This period vector can be uniquely given in terms of (294),(291) by adapting the leading log behavior. The A_{ab} are further restricted by the requirement that the Peccei-Quinn symmetries $t^a \to t^a + 1$ act as integral $\mathrm{Sp}(2h^{11}+2,\mathbf{Z})$ transformations on Π. Note that $\mathcal{F}^{(0)}$ can be read off from the periods and since t^a are flat coordinates, we have

$$C_{abc}(q) = \partial_{t^a}\partial_{t^b}\partial_{t^c}\mathcal{F}^{(0)} = \kappa_{abc} + \sum_{d_a,d_b,d_c\geq 0} n_d^{(0)} d_a d_b d_c \frac{q^d}{1-q^d}, \tag{297}$$

where the sum counts the contribution of the genus zero worldsheet instantons. We defined $q^d = \prod_a e^{-2\pi i d_a t^a}$ where the tuple $(d_1,\ldots,d_h)$ specifies a class β in $H^2(M,\mathbf{Z})$. The expansion predicts the first column in table 4. Higher genus predictions will be discussed in sec. 8.14.

The vectors $l^{(a)}$ are technical core data of mirror symmetry for toric complete intersections, some programs which aid to find these vectors for these manifolds are available at [144]. Let us summarize the multitude of information they contain

- 1.) They contain the degrees of the constraints and the $\mathbf{C}^*$ actions of the toric variety of ambient space and fix thereby M.
- 2.) Equivalently they can be viewed as $U(1)$ charges vectors for the fields in the linear σ model [213].
- 3.) They span the Mori cone of M, which is dual the Kähler cone of M.
- 4.) They specify the point of *maximal unipotent monodromy* in the moduli space of W namely $z^{(a)} = 0$, where the $z^{(a)} = 0$ of (290) are good local coordinates near this points and all mondromies T^a around $z^{(a)} = 0$, $a = 1,\ldots,h_{21}(M)$ satisfy (285) with $N = 1$ and $k = \dim_C(W)$.
- 5.) The periods of M are generalized hypergeometric functions with symplectic basis at $z^{(a)} = 0$ given by (296) and the $l^{(a)}$ are for those functions what the constants a,b,c are for ordinary hypergeometric functions ${}_2F_1(a,b,c,z)$ (291).

Similar, in fact simpler, solutions can be obtained for the toric local Calabi-Yau Calabi-Yau manifolds, see [45].

8.12 Rational expressions for the threepoint couplings in generic complex structure parameters

In the previous section we have focused on expressions of the genus 0 prepotential $\mathcal{F}$, which are be expanded around the large complex structure point. The expansion parameter $q = \exp(2\pi i)$ contains t, which maps in the A-model to the complexified area of curves in the Calabi-Yau. The phase in t is so that $q \to 0$ if the real area in t goes to infinity. This is the natural expansion for the Gromow-Witten invariants, where small q corresponds to large areas and hence supressed instanton corrections.

For global considerations and the calculation of the holomorphic anomaly it is necessary to have expressions for the three point couplings in terms of the complex structure parameters.

One way to derive them is to start with full system of Picard-Fuchs operators $\mathcal{D}_i\Pi(Z) = 0, i = 1,\ldots,r$. With reference to (242,254,255) we now define

$$\begin{aligned} W^{(k_1,\cdots,k_d)} &= \textstyle\sum_l (z^l \partial_{z_1}^{k_1}\cdots\partial_{z_d}^{k_d} F_l - F_l \partial_{z_1}^{k_1}\cdots\partial_{z_d}^{k_d} z^l) \\ &:= \textstyle\sum_l (z^l \partial^{\mathbf{k}} F_l - F_l \partial^{\mathbf{k}} z^l) \ . \end{aligned} \tag{298}$$

In this notation, $W^{(\mathbf{k})}$ with $\sum k_i = 3$ describes the various types of triple couplings and by (252) and consideration of type $W^{(\mathbf{k})} \equiv 0$ for $\sum k_i = 0,1,2$. If we now write the Picard-Fuchs differential operators in the form

$$\mathcal{D}_\alpha = \sum_{\mathbf{k}} A_\alpha^{(\mathbf{k})} \partial^{\mathbf{k}} \quad , \tag{299}$$

then we immediately obtain the relation

$$\sum_{\mathbf{k}} A_\alpha^{(\mathbf{k})} W^{(\mathbf{k})} = 0 \quad . \tag{300}$$

Further relations are obtained from operators $\partial_{z_i}\mathcal{D}_\alpha$. If the system of PF differential equations is complete, it is sufficient for deriving linear relations among the triple couplings and their derivatives, which can be integrated to give the Yukawa couplings up to an overall normalization. In the derivation, we need to use the following relations which are easily derived

$$\begin{aligned} W^{(4,0,0,0)} &= 2\partial_{z_1} W^{(3,0,0,0)} \\ W^{(3,1,0,0)} &= \tfrac{3}{2}\partial_{z_1} W^{(2,1,0,0)} + \tfrac{1}{2}\partial_{z_2} W^{(3,0,0,0)} \\ W^{(2,2,0,0)} &= \partial_{z_1} W^{(1,2,0,0)} + \partial_{z_2} W^{(2,1,0,0)} \\ W^{(2,1,1,0)} &= \partial_{z_1} W^{(1,1,1,0)} + \tfrac{1}{2}\partial_{z_2} W^{(2,0,1,0)} + \tfrac{1}{2}\partial_{z_3} W^{(2,1,0,0)} \\ W^{(1,1,1,1)} &= \tfrac{1}{2}(\partial_{z_1} W^{(0,1,1,1)} + \partial_{z_2} W^{(1,0,1,1)} + \partial_{z_3} W^{(1,1,0,1)} + \partial_{z_4} W^{(1,1,1,0)}) \quad . \end{aligned} \tag{301}$$

Exercise: Show that for Calabi-Yau d-folds one gets the relation $W^{(d+1,0,\dots)} = \frac{d+1}{2}\partial_{z_1} W^{(d,0,\dots)}$.

From the Picard-Fuchs equation for the quintic (273) we get $A^{(4)} = z^3(5^5 z - 1)$ and[40] $A^{(3)} = 2z(2^2 \cdot 5^5 z - 3)$. Using (300) and from (301) $W^{(4)} = 2\partial_z W^{(3)}$ we can integrate

$$C_{zzz} = \exp\left(-\frac{1}{2}\int_c^z \mathrm{d}z' \frac{A^{(3)}}{A^{(4)}}\right) = \frac{5}{z^3(1-5^5 z)} \ , \tag{302}$$

where we fixed c to match the A-model normalization $C_{ttt} = 5 + \mathcal{O}(q)$.

For the system (279) we consider first $\mathcal{D}_1, \partial_{z_1}\mathcal{D}_2, \partial_{z_2}\mathcal{D}_2$ in (300) to express e.g. $W^{(3,0)} = C_{z_1,z_1,z_1}$ in terms of $C_{z_1z_1z_2}$, $C_{z_1z_2z_2}$ and $C_{z_2z_2z_2}$. Using $\partial_{z_1}\mathcal{D}_1, \partial_{z_2}\mathcal{D}_1, \partial^2_{z_1}\mathcal{D}_2, \partial_{z_1}\partial_{z_2}\mathcal{D}_2 = \partial^2_{z_2}\mathcal{D}_2$ in (300) we may express $W^{(4,0)}$ in terms of $W^{(3,0)}$ and integrate[41] w.r.t. z_1. Proceeding this way we get after rescaling of $a = 1728 z_1$ and $b = 4z_2$ the triple couplings

$$\begin{aligned} C_{aaa} &= \tfrac{4}{a^3\Delta_1} \ , \qquad & C_{aab} &= \tfrac{2(1-a)}{a^2 b\Delta_1} \ , \\ C_{abb} &= \tfrac{(2a-1)}{ab\Delta_1\Delta_2} \ , \qquad & C_{bbb} &= \tfrac{1+b-a(1+3b)}{2b^2\Delta_1\Delta^2} , \end{aligned} \tag{303}$$

where we defined the components of the discriminant as

$$\Delta_1 = 1 - 2a - a^2(1-b) \ , \qquad \Delta_2 = (1-b). \tag{304}$$

The 3 point couplings (297) can now be recovered using the mirror map (292) in a special gauge $\int_{A_0}\Omega = 1$ in the bundle $\mathcal{L}^{-2}$ as

$$C_{abc}(q) = \frac{1}{X_0^2}\sum_{ijk}\frac{\partial z_i}{\partial t_a}\frac{\partial z_j}{\partial t_b}\frac{\partial z_k}{\partial t_c} C_{z_iz_jz_k}(z(q)) \ . \tag{305}$$

8.13 Coupling the B model to topological gravity

We consider again the moduli space introduced in Sec. 5.2

$$\mathcal{M}_g = \text{large gauge transf.}\backslash\mathcal{H}_g/(\text{diff}\times\text{Weyl})_g \ .$$

with expected dimension $3g-3$ (366). In the covariant quantization of string theory the metric independence of the theory, up to this finite dimensional space (42) we presently discuss, is expressed by a

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[40] For reference we note also $A^{(2)} = z(2^2\cdot 3^2\cdot 5^4 z - 7)$, $A^{(1)} = z(2^3\cdot 35^4 - 1)$ and $A^{(0)} = 120$.

[41] To fi x the function $c(z_2)$ in the z_1 integration, we have to calculate $W^{(3,1)}$ and $W^{(2,1)}$ in a similar fashion.

nilpotent BRST operator just like in (40). Conformal invariance is maintained for σ models on Calabi-Yau spaces. To take advantage of this extra bonus of the B-model note that in a conformal fields theory $T^\mu_\mu = 0$ and (40) splits in the following two components corresponding to $T_{zz} = T(z)$ and $T_{\bar z\bar z} = \bar T(z)$. Now we can borrow literally the treatment of the measure from the critical bosonic string. In the the case of the bosonic string the situation is exactly as in the topological B-model on a Calabi-Yau 3 fold (61), where the ghost number is identified with the $U(1)$ axial charge of the B-model. The geometrical reason for this equivalence is that (367) and (368) give the same anomaly if $\dim_C(M) = 3$ and $c_1(TM) = 0$. As we saw in Sec. 5.4 the $b(z)$ and the Q_{BRST} have ghost number -1 and 1 respectively and there is a ghost number anomaly of $6g - 6 = -3\chi(\Sigma_g)$ on a higher genus wordsheet, which corresponds to the axial current anomaly $6g - 6 = -3\chi(\Sigma_g)$.We can use therefore the same measure over the complex moduli space is in the bosonic string. From the Beltrami-Differentials $\mu^k = \mu^{k\ z}_{\bar z} \mathrm{d}\bar z\partial_z$, $k = 1, \ldots, 3g - 3$ in $H^1(T\Sigma)$, which represent tangent directions of $\mathcal{M}_g$, we define

$$B^k := \int_{\Sigma_g} \sqrt{h} h^{\alpha\gamma} h^{\beta\delta} \delta^{(k)} h_{\alpha\beta} G_{\gamma\delta} = \int_{\Sigma_g} \mathrm{d}^2 z (G_{zz} \mu^{k\ z}_{\bar z} + G_{\bar z\bar z} \bar\mu^{k\ \bar z}_{z}) = \beta^k + \bar\beta^k \ , \tag{306}$$

The definition of $B^{(k)}$ in itself does not require conformal invariance but just (40). We used after the second equality the standard metric in a conformal gauge and the expressions for the Beltrami-Differentials. In the last equality we used $(2, 2)$ supersymmetry and the fact that $G^-, \bar G^-$ are $h = 2$ fields after the B-twist to define

$$\beta^k = \int_{\Sigma_g} \mathrm{d}^2 z\, G^- \mu^k, \qquad \bar\beta^k = \int_{\Sigma_g} \mathrm{d}^2 z\, \bar G^- \bar\mu^k \ . \tag{307}$$

Because of the antisymmetry of G and the Kähler structure on the moduli space $\mathcal{M}_g$ the quantity

$$\mu_g = \langle \prod_{k=1}^{6g-6} B^k \rangle \cdot [\mathrm{d}M] = \left\langle \prod_{k=1}^{3g-3} \beta^k \bar\beta^k \right\rangle \cdot [\mathrm{d}m \wedge \mathrm{d}\bar m] \tag{308}$$

is a top-form on $\mathcal{M}_g$. Here $\cdot[\mathrm{d}M]$ or $\cdot[\mathrm{d}m \wedge \mathrm{d}\bar m]$ means contraction with $\mathrm{d}M_{i_1} \wedge \ldots \wedge \mathrm{d}M_{i_{6g-g}}$ or $\mathrm{d}m_{i_1} \wedge \mathrm{d}\bar m_{i_1} \wedge \ldots \wedge \mathrm{d}m_{i_{3g-3}} \wedge \mathrm{d}\bar m_{i_{3g-3}}$ and suitable normalization. That is we inserted $6g - 6$ times $\beta^{(k)}$ to compensate the ghost or axial anomaly, which is by the index theorems (cff section9.3) identified with the dimension of $\mathcal{M}_g$. The integral

$$\mathcal{F}^{(g)} = \int_{\mathcal{M}_g} \mu_g \tag{309}$$

is the central observable of the topological B model. How does this discussion of the dimension of the moduli space relate to (149). In the A-model we counted the geometrical virtual dimension of the moduli space of non-trivial maps and found that the deformations of the metric $\mathcal{M}_g$ are offset by the obstructions of having a a nontrivial holomorphic map to M, so that the virtual dimension of the moduli space of maps is zero. Here we kill the deformation space of $\mathcal{M}_g$ by viewing the B-model fields as ghost system from which we construct a top form to integrate over $\mathcal{M}_g$. The topological B-model is one of those examples of string theories, where general covariance (40) is maintained by an Q_{BRST} operator, whose charge violation measure the dimension of the moduli space, but the decoupling of ghost and matter sector is not imposed [199].

As part of the prerequisite for coupling topological theories to gravity [202] the measure μ_g must be closed $\mathrm{d}\mu_g = 0$. To see that consider

$$0 = \langle \{Q, \prod_{k=1}^{6g-5} B^k\} \rangle = \sum_{j=1}^{6g-5} (-1)^{j-1} \langle B^1 \ldots \{Q, B^j\}, \ldots B^{5g-5} \rangle \tag{310}$$

and use the fact that $\{Q, B^i\}$ yields the $T^i = \int_{\Sigma_g} \mathrm{d}^2 z T\mu^i$, whose insertions can be interpreted as derivative on $\mathcal{M}_g$ according to (46). A second prerequisite is that μ_g is basic, i.e. that it vanishes for all variations of the metric induced by infinitesimal diffeomorphism. These correspond to the last two terms in (43) and the property is easily checked. We will show below explicitly by manipulations similar to the one that lead to (310) that the Q commutator of the measure is exact. The metric dependence comes from the boundaries of $\mathcal{M}_g$. Combinatorial the calculation is like non-topological higher string loop calculations, apart from the much more sophisticated integrals over $\mathcal{M}_g$. The compactifications of $\mathcal{M}_{g,n}$ is identical to the one discussed in Sec. 6.2. Its boundary components come from pairwise collision of inserted points and nodes. In $2d$ gravity we got from these boundaries the topological recursion relations. In the case of the B-model there is an interesting modification namely that the boundary components contribute only in anti-holomorphic derivatives of $\mathcal{F}_g$, which gives rise to recursion relations involving antiholomorhic derivatives. Since without boundary component contributions the $\mathcal{F}^{(g)}$ would be holomorhic one calls these recursions the *holomorphic anomaly equations*. They are no more anomalous then the topological recursion relations.

8.14 The holomorphic anomaly

We want to consider in this section perturbations of a more general form then in Sec. 5.3 namely

$$S = \int_\Sigma \mathrm{d}^2 z \mathcal{L}_0 + \sum_i t^i \int_\Sigma \mathcal{O}_i + \sum_i \bar{t}^i \int_\Sigma \bar{\mathcal{O}}_i \ . \tag{311}$$

Here the WS two-form field $\mathcal{O} = \mathcal{O}^{(2)}$ is the B-model field (38) which comes from a $\phi = \mathcal{O}^{(0)}$ in the (c, c) ring. We will use here the CFT notation introduced in Sec. 5.5, i.e. $\mathcal{O}_i := \{Q_+, [Q_-, \phi_i]\} \sim \{G_0^-, [\bar{G}_0^-, \phi_i]\}$ and $\bar{\mathcal{O}}_{\bar{\imath}} := \{\bar{Q}_+, [\bar{Q}_-, \bar{\phi}_{\bar{\imath}}]\} \sim \{G_0^+, [\bar{G}_0^+, \bar{\phi}_{\bar{\imath}}]\}$. In an unitary theory $\bar{t}^i = (t^i)^*$, but it will be important in the following to view $\bar{t}^i$ as an independent parameter. As explained in Sec. 5.3 the WS two-form fields in (311) are neutral. Therefore we can expect that arbitrary $n - point$ functions like for $g > 1$

$$C^{(g)}_{i_1,\ldots,i_n} = \int_{\mathcal{M}_g} \langle \int \mathcal{O}_{i_1} \ldots \int \mathcal{O}_{i_n} \prod_{k=1}^{3g-3} \beta^k \bar{\beta}^k \rangle \tag{312}$$

do not vanish. Is it stands (312) is not well defined. We first have to specify how to deal with the contact terms, which are necessarily present in an interacting supersymmetric theory, see (93) or (100). Now in the case $g = 0$ there are the three $PSL(2, \mathbf{C})$ conformal Killing fields. The zero mode integral of their superpartners compensates for three descendant operations and with the $PSL(2, \mathbf{C})$ symmetry we set three points to $0, 1, \infty$. The generic genus zero correlation is then

$$C^{(0)}_{i_1,\ldots,i_n} = \int_{\mathcal{M}_0} \langle \phi_{i_1}(0) \phi_{i_2}(1) \phi_{i_3}(\infty) \int \mathcal{O}_{i_4} \ldots \int \mathcal{O}_{i_n} \rangle \tag{313}$$

This has no contact interaction among the first 3 fields. It is natural to make this function symmetric in its indices. Therefore we exclude *all* contact interactions from the regions of the integrations. This the regularization we adopt for general g.

In view of (311) we can insert $\int_\Sigma \mathcal{O}_i$ operators by taking t^i derivatives ∂_i of $C^{(g)}_{i_1,\ldots,i_n}$ in an the attempt to obtain $C^{(g)}_{i,i_1,\ldots,i_n}$. In order to achieve our short distance regularization we have to subtract the would be contact terms in the integration over Σ. This is *very naturally* achieved by taking covariant derivatives w.r.t. the Weil-Peterson metric, i.e. $\partial_i \to \partial_i - \Gamma_i$. In the tt^* formalism we can isolate the contact term as the difference between $\partial_i (Q_+ Q_- |j\rangle) - \mathcal{O}_j \partial_i |0\rangle = [(A_i)^k)_j \mathcal{O}_k - (A_i)^0_0 \mathcal{O}_j] |0\rangle$. The logic is that in the term $\partial_i (Q_+ Q_- |j\rangle)$ the field $\mathcal{O}_i$ in the integral $\int_\Sigma \mathcal{O}_i$ explores the region near $\mathcal{O}_j$ in (76), while in the second it does not. The Q_+Q_- generate the descendant field from ϕ_j in (76) in order to compare the two terms. In particular applying this to $|j\rangle = |0\rangle$ and using (97,98) we get a contact term with the $\mathbf{1}$ operator $(A_i)^0_0 \cdot \mathbf{1} = -\partial K \cdot \mathbf{1}$. Roughly speaking this non triviality of the vacuum comes from the coupling of ϕ_j to the $U(1)_R$ current (30). One can argue that the above contact term is proportional to the integral of R integrated over the Riemann surface. The above consideration for the half sphere (76) , fixes the normalization and in general gives the Euler number χ of Σ. Subtracting both contact terms one concludes that the insertion of $\int_\Sigma \mathcal{O}^{(2)}_i$ into on a genus g correlation function with the right short distance prescription is given by the covariant derivative of $C^{(g)}_{i_1,\ldots,i_n}$

$$D_i = \partial_i - \Gamma_i - (2-2g)\partial_i K, \tag{314}$$

This reflects the fact that $C^{(g)}_{i_1,\ldots,i_n}$ is a tensor over the complex moduli space of the Calabi-Yau $\mathcal{M}$ transforming in $\mathrm{Sym}^n(T^*\mathcal{M}) \otimes \mathcal{L}^{2-2g}$ in as a generalization of the genus zero discussion in Sec. 8.6. The last factor can also be understood by building the higher genus Riemann surface Σ_g by sewing it from a sphere. This involves g times a $|i\rangle\eta^{ij}\langle j| \in \mathcal{L}^{-2}$ insertion as we will see shortly, which results in $\mathcal{F}^{(g)}$ transforming as section of $\mathcal{L}^{2-2g}$ w.r.t. to Kähler transformations. To summarize the contact algebra analysis yields that all correlators can be obtained from the vacuum correlators $\mathcal{F}^g$ as

$$C^{(g)}_{i_1,\ldots,i_n} = D_{i1}\ldots D_{i_n}\mathcal{F}^{(g)}\ . \tag{315}$$

They are symmetric, because of the vanishing of the corresponding curvature terms in Kähler connections.

Let us therefore investigate similarly as in Sec. (91) the derivative w.r.t. $\bar t_i$ of the correlator

$$\begin{aligned}
\frac{\partial}{\partial \bar t_i}\mathcal{F}^g &= \int_{\mathcal{M}_g}\left\langle \oint_{C_w} G^+ \oint_{C'_w} \bar G^+ \bar\phi_{\bar i}(w) \prod_{k,\bar k=1}^{3g-3} \beta^k\beta^{\bar k}\right\rangle \cdot [\mathrm{d}m\wedge \mathrm{d}\bar m]\\
&= \int_{\mathcal{M}_g} 4\sum_{i\bar\imath=1}^{3g-3}\frac{\partial^2}{\partial m_i\partial\bar m_i}\left\langle \phi_{\bar i}(w)\prod_{k\neq i}\beta^k\prod_{\bar k\neq\bar\imath}\beta^{\bar k}\right\rangle\cdot[\mathrm{d}m\wedge\mathrm{d}\bar m]\\
&= \int_{\mathcal{M}_g}\partial\bar\partial\lambda^{6g-8} = \int_{\partial\mathcal{M}_g}\lambda^{6g-8}
\end{aligned}\tag{316}$$

The contour of $G^+, \bar G^+$ are originally as in Fig. 7 encircling $\bar\phi(w)$. The deformation and splitting of the contour yields a sum of terms in which the G^+ and $\bar G^+$ encircle one $\oint_{C_u}\mathrm{d}w G^+(w)G^-(u)\mu^k = 2T(u)\mu^k$ and one $\oint_{C_u}\mathrm{d}w\bar G^+(w)\bar G^-(u)\mu^{\bar k} = 2\bar T(u)\mu^{\bar k}$ in each summand. Together with the integral in the definition of the β^k and $\bar\beta^k$ and the charges Q_+ and Q_- associated to $G^+(z)$ and $\bar G^+(z)$ we can write the result of the contour deformation as

$$\begin{aligned}
\{Q_-,\beta^k\} &= \textstyle\int_{\Sigma_g}\mathrm{d}^2 z T\mu^k =: T^k\\
\{Q_+,\bar\beta^k\} &= \textstyle\int_{\Sigma_g}\mathrm{d}^2 z\bar T\bar\mu^k =: \bar T^k\ .
\end{aligned}\tag{317}$$

In Sec. 5.5 where the $G^-(u), \bar G^-(u)$ are integrated over a contour we got the L_{-1} mode of the T, which corresponds to derivative of a insertion position. Here we get the T^k and $\bar T^k$, which convert according to (46) into a derivative in the moduli space. Both effects are related and lead to exact forms on $\mathcal{M}_g$ and $\mathcal{M}_{g,n}$. The boundary components $\partial\mathcal{M}_g$, where the integral in the last line of 316 contributes according to Cauchy's theorem are in real codimension two as indicated by the form degree of λ. They are the standard stable degenerations encountered in Sec 6.2 Fig 15. The whole point specific of the B-model is to now figure out what the P_{ij}, A_{ij} and B_{ij} are. This turns out to be much easier then in the $2d$ gravity case. It is a bosonic string higher loop sewing consideration [173] with simplifications. There will be no new information in the P_{ij} above what we summarized in (315). Since $\int_\Sigma \mathcal{O}^{(i)}$ operators correspond to functions on $\mathcal{M}_g$ as opposed to the Ψ classes there is no interesting recursion to expected.

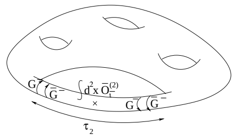

Fig. 35 A-type sewing

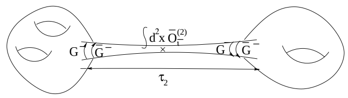

Fig. 36 B-type sewing

It remains to analyze the A and B degeneration depicted in Fig. 35 and 36 respectively. Near the boundary component in the moduli space corresponding to the degenerate surface in the figures the normal direction to the boundary can be parametrized by the length of the tube τ_2. The moduli space of the boundary components consist of the $3g-6$ dimensional moduli space of the irreducible curves of genus $g-1$ in case A or h and $g-h$ in case B respectively with measure $[\mathrm{d}\hat{m} \wedge \mathrm{d}\hat{\bar{m}}]$. That is we loose three complex dimensions in the moduli space of the irreducible components and hence three $\beta\bar\beta$. As we make the tube infinitely long or equivalently infinitesimal thin the data remembered about the shape are merely the two insertion points w and u, the length and the twist of the tube. In particular two $\beta\bar\beta$ are replaced by $(\oint_{C_x} G^- \oint_{C'_x} G^- \phi_X(x))$ with $x = u, w$ and since we want calculate a string amplitude we have to insert a complete set of states for the ϕ_X. The contribution of the boundary is hence

$$\int_{\partial M_g} [\hat{\mathrm{d}}m \wedge \hat{\mathrm{d}}\bar{m}][dw][du] \frac{\partial}{\partial \tau_2} \left\langle \int \bar\phi_{\bar\jmath} (\oint_{C_u} G^- \oint_{C'_u} G^- \phi_i) \eta^{ij} (\oint_{C_w} G^- \oint_{C'_w} G^- \phi_j) \prod_{a=1}^{3g-6} \hat\beta^a \hat{\bar\beta}^a \right\rangle \tag{318}$$

The integration over $[du]$ and $[dw]$ is over the fibre Σ_g of the universal curve. We can hence convert, e.g. the $\oint_{C_u} G^- \oint_{C'_u} G^- \phi_i$ insertions in a descendant field $\mathcal{O}^{(2)}_j$ integrated over Σ_g. Only if the $\int \bar\phi_{\bar\jmath}$ integral extends over the tube one gets a contribution proportional to τ_2 which does not cancel under the derivative in 318 and one can focus on this integration domain. The correlation function factorizes upon complete insertion of states in operator approach, which gives

$$\int_{\partial M_g} [\mathrm{d}\hat{m} \wedge \mathrm{d}\hat{\bar{m}}] \frac{\partial}{\partial \tau_2} \langle k| \int_{tube} \bar\phi_{\bar\jmath} |l\rangle \eta^{ik} \eta^{lj} \left\langle (\int_\Sigma \mathcal{O}_i)(\int_\Sigma \mathcal{O}_j) \prod_{a=1}^{3g-6} \hat\beta^a \hat{\bar\beta}^a \right\rangle . \tag{319}$$

Here we also used the fact that propagation on the tube projects on the groundstate. With the manipulations from the Sec. 5.5 and the normalizing the perimeter of the tube to one we get

$$\begin{aligned} \langle k| \int_{tube} \bar\phi_{\bar\jmath} |l\rangle \eta^{ik}\eta^{lj} &= \langle \bar k| \int_{tube} \bar\phi_{\bar\jmath} |\bar l\rangle M^{\bar k}_k \eta^{ik} M^{\bar l}_l \eta^{lj} \\ &= \tau_2 \langle \bar k| \bar\phi_{\bar\jmath} |\bar l\rangle e^{2K} G^{i\bar k} G^{j\bar l} = \tau_2 \bar C_{\bar k \bar\jmath \bar l} e^{2K} G^{i\bar k} G^{j\bar l} =: \tau_2 C^{ij}_{\bar k} \end{aligned} \tag{320}$$

Using this result in the boundary contribution of the A or B type degeneration and (315) one gets the contributions from the boundaries

$$\bar\partial_{\bar k} \mathcal{F}^{(g)} = \frac{1}{2} \bar C^{ij}_{\bar k} \left(D_i D_j \mathcal{F}^{(g-1))} + \sum_{r=1}^{g-1} D_i \mathcal{F}^{(r)} D_j \mathcal{F}^{(g-r)} \right) \tag{321}$$

The factor $\frac{1}{2}$ comes the fact that we over count the
integration over $\mathcal{O}_i$ and $\mathcal{O}_j$ in (319) by two in the A degeneration, as the $\mathcal{O}_i \leftrightarrow \mathcal{O}_j$ does not change the complex structure and in the B degeneration we doubled the non symmetric terms.

For $g = 1$ the situation is more tricky and interesting. Because of $h^0(T^2) = 1$ we have to kill the infinite automorphism by the insertion of one operator to start with a stable curve. Hence we have to consider $\bar\partial_{\bar k}\partial_m \mathcal{F}^{(1)}$. That leads in addition to the A degeneration to a contact term between $\mathcal{O}_i \bar{\mathcal{O}}_{\bar\jmath}$

$$\bar\partial_{\bar k}\partial_m \mathcal{F}^{(1)} = \frac{1}{2}\bar C_{\bar k}^{ij} C_{mij} + \left(\frac{\chi}{24} - 1\right) G_{\bar k m}\ . \tag{322}$$

The first term above is from the A type degeneration. The contact term sees global properties of the Calabi-Yau and is the most interesting one have encountered. There are two ways to normalize the contact term. Compare with the operator

$$\mathcal{F}_1(t,\bar t) = \frac{1}{2}\int \frac{\mathrm{d}^2}{\tau_2} \mathrm{Tr}(-1)^F F_L F_R q^H \bar q^{\bar H}\ . \tag{323}$$

formulation $\mathcal{F}^{(1)}$ [19] and calculate the $t\bar t$ term as in [43].

As connection explained further in [20] the topological or holomorphic limit of the genus one free energy $F^{(1)\ top}$ is related to the holomorphic Ray-Singer torsion [175]. The latter describes aspects of the spectrum of the Laplacians of $\Delta_{V,q} = \bar\partial_V \bar\partial_V^\dagger + \bar\partial_V^\dagger \bar\partial_V$ of a del-bar operator $\bar\partial_V : \wedge^q \bar T^* \otimes V \to \wedge^{q+1}\bar T^* \otimes V$ coupled to a holomorphic vector bundle V over M. More precisely with a regularized determinante over the non-zero mode spectrum of $\Delta_{V,q}$ one defines[42] [175] $I^{RS}(V) = \prod_{q=0}^n \det{}' \Delta_{V,q}^{\frac{q}{2}(-1)^{q+1}}$. One case of interest, $V = \wedge^p T^*$ with $\Delta_{p,q} := \Delta_{\wedge^p T^*,q}$, leads to the definition of a family index $F^{(1)\ top} = \frac{1}{2}\log \prod_{p=0}^n \prod_{q=0}^n (\det{}' \Delta_{pq})^{(-1)^{p+q} pq}$ depending only on the complex structure of M. As was shown in [20] the holomorphic and antiholomorphic dependence of this object on the complex structure [24] yields , which can be integrated using special geometry to $\mathrm{F}^{(1)\ top} = \frac{1}{2}\log\left[\frac{f(z)\det\left(\frac{\partial z}{\partial t}\right)}{(X^0)^\kappa}\right] .Uptothenormalisationfactor 1_{\overline{2}}$
this is the same expression that was derived in [19] using world-sheet arguments. Global topological data enter (8.14) via $\kappa = 3 + h_{11} - \frac{\chi}{12}$ and its large volume behaviour $F^{(1)\ top} \sim \sum_{i=1}^{h_{11}} t_i \int_M c_2(T) \wedge J_i$. The latter as well as local topological data of other singular limits in the complex structure moduli space of M determining the leading behaviour of $F^{(1)\ top}$ and fix the holomorphic ambiguity $f(z)$.

The counting functions for the GW invariants are obtained as a holomorphic limit of the result of the integration $\mathcal{F}^{g\ top}(t) = \lim_{\bar t \to \infty} \mathcal{F}^g(t,\bar t)$ of (8.14). One difficulty in integrating $\mathcal{F}^g(t,\bar t)$ is the possibility of adding an holomorphic piece to it. Its from is however restricted to

$$f_g(z) = \sum_{i=1}^{D} \sum_{k=0}^{2g-2} \frac{p_i^{(k)}(z)}{{\Delta_i}^k} \tag{324}$$

where D is the number of components Δ_i of the discriminant, and $p_i^{(k)}(z)$ are polynomials of degree k. Using the expansion (180) and the genus one data of the quintic discussed in (8.9) one obtains the BPS numbers in table 4 and 5.

9 Complex-, Kähler- and Calabi-Yau manifolds.

Let us describe in the following the definitions and key properties of the manifolds mentioned above. A quick introduction from the physics point of view is [117], a more extensive one is [32]. A good introduction of supersymmetric compactifications with emphasis on Calabi-Yau manifolds and orbifolds is [97][72]. One purpose of this section is to give a guide to further mathematical references which are given as we go along.

[42] [172] reviews these facts and relates the Ray-Singer torsion to Hitchins generalized 3-form action at one loop.

d	arith. genus 2	3	4
1	0	0	0
2	0	0	0
3	0	0	0
4	534 750	8 625	0
5	75 478 987 900	−15 663 750	15 520
6	871 708 139 638 250	3 156 446 162 875	−7845381850
7	5 185 462 556 617 269 625	111 468 926 053 022 750	243 680 873 841 500
8	22 516 841 063 105 917 766 750	1 303 464 598 408 583 455 000	25 509 502 355 913 526 750

Table 5 BPS degeneracies $n^{(g)}_{\beta=d}$ associated to genus 2,3,4 curves on the Quintic in $\mathbf{P}^4$

9.1 Complex manifolds

Consider a real $2n$ dimensional manifold M with a covering by coordinate patches $\mathcal{U}_i$, $i = 1, \ldots, r$, which are homeomorphic to a neighborhood $U_i \in \mathbf{C}^n$. Then we can pick $x^{(i)}_\alpha(p)$, $\alpha = 1, \ldots, n$ complex coordinates on each $\mathcal{U}_i$. M is a complex manifold, if all transition functions

$$f^{(jk)} : x^{(k)}(p) \to x^{(j)}(p) \ , \tag{325}$$

defined for $p \in \mathcal{U}_j \cap \mathcal{U}_k$, are biholomorphic.

Obviously $\mathbf{C}^n$ is a non-compact complex manifold with one chart. It is also Kähler. One may hope to get examples of compact complex manifolds by considering constraints like $f(x_1, \ldots, x_n) = 0$, which are holomorphic in all variables. While this leads indeed to a complex manifold, it fails to define compact ones, because of the maximum modulus theorem, which states that the maximum value of the modulus of a non constant differential function on an arbitrary domain D is taken at the boundary of D. If now $f = 0$ is solved for some x_i in a compact domain D of the other variables, x_i takes its maximal modulus on the boundary of D and the construction fails to define a differentiable compact manifold.

A way out is to use identifications on $\mathbf{R}^{2n}$ by discrete shift symmetries, i.e. consider tori $T^{2n} = \mathbf{R}^{2n}/\Gamma_{2n}$, where the lattice $\Gamma_{2n} \cong \mathbf{Z}^{2n}$ as abelian groups. If one chooses a complex structure on $\mathbf{R}^{2n}$ by aligning real and imaginary directions of $T^*\mathbf{R}^{2n} \cong \mathbf{R}^{2n}$ with the basis of Γ_{2n} one gets compact complex tori $T^3_{\mathbf{C}}$. They are flat and have hence trivial holonomy. Dividing by discrete rotations G of the lattice Γ_{2n} leads to orbifold compactifications. If G acts as a discrete irreducible subgroup of $SU(3)$ in the fundamental representation on the complex coordinates of $T^3_{\mathbf{C}}$ then one gets a complex orbifold with curvature singularities at the fixset of G. The corresponding lattice automorphisms have been classified [67]. Remarkably one can prove that this curvature singularities can be smoothed to get a Kähler manifold with $SU(3)$ holonomy.

An alternative route to construct simple compact complex manifolds is by dividing by $\mathbf{C}^* := \mathbf{C} \setminus \{0\}$ actions. E.g. $\mathbf{P}^n$ is defined as the space of complex lines through the origin in $\mathbf{C}^{n+1}$. This is the space of equivalence classes of $[x_1, \ldots, x_{n+1}]$ in $\mathbf{C}^{n+1} \setminus \{\underline{0}\}$ with the equivalence relation

$$(x_1, \ldots, x_{n+1}) \sim \lambda(x_1, \ldots, x_{n+1}), \tag{326}$$

where $\lambda \in \mathbf{C}^*$. For the charts we take

$$\mathcal{U}_i = \{x_i \neq 0 | x_i \in \mathbf{P}^n\}$$

and as their coordinates $x^{(i)}_m = x_m/x_i$. On $\mathcal{U}_j \cap \mathcal{U}_k$ we have the transition functions

$$x^{(i)}_m = \frac{x_m}{x_k} / \frac{x_i}{x_k} = \frac{x^{(k)}_m}{x^{(k)}_i} \ , \tag{327}$$

which are biholomorphic. $\mathbf{P}^n$ is a obviously compact and a Kähler manifold as we shall see.

A hypersurface constraint in $\mathbf{P}^n$ of the type $P(x_1, \ldots, x_{n+1}) = 0$ must be homogeneous of some degree d in the x_i, i.e. $P(\lambda x_1, \ldots, \lambda x_{n+1}) = \lambda^d P(x_1, \ldots, f_{n+1})$, to be well defined on the equivalence classes. It defines a compact complex Kähler manifold. This manifold is smooth if f is transversal, i.e. $\mathrm{d}P \neq 0$ for $P = 0$. We will give a short overview about the application of this construction and generalizations to Calabi-Yau manifolds in Sec. 9.10.

Conceptional it is an important question if and how many complex structures an even dimension real manifold possesses. A necessary prerequisite to have a complex structure is a differentiable endomorphism of the tangentbundle $J : TM \to TM$ with $J^2 = -\mathbf{1}$. J corresponds to multiplication of the tangentbundle by $i = \sqrt{-1}$ and manifold with this structure is called an almost complex manifold [43]. With J we can define projectors

$$P = \frac{1}{2}(\mathbf{1} - iJ)$$

on the holomorphic sub-bundle and the antihomlomophic sub-bundle of the tangents bundle

$$\bar{P} = \frac{1}{2}(\mathbf{1} + iJ)$$

respectively. According to a theorem of Niremberg and Newlander a necessary and sufficient[44] condition for the existence of complex coordinates, i.e. a complex structure, is that the Lie bracket (230) of two holomorphic vector fields X, Y is always a holomorphic vector field [166] (see [105] and [32] Chap. V. for physicists review). Written with the projectors one formulates this condition as

$$\bar{P}[PX, PY] = 0. \tag{328}$$

This integrability condition leads to $[JX, JY] - J[X, JY] - J[JX, Y] - [X, Y] = 0$. In local flat coordinates $J(\partial_b) = J_b^e \partial_e$ and with $J_c^b J_d^c = -\delta_d^b$, i.e. $(\partial_a J_c^b) J_d^c = -J_c^b(\partial_a J_d^c)$, this means that the so called Nijenhuis tensor vanishes identically [166]

$$N_{bd}^c := J_b^a(\partial_a J_d^c - \partial_d J_a^c) - J_d^a(\partial_a J_b^c - \partial_b J_a^c) \equiv 0\ . \tag{329}$$

Once complex coordinates $x^k = u^k + iv^k$ with

$$\partial_k := \frac{\partial}{\partial x^k} = \frac{1}{2}\left(\frac{\partial}{\partial u^k} - i\frac{\partial}{\partial v^k}\right), \qquad \partial_{\bar{k}} := \frac{\partial}{\partial \bar{x}^k} = \frac{1}{2}\left(\frac{\partial}{\partial u^k} + i\frac{\partial}{\partial v^k}\right) \tag{330}$$

are defined, we can split $T_{\mathbf{C}}M = T_{\mathbf{R}}M \otimes \mathbf{C}$, which is spanned over $\frac{\partial}{\partial w_k}$, $k = 1, \ldots, 2n$ with complex coefficients v^i as $T_{\mathbf{C}}M = T^{1,0}M \oplus T^{0,1}M$. Here $\{u_k, v_k\} =: \{w_k, w_{k+n}\}$ and each vector V in $T_{\mathbf{C}}M$ decomposes as

$$V = \sum_{k=1}^{2n} V^k \frac{\partial}{\partial w_k} = \sum_{k=1}^{n} \left[(V^k + iV^{n+k})\partial_k + (V^k - iV^{n+k})\partial_{\bar{k}}\right] =: V^{1,0} + V^{0,1}\ . \tag{331}$$

The transition function of $T^{1,0}M$ $[T^{0,1}]$ spanned by ∂_k, $[\partial_{\bar{k}}]$ are [anti-]holomorphic and we call it the [anti]holomorphic tangent bundle. Obviously under complex conjugation $T^{0,1}M = \overline{T^{1,0}M}$. Similarly the cotangent bundle splits $T^*_{\mathbf{C}}M = T^{*1,0}M \oplus T^{*0,1}M$ into a holomorphic and an anti-holomorphic sub

[43] A complex manifold is almost complex, because multiplying the basis of TM of a complex manifold with coordinates $x^k = u^k + iw^k$ by $i = \sqrt{-1}$ maps $\begin{pmatrix} \frac{\partial}{\partial y^k} \\ \frac{\partial}{\partial v^k} \end{pmatrix} \mapsto \begin{pmatrix} \frac{\partial}{\partial v^k} \\ -\frac{\partial}{\partial u^k} \end{pmatrix}$, i.e. $J = \mathrm{d}u^i \otimes \frac{\partial}{\partial v^i} - \mathrm{d}v^i \otimes \frac{\partial}{\partial u^i}$. In holomorphic and anti-holomorphic coordinates this means $J_j^i = i\delta_j^i$, $J_{\bar{j}}^{\bar{i}} = -i\delta_{\bar{j}}^{\bar{i}}$ and $J_{\bar{j}}^i = J_j^{\bar{i}} = 0$

[44] That is the nontrivial part.

bundle spanned by $\mathrm{d}x^k$ and $\mathrm{d}\bar{x}^k := \mathrm{d}x^{\bar{k}}$ respectively[45]. Sections of $\wedge^r T^*_{\mathbf{C}}M$ are called r-forms Ω^r and can be decomposed into sections of $\wedge^p T^{*1,0}M \wedge^q T^{*0,1}$, which are called (p,q)-forms $\Omega^{p,q}$, i.e the space A^r of r forms splits into the space $A^{p,q}$ of (p,q)-forms $A^r = \bigoplus_{r=p+q} A^{p,q}$. If J is integrable[46], the de Rham exterior derivative splits likewise into

$$\mathrm{d} = \partial + \bar{\partial}\,, \tag{332}$$

i.e. for $\omega = \omega_{i_1,\ldots,i_p,\bar{\jmath}_1\ldots,\jmath_q}\mathrm{d}x^{i_1}\wedge\ldots\wedge\mathrm{d}x^{i_p}\mathrm{d}x^{\bar{\jmath}}\wedge\ldots\wedge\mathrm{d}x^{\bar{\jmath}_q}\in A^{p,q}$ one has

$$\begin{aligned}\partial\omega &= (\partial_k\omega_{i_1,\ldots,i_p,\bar{\jmath}_1\ldots,\jmath_q})\mathrm{d}x^k\wedge\mathrm{d}x^{i_1}\wedge\ldots\wedge\mathrm{d}x^{i_p}\wedge\mathrm{d}x^{\bar{\jmath}}\wedge\ldots\wedge\mathrm{d}x^{\bar{\jmath}_q}\in A^{p+1,q}\\ \bar{\partial}\omega &= (\partial_{\bar{k}}\omega_{i_1,\ldots,i_p,\bar{\jmath}_1\ldots,\jmath_q})\mathrm{d}x^{\bar{k}}\wedge\mathrm{d}x^{i_1}\wedge\ldots\wedge\mathrm{d}x^{i_p}\wedge\mathrm{d}x^{\bar{\jmath}}\wedge\ldots\wedge\mathrm{d}x^{\bar{\jmath}_q}\in A^{p,q+1}\end{aligned} \tag{333}$$

so that $\mathrm{d}\Omega^{p,q}\in A^{p+1,q}\oplus A^{p,q+1}$. It follows by consideration of the (p,q) type that the equation $\mathrm{d}^2=0$ on A^* implies $\partial^2=0$, $\bar{\partial}^2=0$ and $\bar{\partial}\partial+\partial\bar{\partial}=0$. Since $\bar{\partial}$ is nilpotent we can define the cohomology $H^*_{\bar{\partial}}=\frac{\mathrm{Kern}\bar{\partial}}{\mathrm{Im}\bar{\partial}}$.

A central result is the Čech-Dolbault isomorphism, which follows from the Čech-deRham isomorphism see [100] page 43-44 and the $\bar{\partial}$-Poincaré Lemma. It states for sheaves of vectors fields F that

$$H^q(M,\Omega^p(F))\cong H^{p,q}_{\bar{\partial}}(M,F)\,. \tag{334}$$

For example $H^q(M,\wedge^p T^*M)\cong H^{p,q}(M,TM) =: H^{p,q}(M)$.

9.2 Kähler manifolds

A *hermitian metric* is a positive-definite inner product $TM\otimes\bar{T}M\to\mathbf{C}$. Locally it can be given by a covariant tensor $\sum_{i,j}^n g_{i\bar{\jmath}}(w)\mathrm{d}x^i\otimes\mathrm{d}x^{\bar{\jmath}}$ such that $\overline{g_{i\bar{\jmath}}}=g_{j\bar{\imath}}$ and $\forall v^i\in\mathbf{C}$ one has $v^i g_{i\bar{\jmath}}v^{\bar{\jmath}}>0$, if not all $v^i=0$. Note that the first index of $g_{i\bar{\jmath}}$ is only summed over the unbarred $i=1,\ldots,n$ and the second only over barred $\bar{\jmath}=\bar{1},\ldots,\bar{n}$ indices respectively. To define an hermitian metric an almost complex structure is sufficient. Hermiticity is the condition $g(X,Y)=g(JX,JY)$ on the real metric, which becomes

$$g_{mn} = J^a_m J^b_n g_{ab} \tag{335}$$

in coordinates. It does not constraint M further then admitting J and any metric say g', because for any such g' the metric $g_{mn}=\frac{1}{2}(g'_{mn}+J^a_m J^b_n g'_{ab})$ is hermitian. In particular on any complex manifold we can define a *hermitian metric* see [139] Chap 3.5. Multiplying (335) with J^m_p, defining $J_{nm}=J^a_n g_{am}$ and using $J^m_p J^a_m=-\delta^a_p$ we see that $J_{nm}=-J_{mn}$. Hence we can define a 2-form $\omega=J_{nm}\mathrm{d}w^n\wedge\mathrm{d}w^m$. In complex notation this becomes

$$\omega = i\sum_{i,j=1}^n g_{i\bar{\jmath}}\mathrm{d}x^i\wedge\mathrm{d}x^{\bar{\jmath}}\,. \tag{336}$$

This is a real form $\bar{\omega}=\omega$ of type $(1,1)$ and is called the *fundamental form* associated to the *hermitian metric*. Because[47] $g:=\det(g_{i\bar{\jmath}})>0$ one gets by wedging ω n-times

$$\mathrm{vol} = \frac{\omega^n}{n!} = i^n\det(g_{i\bar{\jmath}})\mathrm{d}x^1\wedge\mathrm{d}\bar{x}^1\wedge\ldots\wedge\mathrm{d}x^n\wedge\mathrm{d}\bar{x}^n = 2^n\det(g_{ij})^{\frac{1}{2}}\mathrm{d}w^1\wedge\ldots\wedge\mathrm{d}w^{2n} \tag{337}$$

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[45] To avoid too complicated notations TM (T^*M) will mean in the following the holomorphic tangent bundle $T^{1,0}M$ (cotangent bundle $T^*M=T^{*\,1,0}M$).

[46] On an almost complex manifold one can project r-forms Ω with p P's and q $\bar{P}$'s ($r=p+q$) to (p,q)-forms $\Omega^{p,q}$. As J depends on the coordinates one gets $\mathrm{d}\Omega^{p,q}=(\mathrm{d}\Omega)^{p-1,q+2}+(\mathrm{d}\Omega)^{p,q+1}+(\mathrm{d}\Omega)^{p+1,q}+(\mathrm{d}\Omega)^{p+2,q-1}$ and one may define $\partial\Omega^{p,q}=(\mathrm{d}\omega)^{p+1,q}$ and $\bar{\partial}\Omega^{p,q}=(\mathrm{d}\Omega)^{p,q+1}$. One can check that the condition $\bar{\partial}^2=0$ is equivalent to $N^b_{cd}\equiv 0$.

[47] Note in coordinates $x^i, x^{\bar{i}}$ one has the block form $g_{nm}=\begin{pmatrix}0 & g_{\mu\bar{\nu}}\\ g_{\sigma\bar{\rho}} & 0\end{pmatrix}$ and e.g. [32] defines $g:=\det(g_{nm})=\det^2 g_{\mu\bar{\nu}}$.

a positive volume form on M, which implies also that M is orientable.

An hermitian metric whose fundamental form is closed $\mathrm{d}\omega = 0$ is called a Kähler metric. An complex manifold endowed with a Kähler metric is called a Kähler manifold. $\mathrm{d}\omega = 0$ implies $\partial\omega = \bar{\partial}\omega = 0$, which is equivalent to $\partial_k g_{i\bar{\jmath}} = \partial_i g_{k\bar{\jmath}}$ and $\bar{\partial}_{\bar{k}} g_{i\bar{\jmath}} = \bar{\partial}_{\bar{\jmath}} g_{i\bar{k}}$. The latter equations are integrability conditions for the existence of a local Kähler potential $K(x,\bar{x})$ which is real and yields the metric as follows

$$g_{i\bar{\jmath}} = \partial_i \partial_{\bar{\jmath}} K(x,\bar{x}) = -\frac{1}{2}\mathrm{d}(\partial - \bar{\partial})K(x,\bar{x}) \ . \tag{338}$$

Note that despite the form above ω cannot be exact. For if $\omega = \mathrm{d}A$ would have been exact (337) could not be true, because using Stokes theorem the integral $\int \omega^n$ would be zero. That means that $(\partial - \bar{\partial})K$ is not globally defined. Indeed as far as the definition of ω goes $K(x,\bar{x})$ only needs to be defined up to a Kähler transformation $K(x,\bar{x}) \to K(x,\bar{x}) + f(x) + \bar{f}(\bar{x})$, so e^K will be a section of a nontrivial line bundle over M. In general two Kähler forms ω and ω' are in the same class in $H^2(M,\mathbf{R})$, if we can find a smooth global real function ϕ on M and

$$\omega' = \omega + \partial\bar{\partial}\phi(x,\bar{x}) \tag{339}$$

Above property (338) simplifies the expressions for the Christoffel symbols and the curvature tensors

$$\begin{array}{lll} a.) & \Gamma^k_{ij} = g^{k\bar{l}}\partial_i g_{j\bar{l}}, \qquad \Gamma^{\bar{k}}_{\bar{i}\bar{\jmath}} = g^{l\bar{k}}\bar{\partial}_{\bar{i}} g_{l\bar{\jmath}} & \\ b.) & R_{i\bar{\jmath}k\bar{l}} = -\partial_i\bar{\partial}_{\bar{\jmath}} g_{k\bar{l}} + g^{m\bar{n}}(\partial_i g_{k\bar{n}})(\bar{\partial}_{\bar{\jmath}} g_{m\bar{l}}), & R_{i\bar{\jmath}k}{}^{l} = -\bar{\partial}_{\bar{\jmath}}\Gamma^l_{ik} \\ c.) & R_{i\bar{\jmath}} \equiv g^{k\bar{l}} R_{i\bar{\jmath}k\bar{l}} = -\partial_i\bar{\partial}_{\bar{\jmath}} \log\det(g_{i\bar{\jmath}}) \ . & \end{array} \tag{340}$$

Note that the pure index Christoffel symbols are the only non-vanishing ones and that $R_{i\bar{\jmath}k\bar{l}} = R_{k\bar{\jmath}i\bar{l}} = R_{i\bar{l}k\bar{\jmath}}$, because of the integrability condition. The other non vanishing components of the Ricci tensor are of type $R_{\bar{\jmath}ik\bar{l}}$, $R_{i\bar{\jmath}i\bar{l}}$ and $R_{\bar{\jmath}i\bar{l}k}$. From the Ricci tensor one defines the Ricci form

$$\mathcal{R} = iR_{i\bar{\jmath}}\mathrm{d}x^j \wedge \mathrm{d}x^{\bar{\jmath}} = -i\partial\bar{\partial}\log\det(g_{i\bar{\jmath}}) = \frac{i}{2}\mathrm{d}(\partial - \bar{\partial})\log\det(g_{i\bar{\jmath}}) \ . \tag{341}$$

It satisfies $\mathrm{d}\mathcal{R} = 0$, but is not exact, despite the form it is written above, because $\log\det(g_{i\bar{\jmath}})$ is a density and not a function.

We now turn to harmonic theory for complex manifolds. On (p,q)-forms $\phi = \frac{1}{p!q!}\phi_{i_1,\ldots,i_p,\bar{\jmath}_1\ldots,\bar{\jmath}_q}\mathrm{d}x^{i_1} \wedge \ldots \wedge \mathrm{d}x^{i_p} \wedge \mathrm{d}x^{\bar{\jmath}_1} \wedge \ldots \wedge \mathrm{d}x^{\bar{\jmath}_q}$ we have an *local inner product* defined by a hermitian metric

$$(\phi,\psi)(x) = \frac{1}{p!q!}\phi_{i_1\ldots i_p\bar{\jmath}_1\ldots\bar{\jmath}_q}\psi^{i_1\ldots i_p\bar{\jmath}_1\ldots\bar{\jmath}_q} \tag{342}$$

where $\psi^{i_1\ldots i_p\bar{\jmath}_1\ldots\bar{\jmath}_q} = g^{i_1\bar{l}_1}\ldots g^{i_p\bar{l}_p}g^{k_1\bar{\jmath}_1}\ldots g^{k_q\bar{\jmath}_q}\overline{\psi_{k_1\ldots k_q\bar{l}_1\ldots\bar{l}_p}}$. With this we can define an *global inner product* $A^{p,q} \times A^{p,q} \to \mathbf{C}$

$$(\phi,\psi) = \int_M (\phi,\psi)(x)\mathrm{vol}, \tag{343}$$

with

$$(\phi,\psi) = \overline{(\psi,\phi)}, \qquad (\phi,\phi) > 0 \text{ unless } \phi = 0, \tag{344}$$

which makes $A^{p,q}$ in a pre-Hilbert space. One can define the Hodge operator[48] $* : A^{p,q} \to A^{n-q,n-p}$ i.e. $* : \psi \mapsto *\psi$ by

$$(\phi,\psi)\mathrm{Vol} = \phi \wedge *\bar{\psi} \ , \tag{345}$$

[48] Here the conventions are as in [139]. The $*$ operator in [100] maps $*_{gh} : A^{p,q} \to A^{n-p,n-q}$, so it involves an additional complex conjugation $*_{gh}\psi = *_{ko}\bar{\psi}$.

with $\bar\psi = \frac{1}{p!q!}\overline{\psi_{i_1\ldots i_p,\bar\jmath_1\ldots\bar\jmath_q}\mathrm{d}x^{i_1}\wedge\ldots\wedge\mathrm{d}^{i_p}\wedge\mathrm{d}x^{\bar\jmath_1}\wedge\ldots\wedge\mathrm{d}x^{\bar\jmath_q}} = \frac{1}{p!q!}\overline{\psi}_{j_1\ldots j_q,\bar\imath_1\ldots\bar\imath_p}\mathrm{d}x^{j_1}\wedge\ldots\wedge\mathrm{d}^{j_q}\wedge\mathrm{d}x^{\bar\imath_1}\wedge\ldots\wedge\mathrm{d}x^{\bar\imath_p}$ and $\overline{\psi_{i_1\ldots i_p,\bar\jmath_1\ldots\bar\jmath_q}} = (-1)^{pq}\bar\psi_{j_1\ldots j_q,\bar\imath_1\ldots\bar\imath_p}$. Explicetly

$$*\psi = \frac{i^n(-1)^{n(n-1)/2+np}}{p!q!(n-p)!(n-q)!}g\epsilon^{k_1\ldots k_p}_{\bar\jmath_1\ldots\bar\jmath_{n-p}}\epsilon^{\bar l_1\ldots\bar l_q}_{i_1\ldots\bar\imath_{n-q}}\psi_{k_1\ldots k_p,\bar l_1\ldots\bar l_q}\mathrm{d}x^{i_1}\wedge\ldots\wedge\mathrm{d}^{i_{n-q}}\wedge\mathrm{d}x^{\bar\jmath_1}\wedge\ldots\wedge\mathrm{d}x^{\bar\jmath_{n-p}}\,. \tag{346}$$

One checks $*\bar\psi = \overline{*\psi}$ and $**\psi = (-1)^{pq}\psi$ for ψ a (p,q)-form.

With the norm $(\cdot,\cdot)$ we can define the *adjoint operators* $\partial^*: A^{p,q}\to A^{p-1,q}$ and $\bar\partial^*: A^{p,q}\to A^{p,q-1}$ by

$$(\partial^*\psi,\phi) := (\psi,\partial\phi), \qquad \text{and} \qquad (\bar\partial^*\psi,\phi) := (\psi,\bar\partial\phi) \tag{347}$$

respectively. On a compact manifold one has $\bar\partial^* = -*\partial*$. With the adjoint operator one can define beside the de Rham Laplacian $\Delta_\mathrm{d} = \mathrm{dd}^* + \mathrm{d}^*\mathrm{d}$ the Laplacians $\Delta_\partial = \partial\partial^* + \partial^*\partial$ and $\Delta_{\bar\partial} = \bar\partial\bar\partial^* + \bar\partial^*\bar\partial$. The Hodge theorem states that every element $\phi\in A^{p,q}$ has an unique orthogonal decomposition into a harmonic form h, an exact piece $\bar\partial\xi$ with $\xi\in A^{p,q-1}$ and a co-exact piece $\bar\partial^*\eta$ with $\eta\in A^{p,q+1}$ i.e.

$$A^{p,q} = \mathcal{H}^{p,q}\oplus\bar\partial A^{p,q-1}\oplus\bar\partial^* A^{p,q+1}\,. \tag{348}$$

This is in analogy with the de Rham decomposition $A^p = \mathcal{H}^p\oplus\mathrm{d}A^{p-1}\oplus\mathrm{d}^*A^{p+1}$. The usual argument shows that if ϕ is closed, i.e. $\bar\partial\phi = 0$, then the $\bar\partial^*\eta$ piece in the decomposition is zero, because $\bar\partial\phi = \bar\partial\bar\partial^*\eta$ and thus $0 = (\bar\partial\phi,\eta) = (\bar\partial^*\eta,\bar\partial^*\eta)$, which implies $\bar\partial^*\eta = 0$. This in turn means that every $\bar\partial$ closed form can be uniquely decomposed into a harmonic form w.r.t. $\Delta_{\bar\partial}$ and a $\bar\partial$ exact piece, which implies $H^{p,q}_{\bar\partial}(M)\cong\mathcal{H}^{p,q}(M)$.

Using $(\bar\partial^*\psi)_{i_1\ldots i_p\bar\jmath_2\ldots\bar\jmath_p} = (-1)^{p+1}\nabla^{\bar\jmath_1}\psi_{i_1\ldots i_p\bar\jmath_1\bar\jmath_2\ldots\bar\jmath_q}$ one can show that the Kähler ω form is harmonic. Hence $h^{1,1}(M)\geq 1$ on a Kähler manifold. Similarly one shows that all ω^m, $m=1,\ldots,n$ are nontrivial elements in $H^{m,m}(M)$. A very important result for Kähler manifolds is the Laplacians are all equivalent

$$\Delta_\partial = \Delta_{\bar\partial} = \frac{1}{2}\Delta_d\,, \tag{349}$$

where $\Delta_{\bar\partial} = \bar\partial\bar\partial^* + \bar\partial^*\bar\partial$, $\Delta_\partial = \partial\partial^* + \partial^*\partial$ and $\Delta_\mathrm{d} = \mathrm{dd}^* + \mathrm{d}^*\mathrm{d}$. As a consequence of (349) Δ_d like Δ_∂ and $\Delta_{\bar\partial}$ does not change the (p,q)-type and taking the harmonic forms as unique representatives we get the Hodge decomposition of the deRham cohomology groups

$$H^r(M) = \bigoplus_{p+q=r} H^{p,q}(M)\,. \tag{350}$$

On the cohomology of a Kähler manifold with $n = \dim_C(M)$ one can define the exterior product with the standard Kähler form ω defined on $\mathbf{C}^n$, i.e. $\omega = \frac{i}{2}\sum_i \mathrm{d}x_i\wedge\mathrm{d}\bar x_i$, as lowering operator S^-, the adjoint operator as raising operator S^+ and the diagonal operator, which associates to each form of degree r the eigenvalue $(n-r)/2$, as H. Then $H, S^\pm$ fullfill the Lie algebra of $sl(2,C)$, $[S^+,S^-] = 2H$, $[H,S^\pm] = \pm S^\pm$ and the cohomology decomposes into irreducible representations. More precisely the *Hard Lefshetz Theorem* [101] says the following: $(S^-)^k: H^{n-k}\to H^{n+k}$ is an isomorphism and with $P^{n-k} := (\mathrm{Ker}(\mathrm{S}^-)^{\mathrm{k}+1}: H^{n-k}\to H^{n+k+2}) = (\mathrm{Ker}S^+)\cap H^{n-k}$ the *primitive cohomology* one has the *Lefshetz decomposition*

$$H^r(M) = \bigoplus_k (S^-)^k P^{r-2k}(M)\,. \tag{351}$$

The primitive parts of the cohomology play the rôle of highest weight vectors.

Examples: The cohomology of $\mathbf{P}^n$ forms a representation $\left(\frac{\mathbf{n}}{\mathbf{2}}\right)$. The cohomology of the two torus ($\dim_R(T^2) = 2$) decomposes as $2(\mathbf{0}) + \left(\frac{\mathbf{1}}{\mathbf{2}}\right)$, where the two $(\mathbf{0})$ representations are $\mathrm{d}x$ and $\mathrm{d}\bar{x}$ while $[1, \mathrm{d}x \wedge \mathrm{d}\bar{x}]$ form the $\left(\frac{\mathbf{1}}{\mathbf{2}}\right)$ representation. Check that the cohomology of the T^{2n} torus has the $sl(2,C)$ decomposition $\left(2(\mathbf{0}) + \left(\frac{\mathbf{1}}{\mathbf{2}}\right)\right)^{\otimes n} = \bigoplus_{r=1}^{n} \left(\binom{2n}{n-r} - \binom{2n}{n-r-2}\right)\left(\frac{\mathbf{r}}{\mathbf{2}}\right)$.

Let us note for further reference that the action of Δ_{d} on p-forms ω can be expressed in terms of covariant derivatives and the curvature tensors as

$$(\Delta_{\mathrm{d}}\omega)_{\mu_1\ldots\mu_p} = -\nabla^\nu\nabla_\nu\omega_{\mu_1\ldots\mu_p} - pR_{\nu[\mu_1}\omega^{\nu}_{\mu_2\ldots\mu_p]} - \frac{1}{2}p(p-1)R_{\nu\rho[\mu_1\mu_2}\omega^{\nu\rho}_{\mu_3\ldots\mu_p]} \tag{352}$$

By consideration of type follows that every holomorphic $(p,0)$-form ω is harmonic and vice versa. We have $\bar\partial^*\omega = 0$ as it maps to $A^{p,-1}$ which is trivial. If $\Delta_{\bar\partial}\omega = 0$ then from $\bar\partial^*\bar\partial\omega = 0$ follows $\bar\partial\omega = 0$.

Forms of Kähler manifolds are related by complex conjugation $\overline{A^{p,q}} = A^{q,p}$, which implies for the cohomology groups $H^{p,q}(M) = H^{q,p}(M)$, since complex conjugation commutes with Δ_{d}. The star operator $* : A^{p,q} \to A^{n-q,n-p}$ is another bijection which commutes with Δ_{d} and hence

$$H^{q,p}(M) = H^{p,q}(M) = H^{n-q,n-p}(M)\,. \tag{353}$$

Let us mention briefly further important facts about Kähler manifolds. The property of the Christoffel symbol to have only pure indices leads to the fact that parallel transport of a vector generates only the holonomy group $U(n) \in SO(2n)$ rather then $SO(2n)$, which would be the holonomy of a generic orientable manifold.

Another well known fact is that $\mathbf{P}^n$ is a Kähler manifold. This can be established by giving with the *Fubini-Study metric* an explicit. In the $\mathcal{U}_i$, $i = 0,\ldots,n$ patches the Kähler potential is given by $K^{(i)}(x^{(i)}, \bar{x}^{(i)}) = \log(1+|x^{(i)}|^2)$, where $|x^{(i)}|^2 = \sum_{j\neq i}|x^{(i)}_j|^2$. Using (327) we see that $K^{(i)}(x^{(i)}, \bar{x}^{(i)}) = K^{(j)}(x^{(j)}, \bar{x}^{(j)}) - \log\frac{x_i}{x_j} - \log\frac{\bar{x}_i}{\bar{x}_j}$. The latter two terms are holomorphic and antiholomorphic functions respectively on $\mathcal{U}_i \cap \mathcal{U}_j$. Hence they do not affect the metric $g_{i\bar{j}} = \partial_i\partial_{\bar{j}}K(x,\bar{x})$, which is globally well defined. Dropping the index for the patch we get

$$\omega = i\partial\bar\partial\log(1+|x|^2) = i\left(\frac{\mathrm{d}x^i \wedge \mathrm{d}x^{\bar\imath}}{1+|x|^2} - \frac{\bar{x}^i\mathrm{d}x^i \wedge x^j\mathrm{d}x^{\bar\jmath}}{(1+|x|^2)^2}\right)\,. \tag{354}$$

This defines a positive-definite metric. With $\det(g_{i,\bar{j}}) = \frac{1}{(1+|x|^2)^{n+1}}$ one calculates the Ricci tensor $R_{i\bar{j}} = -\partial_i\partial_{\bar{j}}\log\det(g_{i\bar{j}}) = (n+1)g_{i\bar{j}}$. If the Ricci tensor is proportional to the Kähler metric one calls the metric Kähler-Einstein.

9.3 Characteristic classes of holomorphic vector bundles

In the last section we encountered the holomorphic tangent bundle of M as an example of a holomorphic vector bundle E with a hermitian metric, which we call h_{ab} in the general case. The connection one form

$$A_k = (\partial_k h)h^{-1}, \qquad A_{\bar{k}} = 0 \tag{355}$$

defines the unique affine connection, which is compatible with the hermitian metric, i.e $\nabla h = 0$, and compatible with the complex structure. One defines the curvature two form as $F = \mathrm{d}A + A \wedge A$. The differential geometry approach to Chern classes $c_i(E) \in H^{2i}(M, \mathbf{R})$ of a rank r holomorphic vector bundle is to define them in terms symmetric function of the eigenvalues of the curvature form as

$$c(E) = \det(1 + \frac{i}{2\pi}F) = 1 + \sum_i c_i(E) = 1 + \frac{i}{2\pi}\mathrm{Tr}F + \ldots \tag{356}$$

and to prove then that they do not depend on the metric[22][190].

Topologically one can represent the Chern class c_k as the Poincaré dual to the degeneracy cycle

$$D_{r-k+1}(\sigma) = \{x : \sigma_1(x) \wedge \ldots \sigma_{r-k+1}(x) = 0\}, \tag{357}$$

where $r-k+1$ generic $\mathcal{C}^\infty$ sections σ_i of E become linearly dependent. This is described as Gauss Bonnet formula II in Chap 3.3 of [100], see also [80][111] for the approach using classifying spaces. The simplest example of the above dual descriptions arise for line bundles $\mathcal{L}$. Let $|\sigma|^2$ be a metric on a line bundle L, where σ is a section of L. Local trivialization of L are $\phi : L|_U \to U \times \mathbf{C}$, where s_U is a holomorphic function and $|\sigma|^2 = h(x)|s_U|^2$ for some function $h(x)$, which is positive if the metric is. The curvature 2-form given by

$$\mathcal{R} = -\bar\partial\partial \log h(x) \tag{358}$$

defines the Chern-class of L represented by $c_1(L) = \frac{i}{2\pi}[\mathcal{R}] \in H^2(M)$. This class is Poincaré dual to the divisor class $[D]$ which defines L and is uniquely recovered from L as the locus where the generic section vanishes. As a corollary the first Chern class of a holomorphic vector bundle is also the first Chern class of the determinant bundle $L_D = \wedge^r E$

$$c_1(E) = c_1(L_D)\ . \tag{359}$$

For the tangent bundle we identify the curvature 2-form F with $\Theta^j_{\bar i} = g^{j\bar p} R_{i\bar p k \bar l} \mathrm{d}x^k \wedge \mathrm{d}x^{\bar l}$ and get a representative for $c_1(TM)$ (which we also call $c_1(M)$)

$$c_1(M) = \frac{i}{2\pi}\Theta^i_i = R_{k\bar l}\mathrm{d}x^k \wedge \mathrm{d}x^{\bar l} = -\frac{i}{2\pi}\partial\bar\partial \log\det(g_{k\bar l})\ . \tag{360}$$

The canonical line bundle is the determinant line bundle of the holomorphic tangent bundle $K_M = \wedge^n T^{*\,1,0}M$. By (359) and (364) we have therefore

$$-2\pi c_1(K_M) := -2\pi c_1(\wedge^n T^{*\,1,0}M) = -2\pi c_1(T^*M) = 2\pi c_1(TM)\ . \tag{361}$$

Let us derive this also using as an explicit representative of the Chern class the curvature 2-form. Given an complex structure and a Kähler metric $g_{i\bar j}$ we have a connection on $T^{*\,1,0}M$ described by the holomorphic Christoffel symbols. This connection induces a connection on the line bundle K_M and a straightforward calculation shows on total antisymmetric forms $[\nabla_i, \nabla_{\bar j}]\omega_{i_1\ldots,i_n} = -R_{i\bar j}\omega_{i_1\ldots,i_n}$ Therefore we can identify $h(x)$ of (358) with $\det^{-1}(g_{i\bar j})$ and by (358) the first Chern class of K_M is

$$-2\pi c_1(K_M) = [\mathcal{R}] = 2\pi c_1(TM)\ . \tag{362}$$

If one uses the Poincaré Hopf theorem that the Euler number $\chi(M)$ of a manifold of dim n is given by the sum of indices of zeros of a generic vector field, i.e. a section of the tangent bundle, then by (357) the dual to $c_n(TM)$ is D_1. Counting these zeros leads then to the Gauss-Bonnet formula

$$\chi(M) = D_1 \cap M = \int_M c_n(TM)\ . \tag{363}$$

Let us discuss further properties of the Chern classes. By (356) one has $c_0(E) = 1$, $c_{k>r}(E) = 0$ and the Whitney product formula $c(E \oplus F) = c(E)C(F)$ from the properties of the determinant, see [26] for a proof from the topological definition. It is also easy to see[100] that

$$c_k(E^*) = (-1)^k c_k(E) \tag{364}$$

and $c_k(f(E)) = f^* c_k(E)$ for $f : M \to M'$ a differentiable mapping. A further important property is the *splitting principle* [26]. For an exact sequence of holomorphic vector bundles or sheaves one has $0 \to E \to$

$F \to G \to 0$ one has $c(F) = c(E)c(G)$. One considers often classes x_i such that $c(E) = \prod_{i=1}^r (1 + x_i)$ where x_i are Chern classes of line bundles. One reason that this is useful is that the *splitting principle* implies that if one wants to derive polynomial identities among Chern classes of vector bundles, one may replace the vector bundles by direct sums of line bundles. This opens up a calculational machinery with classes, which behave e.g. more natural on direct products as the *Chern character* $\mathrm{Ch}(E) = \sum_{i=1}^r e^{x_i}$. All expression are polynomial, defined by expanding up to degree r in x_i. Obviously $\mathrm{Ch}(E \oplus F) = \mathrm{Ch}(E) + \mathrm{Ch}(F)$ and $\mathrm{Ch}(E \otimes F) = \mathrm{Ch}(E)\mathrm{Ch}(F)$. A little playing with symmetric functions reveals $\mathrm{Ch}(E) = r + c_1 + \frac{1}{2}(c_1^2 - 2c_2) + \frac{1}{6}(c_1^3 - 3c_1c_2 + 3c_3) + \ldots$, where we set $c_k = c_k(E)$. Similar is the Todd genus defined $\mathrm{td}(E) = \prod_{i=1}^r \frac{x_i}{1-e^{x_i}} = 1 + \frac{1}{2}c_1 + \frac{1}{12}(c_1^2 + c_2) + \frac{1}{24}c_1c_2 + \ldots$. A central theorem is the Hirzebruch-Riemann-Roch formula, which gives the arithmetic genus $\chi(E) = \sum_k (-1)^k h^k(E)$ of a vector bundle over a manifold M, see [111] for the proof

$$\chi(E) = \int_M \mathrm{ch}(E) \wedge \mathrm{td}(TM) \,. \tag{365}$$

In sections 6.1,6.2 and 8.13 we needed applications of (365). Namely to count the deformation space (42) of a Riemann surface[49] Σ_g. As seen in section 8.2 the complex structure moduli of the metric are given by elements in the Čech cohomology group $H^1(T)$ with $T = T\Sigma$ and for $g > 1$ there are no conformal Killing vectors generating global diffeomorphims i.e. one has $h^0(T) = 0$. However for $g = 1$ the shift $z \to z + \lambda$ on the torus accounts for $h^0 = 1$ and for $g = 0$ the three generators of $PSL(2, \mathbf{C})$ $z \to \frac{az+c}{cz+d}$ on S^2 account for $h^0 = 3$. For a vector bundle V of rank $r$over the Riemann surface Σ the formula (365) gives

$$h^0(\Sigma, V) - h^1(\Sigma, V) = \int_\Sigma \mathrm{ch}(V) \wedge \mathrm{td}(T) = \int_\Sigma (r + c_1(V))(1 + \frac{1}{2}c_1(T)) = \int_\Sigma c_1(V) + r(1-g) \,. \tag{366}$$

The *virtual dimension* of the deformation space is obtained by setting $V = T$ with rank 1

$$\dim \mathcal{M}_g = h^1(T) - h^0(T) = -\int_\Sigma \mathrm{ch}(T) \wedge \mathrm{td}(T) = 3g - 3 \,. \tag{367}$$

In the integral over the metric moduli space in string amplitudes one sacrifice in the $g = 0, 1$ cases $h^0 = 3, 1$ additional parameters, the position of insertion points, to offset the negative contributions to (367) from the conformal Killing fields. Another application leads to the formula (115) describing the dimension of the deformation space of holomorphic maps $x : \Sigma \to M$. The movement of the curve in M is described infinitesimal by a vector field $x^i \to x^i + \epsilon \xi^i$ on M. The vector field must be holomorphic $\partial_{\bar z} \xi = 0$ so that the deformed map stays holomorphic. Also we are not counting vector fields which correspond to reparametrizations of Σ. That is we look at elements of $H^0_{\bar\partial}(\Sigma, x^*(TM)) = H^0(x^*(TM))$ and (365) gives us

$$h^0(x^*(TM)) - h^1(x^*(TM)) = \int_\Sigma (\dim_C M + x^*(c_1(TM)))(1 + \frac{1}{2}c_1(T)) = c_1(TM)\cdot\beta + \dim_C M(1-g) \,. \tag{368}$$

Generically the movement of the map is unobstructed and $H^1(x^*(TM)) = 0$. In the case the above is also the dimension of the deformation space. In the case of Calabi-Yau three folds we get for genus 0 that the dimension of the deformation space is 3. We can think about this in two ways. Either we don't

[49] This related by the Atiayh-Singer index formula to the index of the Dirac operator and hence to the ghost zero modes. An overview about index formulas for physicist can be found in [65] and the connections to the zero modes is in explained e.g. in [173].

fix points on S^2, then we have to mod out by the 3 dim automorphism group $PL(2,\mathbf{C})$ of S^2 and the expected dimension of the moduli space is 0. That is the way the corrections in $\mathcal{F}^{(0)}$ are interpreted. Or we kill $PL(2,\mathbf{C})$ by marking three points on the S^2 required to map into three divisors, which put three constraints and yields again a zero dimensional moduli space. That is the interpretation of corrections in $C_{ijk}(t)$.

Let us introduce for reference in the next section the *Pontrjagin classes* for real vector bundles V as the Chern class of the complexification of $V_{\mathbf{C}}$ of V [111]

$$p_k(V)=(-1)^k c_{2k}(V_{\mathbf{C}}) \tag{369}$$

The *Euler class* of the real vector rank r bundle V can now be defined as $e^2(V)=p_{\frac{r}{2}}(V)$. The Gauss-Bonnet formula, e.g. $\int_M e(TM)=\chi(M)$ fixes the sign. The Pontrjagin class of a complex vector bundle E is defined via the Pontrjagin of its *realization* $E_{\mathbf{R}}=E\oplus\bar{E}$ as $p_k(E)=(-i)^k c_{2k}(E_{\mathbf{R}})$. By the splitting principle and Whitneys formula [26] one gets $c_r(E)=e(E_{\mathbf{R}})$. The *A-roof or Dirac genus* is defined as symmetric polynomial in x_i^2 and can therefore be expressed in terms of the Pontrjagin classes $\hat{A}(E)=\prod_{j=1}^r \frac{x_j/2}{\sinh(x_j/2)}=1-\frac{1}{24}p_1+\frac{1}{5760}(7p_1^2-4p_2)+\ldots$. A usefull formula with applications to the Calabi-Yau tangent bundle is that $\mathrm{td}(E)=e^{c_1(E)}\hat{A}(E)$.

9.4 Axial anomaly

Let us consider the functional integral

$$Z_D(M)=\int \mathcal{D}\psi\mathcal{D}\bar{\psi}e^{-S_D}=\int \mathcal{D}\psi\mathcal{D}\bar{\psi}e^{-\int \mathrm{d}^{2n}x i\bar{\psi}D\psi} \tag{370}$$

for fermions on a manifold M with Dirac operator $iD=\begin{pmatrix}0 & \partial^\dagger\\ \partial & 0\end{pmatrix}$. A more detailed treatment of the following couple of paragraphs can be found in [162]. For M to admitt a spin structure it must be orientable $w_1(TM)=0$ and the second Stiefel-Whitney class[50] $w_2(TM)$ must vanish as well[180, 149] for review. We assume also even dimensionality to have a chiral decomposition of the spin representation $S=S^+\oplus S^-$ into two irreducible representation of dim 2^{n-1}, with the usual projector on the chiral sub bundles $P_\pm=\frac{1}{2}(\mathbf{1}\pm\gamma_5)$ with $\gamma_5=\begin{pmatrix}\mathbf{1} & 0\\ 0 & -\mathbf{1}\end{pmatrix}$.

One is interested in the axial or chiral $U(1)_A$ symmetry generated by infinitessimal transformations $\psi'(x)=(\mathbf{1}+i\epsilon(x)\gamma_5)\psi(x)$ and $\bar{\psi}'(x)=\bar{\psi}(x)(\mathbf{1}+i\epsilon(x)\gamma_5)$. By the usual Noether current argument the vanishing of the linear change $\int \mathrm{d}^{2n}\epsilon(x)\partial_\mu j_5^\mu$ of the action S_D under the chiral symmetry transformation implies classically the conservation of the axial current $\partial_\mu\bar{\psi}\gamma^\mu\gamma_5\psi=\partial_\mu j_5^\mu(x)=0$.

Following e.g. [77] it is easy to see at least at a formal level[51] how this fails due to the anomalous transformation of the measure. Let ψ_n an orthonormal eigen system $\langle\psi_k|\psi_l\rangle=\int \mathrm{d}^{2n}x\psi_k^\dagger\psi_l=\delta_{kl}$ of wave solutions to the Dirac operator. The Grassmann nature of $\psi=\sum_n a_n\psi_n$ and $\bar{\psi}=\sum_n b_n\bar{\psi}_n$ is captured in the Grassmann valuedness of the coefficients a_n, b_n and the path integral measure can be written as $\mathcal{D}\psi\mathcal{D}\bar{\psi}=\prod_n \mathrm{d}a_n\prod_n \mathrm{d}b_n$. The Jacobian of the infinitessimal transformation $\psi'(x)=(\mathbf{1}+i\epsilon(x)\gamma_5)\psi(x)=\sum_n a'_n\psi_n$ reads in the a_n parametrisation $a'_n=\langle\psi_n|\psi'\rangle=\langle\psi_n|1+i\epsilon(x)\gamma_5|\psi_m\rangle a_m=(\delta_{nm}+i\langle\psi_n|\epsilon(x)\gamma_5|\psi_m\rangle)a_m$. Using the fact that fermion measure transforms with the inverse Jacobian $\det(\delta_{nm}+i\langle\psi_n|\epsilon(x)\gamma_5|\psi_m\rangle)^{-1}=\exp(-\mathrm{Tr}\,\log(\delta_{nm}+i\epsilon(x)\langle\psi_n|\gamma_5|\psi_m\rangle))\sim\exp(-i\mathrm{Tr}(\langle\psi_n|\epsilon(x)\gamma_5|\psi_m\rangle))=$

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[50] That is the first Pontrjagin class (first Chern class for complex manifolds) must vanish modulo two. In this case all intersections in $H^2(M,\mathbf{Z})$ are even (Wu's Theorem).

[51] That means that we tacitly assume that there will be a suitable regularization of the infinite sums and products below. A discussion of the axial anomalies of 2d $U(1)$ gauge theories can also be found in Chap 19.1 of [171].

$\exp(-i\sum_n\langle\psi_n|\epsilon(x)\gamma_5|\psi_n\rangle)$ and performing the same argument for the b_n we see that the vanishing of the total change of the exponent of (370) linear in $\epsilon(x)$ implies an anomaly term in the j_5^μ current conservation

$$\partial_\mu j_5^\mu + 2iA(x) = 0, \qquad \text{with} \quad \int \mathrm{d}^{2n}x A(x) = \int \mathrm{d}^{2n}x \sum_n \psi_n^\dagger\gamma_5\psi_n = \sum_n \langle\psi_n|\gamma_5|\psi_n\rangle\ . \quad (371)$$

The quantity $A(x)$ is called the *anomaly density*. For the vector $U(1)_V$ symmetry the contribution of the a_n and b_n cancels. Now since iD is hermitian the eigenspaces spanned by $|\psi_n\rangle$ with $iD|\psi_n\rangle = \lambda_n|\psi_n\rangle$ are orthogonal to each other. On the other hand as $\{iD,\gamma_5\} = 0$ the eigenvalues of the states $|\psi_n\rangle$ and $\gamma_5|\psi_n\rangle$ are negatives of each other. Therefore the sum in $A(x)$ has only contributions from the zero modes $\lambda_l = 0$. With the γ_5 in the trace the total current violation evaluates to

$$\begin{aligned}\text{index } D^+ &= \textstyle\int \mathrm{d}^{2n}x A(x) = \#(+0 \text{ modes}) - \#(-0 \text{ modes}) \\ &= \dim\text{Ker}\,\partial - \dim\text{Ker}\,\partial^\dagger = \dim\text{Ker}\,\partial - \dim\text{coker}\,\partial\ ,\end{aligned} \quad (372)$$

where the last equality used that $\partial = D^+ = P^+D$ ($\partial^\dagger = P^-D$) is a Fredholm operator, i.e. kernel and cokernel are finite dimensional, and linear algebra. Nothing about the above principal setting will change if in addition to the spin connection we couple to a gauge bundle as well and consider $D = i\gamma^\alpha e^\mu_\alpha(\partial_\mu + \omega_\mu + A_\mu)$.

More importantly it is obvious that under smooth deformations away from singularties of the background geometry $\omega_\mu + A_\mu$, the spectrum of D^+ will change contineously and once an eigenvalue disappears from the kernel of D^+ it appears on the image of D^+ and hence disappears from the complement of the image (cokernel), see Fig. 37. As a difference one expects therefore the index only to change if we do something really violent to geometry. The precise nature of the topological quantity behind this expectation was found by Atiyah and Singer, as we review in the next chapter. Fig. 37 compares the deformation invariance argument in various disguises. Column one is familiar for Sec. 3.1. The second column showing the cricitical points of a Morse function is included for completness, a discussion can be found in Sec. 10.5 of [105]. The third- and the fourth column can really be made equivalent statements. E.g. for supersymmetric quantum mechanics on a target M, $Q\sim \mathrm{d}\sim D$, $\Delta\sim H$ and the index in both cases is (376).

If the index *does not* vanish we *do* have fermion zero modes and $Z_D(M)$ vanishes due to the Grassmann integration. If the index *does* vanish we don't know yet, since there could be zero modes in equal numbers. In this case we have to analyze the $h^i(E)$ of the Dirac complex described below. Still a topological question, but less protected against background changes. It will tell us what fermion zero modes the operators have to carry which we might wish to insert into $Z_D(M)$ to obtain a non-vanishing result.

9.5 Atiyah-Singer index theorem

The difference of the right hand side of () can be viewed as the *index* of an *elliptic complex* E of complex vector bundles $E^\pm = P\times S^\pm$ over M, where P is the principal $\mathrm{Spin}(2n)$ bundle. Atiyah and Singer [12] define the elliptic complexes in a wider context and obtain a generalisation of (365). As usual a complex E [12] is described by a sequence of maps $d_i : \Gamma(M,E^i)\to\Gamma(M,E^{i+1})$ given by pseudo differential operators d_i of order m with $d_{i+1}d_i = 0$.

$$0\to\Gamma(M,E^0)\xrightarrow{d_0}\Gamma(M,E^1)\xrightarrow{d_1}\dots\xrightarrow{d_{n-1}}\Gamma(M,E^{n-1})\to 0\ . \quad (373)$$

In local coordinates x of M and a local trivialisation of E with coordinates v_i $i = 1,\dots,r = \mathrm{rank}(E)$ we can write $d = A^{ij}_\kappa(x)D_\kappa$, where A is a $r\times r$ matrix and D_α is a differential operator of order m. The symbol of d denoted $\sigma(d)$ is obtained by the Fourier transform of the derivatives in d, i.e. replacing

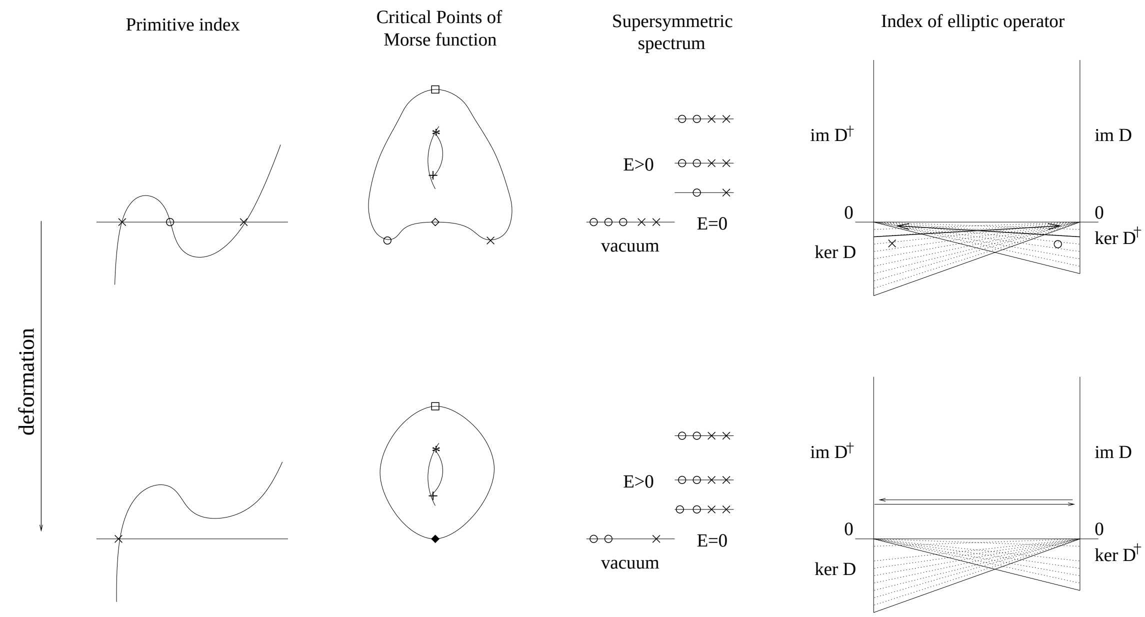

Fig. 37 Four variations of the idea of deformation invariance of indices

$-i\frac{\partial}{\partial x_i} \to p_i$. Then $(\underline{x}, \underline{p})$ are local coordinates of the bundle T^*M. The bundles $E^i \to M$ pull back under $\pi : TM \to M$ to bundles $\pi^* E^i \to T^*M$. The complex is *elliptic* if the symbol complex is exact for $\underline{p} \neq 0$

$$0 \to \pi^* E^0 \xrightarrow{\sigma(d_0)} \pi^* E^1 \xrightarrow{\sigma(d_1)} \ldots \xrightarrow{\sigma(d_{n-1})} \pi^* E^{n-1} \to 0 \,. \tag{374}$$

In particular if there are only two bundles it means that $\sigma(d)$ is invertible. According to [**?**] $H^i(E) = \mathrm{Ker} d_i / \mathrm{Im} d_{i+1}$ is finite dimensional and $\chi(E) = \sum (-1)^i h^i(E)$ exists. With a metric on E^i we can define an adjoint operator $d_i^* : \Gamma(M, E^{i+1}) \to \Gamma(M, E^i)$ and fold the elliptic complex with a single operator $D : \Gamma(\oplus_i E^{2i}) \to \Gamma(\oplus_i E^{2i+1})$, where $D = d_{2i} + d^*_{2i+1}$. Defining $D^* D = \oplus_i \Delta_{2i}$ and $DD^* = \oplus_i \Delta_{2i+1}$ with "Laplacian" $\Delta_i = d_{i-1} d^*_{i-1} + d^*_i d_i$ it is clear from (374) that $\sigma(\Delta_i) : \pi^* E^i \to \pi^* E^i$ is an isomorphism outside $\underline{p} = 0$ (the zero section of T^*M). It follows that Δ_i and D are operators of an elliptic complex and ker $D = \oplus_i H^{2i}(E)$ while coker $D = \oplus_i H^{2i+1}(E)$ so $\chi(E) =$ index D. One can generalize the proof for (365) in [111] to obtain [12]

$$\text{index } D = (-1)^n \int_M \frac{1}{e(TM)} \sum_p (-1)^p \mathrm{ch}(\mathrm{E^p}) \wedge \mathrm{td}(TM_{\mathbf{C}}) \,. \tag{375}$$

Examples:

- *De Rham complex:* If $E^i = \Lambda^i T^* M$ on an even $m = 2l$ dimensional manifold and $D = d$ is the exterior derivative, then using the relation of the Euler class to the top Chern class $e(TM) = \prod_{i=1}^l x_i(TM_{\mathbf{C}})$, see cff (369), we get

$$\text{index } d = \int_M e(M) = \chi(M) \,. \tag{376}$$

- *Dolbeault complex:* If $E^i = \Omega^{0,i}$ on a complex m dimensional manifold and $D = \bar\partial$ then

$$\text{index } \bar\partial = \sum_{k=1}^m (-1)^k h^{0,k} = \int_M \mathrm{td}(TM) \,. \tag{377}$$

is the *arithmetic genus*.

- *Twisted Dolbeault complex:* If $E^i = \Omega^{0,i} \otimes E$ with E a holomorphic vector bundle on a complex m dimensional manifols and $D = \bar{\partial}_V$ then

$$\text{index } \bar{\partial}_V = \chi(E) = \sum_k (-1)^k h^k(E) = \int_M \text{ch}(E)\text{td}(TM) \ . \tag{378}$$

is the Hirzebruch-Riemann-Roch formula.

- Spin complex: If E is the 2-complex $E^{\pm} = P \times S^{\pm}$ over a $2n$ dimensionalmanifold, where P is the principal $\text{Spin}(2n)$ bundle and $D^+ = P^+D$, with D is the Dirac operator coupled to the spin connection then

$$\text{index } D^+ = \int_M \hat{A}(TM) \ . \tag{379}$$

- *Twisted Spin complex:* If $E^{\pm}_E = E^{\pm} \times E$, where E is a gauge bundle $D^+_E = P^+D$, with connection A_μ and D is the Dirac operator coupled to the spin connection and E, i.e. $D = i\gamma^\alpha e^\mu_\alpha(\partial_\mu + \omega_\mu + A_\mu)$

$$\text{index } D^+_E = \int_M \hat{A}(TM)\text{ch}(E) \ . \tag{380}$$

- *bc system:* The following standard example from bosonic string theory [55][173] uses techinques of this and the last section. Let $T^n = T^{q-p}$ be a section of $(\otimes^q_{i=1} T\Sigma) \otimes (\otimes^p_{i=1} T^*\Sigma)$ over a Riemann surface and compare (340)

$$\begin{array}{llll} \nabla^z_n: & T^n \to T^{n+1}, & \nabla^z_n = h^{z\bar{z}}\partial_{\bar{z}}T, & (\nabla^z_n)^\dagger = -\nabla^{n+1}_z, \\ \nabla^n_z: & T^n \to T^{n-1}, & \nabla^n_z = (h^{z\bar{z}})^n\partial_z[(h_{z\bar{z}})^nT], & (\nabla^n_z)^\dagger = -\nabla^{n-1}_z, \end{array} \tag{381}$$

where the inner product is $\langle T_1, T_2\rangle = \int_\Sigma \text{d}^2z\sqrt{h}(h^{z\bar{z}})^n T^*_1 T_2$. In a conformal theory real traceless symmetric tensors transforming as a subbundle of $S^n = T^n \oplus T^{-n}$ are of special interest and of the form $\Phi = (\phi, (h_{z\bar{z}})^n\phi^*)$ with $\phi = \phi^{\overbrace{z,\cdots z}^{n}}$. One defines on them

$$\begin{array}{ll} P_n & = \nabla^z_n \oplus \nabla^{-n}_z : S^n \to S^{n+1} \\ P^\dagger_n & = -(\nabla^{n+1}_z \oplus \nabla^z_{-n-1}) : S^{n+1} \to S^n \ . \end{array} \tag{382}$$

where the inner product is= $\langle \Phi_1, \Phi_2\rangle \int_\Sigma \text{d}^2z\sqrt{h}(h^{z\bar{z}})^n(\phi^*_1\phi_2 + \phi^*_2\phi_1)$. Note that the choice of the metric is $h_{z\bar{z}} = h_{\bar{z}z} = \frac{1}{2}e^{2\sigma}$, $h^{z\bar{z}} = h^{\bar{z}z} = 2e^{-2\sigma}$ with vanishing pure components. P_1 above is as in (43). In particular that $b = (b^{zz}, (h_{z\bar{z}})^2 b^{\bar{z}\bar{z}}), c = (c^z, h_{z\bar{z}}c^{\bar{z}})$ system has the action $S = \frac{1}{\pi}\langle n, P_1 c\rangle = \frac{1}{\pi}\int \text{d}^2z(b_{zz}\partial_{\bar{z}}c^z + b_{\bar{z}\bar{z}}\partial_z c^{\bar{z}})$. We want to calculate the anomaly density of the $U(1)$ $c^z \to e^{-i\theta_z}c^z$, $c^{\bar{z}} \to e^{i\theta_{\bar{z}}}c^{\bar{z}}$, $b_{\bar{z}\bar{z}} \to e^{-i\theta_{\bar{z}}}b_{\bar{z}\bar{z}}$ and $b_{zz} \to e^{i\theta_{\bar{z}}}b_{zz}$ *ghost number current*. The Laplacians above become $\Delta_1 = P^\dagger_1 P_1$ and $\Delta_2 = P_1 P^\dagger_1$ with $\sigma(\delta_i) : \pi^*S^i \to \pi^*S^i$ and isomorphism outside the zero section. One expands $c = \sum_n c_n\psi_n$ and $b = \sum_n b_n\phi^n$ as eigenfunctions of $\Delta_{1/2}$ othonormal w.r.t. the inner product $\langle\Phi_1, \Phi_2\rangle$, repeats the Noether procedure as well as the analysis of the transformation of the fermionic measure as in (9.4). This exercise is made made explicite in [78] and one finds the anomalies of the ghost currents $j_z = b_{zz}c^z$ and $j_{\bar{z}} = b_{\bar{z}\bar{z}}c^{\bar{z}}$ is $\partial_{\bar{z}}j_z = \pi A(z,\bar{z})$ and $\partial_z j_{\bar{z}} = \pi A(z,\bar{z})$ with $\int_\Sigma A(z,\bar{z}) = \sum_n\langle\psi_n, \psi_n\rangle - \sum_m\langle\phi_m, \phi_m\rangle$. Again these sums contribute only if the eigen functions ψ_n of Δ_1 and ϕ_m of Δ_2 are zero modes. E.g. if $\Delta_1\psi_n = \lambda_n\psi_n$, $\lambda > 0$ then $\lambda_n(P_1\psi_n) = P_1\Delta_1\psi_n = \Delta_2(P_1\psi_n)$ is a eigenfunction of Δ_2, so the corresponding contributions to the sum cancel and the integral over the anomaly density is $\ker\Delta_1 - \ker\Delta_2 = \ker P_1 - \text{coker}P_1 = \text{indexP}_1 =$

$\mathrm{index}\nabla^z_1 + \mathrm{index}\nabla^{-1}_z = \frac{3}{2}\chi(\Sigma) + \frac{3}{2}\chi(\Sigma)$. Here we used in the last step (375) with $\mathrm{index}\nabla^{-1}_z = \mathrm{index}\nabla^z_1 = -\int_\Sigma \frac{\mathrm{ch}(T\Sigma)-\mathrm{ch}(T\Sigma\otimes T\Sigma)}{e(\Sigma)}\mathrm{Td}(T\Sigma_C)$ with $e(\Sigma) = c_1(T\Sigma)$ and the expressions of Sec. 9.3. Hence the anomaly density must be $A(z,\bar{z}) = \frac{3}{2\pi}\sqrt{h}R$ and the current anomaly in covariant form is

$$\partial_\mu j^\mu = 3\sqrt{h}R \tag{383}$$

A physics approach to proof (375 is to evaluate the anomaly density integral in (9.5) by a *heat kernel regularization*, see [79] for a review, with further references. For instance the calculation of the last example using the heat kernel i.e. without resorting to the index theorem is an exercise whose solution is found in Appendix B2 of [55]. Interesting are also the proofs by supersymmetric localisation [5] [75], very much in the spirit of Sec. 3.1.

9.6 Family indices

The key idea in Sec. 9.4 and 9.5 is to throw away details of the eigenvalue spectrum of D and concentrate on the roughest topological information, which is of course deformation independent. Trying to keep the full information is maybe overambitous at the current state of understanding and ingenuity is required to formulate addressable questions. A sucessfull stragegy is to throw away this time the zero modes and the take the determinate $\det' D$ of the rest of D.

9.7 Metric Connection and Holonomy

To describe spinor connection on curved spaces one introduces beside the curved indices $M, N, \ldots$ the flat tangent indices $A, B, \ldots$ which are lowered and raised with the flat metric $\eta^{AB} = \mathrm{diag}(-1, \underbrace{1, \ldots, 1}_{D-1})$ and its inverse.

The Clifford algebra is defined by the anti commutator of $\{\Gamma^A, \Gamma^B\} = 2\eta^{AB}$. In the smallest representation the Γ symbols are $2^{[D/2]} \times 2^{[D/2]}$ matrices. The generators of the Lorentz group in the spinor representation ξ of dimension $2^{[D/2]}$ are given by the commutator $T^s_{AB} = -\frac{i}{2}\Gamma_{AB} = -\frac{i}{4}[\Gamma_A, \Gamma_B]$, i.e. $\xi \mapsto \exp(i\omega^{AB}T^s_{AB})\xi$ under the spin group which is a cover of proper, ortochronous Lorentzgroup $SO^+_\uparrow(1, D-1)$. We do not display spinor indices $a, b \ldots$ like in $\tilde{\xi}_a = (\Gamma^A)_a^b\xi_b$, $a, b = 1, \ldots, [D/2]$ explicitly. For more on spin representations in various dimensions, see e.g. [173].

The relation to curved indices $M, N \ldots$, lowered and raised by the curved metric G_{MN} and its inverse G^{MN}, is provide by the D-bein e^A_M and its inverse e^N_B ($e^A_M e^N_A = \delta^N_M$ and $e^A_M e^M_B = \delta^A_B$) which fulfills $G_{MN} = e^A_M e^B_N \eta_{AB}$. One has $\Gamma^A = e^A_M \Gamma^M$ and $\Gamma^M = e^M_A \Gamma^A$ etc., from which follows $\{\Gamma^M, \Gamma^N\} = 2G^{MN}$. A torsion free $\Gamma^P_{MN} = \Gamma^P_{NM}$ Riemann connection leaves the metric invariant

$$\nabla_S G_{MN} = 0 = \partial_S G_{MN} - \Gamma^P_{SM} G_{PN} - \Gamma^L_{SN} G_{PM} \tag{384}$$

which implies the formula for the Christoffel Symbols

$$\Gamma^S_{MN} = \frac{1}{2} G^{SP} \left(\partial_M G_{PN} + \partial_N G_{MP} - \partial_P G_{MN}\right) \ . \tag{385}$$

The *spin connection* ω^A_{MB} is defined as

$$\nabla_M e^A_N = \partial_M e^A_N - \Gamma^P_{MN} e^A_P + \omega^A_{NB} e^B_M, \tag{386}$$

which implies that

$$\omega^{AB}_M = \frac{1}{2}(\Omega_{MNR} - \Omega_{NRM} + \Omega_{RMN}) e^{NA} e^{RB}, \qquad \text{with} \qquad \Omega_{MNR} = (\partial_M e^A_N - \partial_N e^A_M) e_{AR}$$

(387)

The connection on a spinor is then

$$\partial_M \xi = (\partial_M + \frac{i}{2}\omega_M^{AB} T^s_{AB})\xi \tag{388}$$

and for any other representation carrying only flat indices of the tangent space one has to replace T^s_{AB} by the appropriate generator of the Lorentz group, i.e. $T^v_{AB} = \eta_{AC}\delta^D_B - \eta_{BC}\delta^D_A$ for vectors etc.

If a vector V^N is parallel transported around a infinitesimal rectangle along two tangent vectors $\frac{\partial}{\partial X_A}$ and $\frac{\partial}{\partial X_B}$ with area element $\sigma^{AB} = -\sigma^{BA}$ its infinitesimal rotation is $\delta V^L = -\frac{1}{2}\delta\sigma^{MN} R^L_{MN\ \ P} V^P$, which is one way to explain the effect of curvature

$$[\nabla_M, \nabla_N] V_P = -R_{MNP}{}^S V_S, \text{ with } R_{MNP}{}^S = \partial_M \Gamma^S_{NP} - \partial_N \Gamma^A_{MP} + \Gamma^B_{NP}\Gamma^S_{MB} - \Gamma^B_{MP}\Gamma^S_{NB}\ . \tag{389}$$

Note $R^M_{NOP} = -R^M_{NPO}$ and also for a Kähler manifold the only non vanishing elements of $R^k_{i\bar{j}\ \ l}$ is pure in k, l. That means that a holomorphic vector stays holomorphic under parallel transport and $\delta\sigma^{mn} R^k_{mn\ \ l}$ spans the Lie algebra of $U(n)$. Near the identity $U(n) \cong SU(n) \times U(1)$ and the $U(1)$ part is generated by the trace part of the Riemann tensor which is the Ricci tensor $\delta\sigma^{mn} R^k_{mn\ \ k} = -4\delta\sigma^{\mu\bar{\nu}} R_{\mu\bar{\nu}}$.

Once one knows the holonomy group Hol on vectors the transformation properties of tensors, forms and spinors becomes a matter of representation theory. In particular the following holds see e.g. [120]. If Hol is the holonomy group of a connection ∇ on TM on a *simply connected* manifold M then a tensor section $S \in \bigotimes^i TM \otimes \bigotimes^j T^*M$ is covariantly constant (parallel) iff $S|_{x_0}$ is locally fixed by Hol.

The restriction to simply connected is quite important. Non simply connected manifolds can have monodromy even if they are flat. Consider e.g. the easy example of a non-simply connected space which is topological $M = S^1 \times \mathbf{R}^2$ with the metric

$$\mathrm{d}^2 s = R^2 \mathrm{d}^2\theta + (\mathrm{d}x^i + T^i_j x^j \mathrm{d}\theta)^2\ , \tag{390}$$

where $T = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ is the generator of $SO(2)$ rotations in $\mathbf{R}^2$. M is flat, jet a vector parallel transported around the S^1 gets rotated in the $\mathbf{R}^2$ directions. Similar examples a flat connections on tori, with monodromy. In the case of a gauge connection we call such configurations Wilson lines.

9.8 Calabi-Yau manifolds

A *general Calabi-Yau manifold* is a compact Kähler manifold M with vanishing first Chern class $c_1(TM) = 0$. The following statements are essentially equivalent for complex n dimensional Kähler manifolds M, up to some important subtleties for non-simply connected cases, which we discuss below. Together with the Kähler property they are used to define a (general) Calabi-Yau manifold

- a) The canonical class is trivial.
- b) The first Chern class of the tangent bundle vanishes[52] $c_1(TM) = 0$.
- c) It exists a Kähler metric g whose Ricci tensor vanishes $R_{i\bar{j}}(g) = 0$.
- d) There exists an up to a constant unique nowhere vanishing holomorphic $(n, 0)$ form Ω.
- e) The holonomy group Hol of M is a subgroup of $SU(n)$.

[52] We assume that we have a connection without torsion on TM.

- f) M admits a pair of globally defined covariantly constant (parallel) spinors ξ and $\bar{\xi}$ of opposite chirality if n is odd and of the same chirality if n is even.

Complex tori of all dimensions are general Calabi-Yau manifolds with trivial holonomy. In $dim_{\mathbf{C}} = 1$ the torus is the only topological type of a Calabi-Yau manifold. In $dim_{\mathbf{C}} = 2$ the $K3$-surface is the only topological type of a Calabi-Yau manifold with $G = SU(2)$, while in $dim_{\mathbf{C}} = 3$ the number different topological types of Calabi-Yau manifolds is $> 10^5$. This estimate comes from explicit construction mostly of hypersurface and complete intersections in toric ambient spaces, see also Sec. 9.10.

In physical applications one is mainly interested in how many super symmetries are unbroken in compactifications to four dimensions. An important situation is when the number of supercharges is reduced by $1/4$ by a compactification of the ten dimensional supergravity on the six real dimensional internal manifold M. This is the case if ξ and $\bar{\xi}$ are the only covariantly constant spinors [35]. This in turn holds generically, without further non-trivial background fields, if $\mathrm{Hol} = SU(3)$ and in an interesting special case, namley the $T^1_{\mathbf{C}} \times K3/Z_2$ FHSV model with [70] $\mathrm{Hol} = SU(2) \times Z_2$. Important applications emerging from this scheme are the 10d heterotic compactification, which leads to $N = 1$ supersymmetry in 4d and the 10d type II compactifications, which lead to $N = 2$ supersymmetry in 4d. This $\frac{1}{4}$ susy scheme with exactly two spinors excludes cases involving non-simply connected manifolds such as $T^3_{\mathbf{C}}$ and $T^1_{\mathbf{C}} \times K3$ and other products e.g. $K3 \times K3$. On non-simply connected manifolds the relation between c.) and d.) is more subtle as they can have flat metrics, which do have non-trivial holonomy. They lead to interesting supersymmetry reduction by what is called generalized Scherk-Schwarz mechanism or geometrical Wilson lines [153]. Other interesting examples for conceptual questions are compactification of type IIA or IIB to 6d on $K3$, which has $\mathrm{Hol} = SU(2)$. This reduces the number of supercharges by $1/2$ and leads to $(1,1)$ and $(2,0)$ supersymmetry in $6d$ respectively. A phenomenological very interesting compactifiction with $N = 1$ in 4d is F-theory compactification on an elliptically fibred Kähler manifold with $\mathrm{Hol} = SU(4)$.

From the string point of view the important condition is the vanishing of the first Chern class $c_1(TM) = 0$, which would have to be supplemented by the simply connectedness to restrict to the $\frac{1}{4}$ super symmetry scheme. The first reason is that this is the sufficient condition for the unbroken axial $U(1)$ on the world-sheet, necessary to define the B-twist. More importantly it is known that the non-linear σ-model is not conformally invariant for the Ricci-flat metric. The four loop β-function does not vanish in this geometry [102]. However it has be shown in [164][118] by analyzing the form of the possible counter terms that the total perturbative β-function can be set to zero by a change in the metric so that $\log\det g^{string}_{i\bar{\jmath}} = \log\det g^{Rflat}_{i\bar{\jmath}} + \alpha(x,\bar{x})$, where $\alpha(x,\bar{x})$ is a globally defined real function on M, which is not the absolute square $|f(x)|^2$ of a holomorphic $f(x)$. By (341) this implies that the curvature two form becomes non nonzero, but the first Chern class stays trivial $c_1(TM) = 0$. Ricci-flat manifolds are not vacuum solutions of string theory. One may wonder whether the considerations about the covariantly constant spinors ξ, $\bar{\xi}$ make sense. They do, because what is required is that $(\nabla_m - \frac{i}{2}A_m)\xi = (\nabla_m + \frac{i}{2}A_m)\bar{\xi}$ is zero, where A is a form potential for the Ricci-form $\mathcal{R} = \mathrm{d}A$, where $\bar{\partial}_{\bar{\imath}}\alpha = A_{\bar{\imath}}$ and $\partial_i \alpha = A_i$.

On a Calabi-Yau manifold on has two important forms. The Kähler form ω and the $(n,0)$ form Ω. They are linked by the fact that $\Omega \wedge \bar{\Omega}$ is proportional to the volume form and there is a natural normalization which makes $\mathrm{Re}\Omega$ a calibration

$$\frac{\omega^n}{n!} = (-1)^{\frac{n(n-1)}{2}} \left(\frac{i}{2}\right)^n \Omega \wedge \bar{\Omega}\,. \tag{391}$$

Imposing (391) reduces the freedom in the constant in e.) to a phase [120].

Let us now discuss the relation between the statements a.) to f.). In order to connect a.)-d.) to e.) and f.) we will assume that M is simply connected and not of product form.

a.) $\leftrightarrow$ b.) follows from (361).

c.) $\rightarrow$ b.) is a simple consequence of the independence of the Chern classes on the choice of the Kähler metric. Once one knows that there exists a Ricci-flat metric clearly $c_1(TM) = 0$ and that holds for all Kähler metrics.

b.) $\rightarrow$ c.) is a corollary to Yau' theorem [214], which proves the conjecture that E. Calabi formulated in (1956). It states that given the data

- (C.a) a Kähler metric g,a Kähler form ω, a Ricci form $\mathcal{R}$ on M and a real closed $(1,1)$ form $\mathcal{R}'$, which represents the Chern class $[\mathcal{R}] = [\mathcal{R}'] = 2\pi c_1(TM)$

one can construct

- (C.b) a unique metric g' on M with associated Kähler form ω' such that $[\omega'] = [\omega] \in H^2(M, \mathbf{R})$ and the Ricci form of g' is $\mathcal{R}'$.

In particular $c_1(TM) = 0$ can be represented by $\mathcal{R}' \equiv 0$ and then according to the above there exists a unique metric g' whose Ricci form is $\mathcal{R}'$. Therefore its Ricci tensor vanishes.

One can formulate simpler equivalent versions of (C.a) and (C.b) as requirements on the existence of functions on M as follows. $\mathcal{R} - \mathcal{R}'$ is a $\bar{\partial}$ exact and d closed real $(1,1)$ form. By the $\partial, \bar{\partial}$ Lemma one has a real function f on M so that $\mathcal{R} - \mathcal{R}' = i\partial\bar{\partial} f$ up to a constant κ. Recalling (340) how $\mathcal{R}$ is derived from the positive function multiplying $w^1 \wedge \ldots \wedge w^{2n}$ in (337), which is itself determined by $\frac{\omega^n}{n!}$, we conclude that f must make its appearance also in $e^f \omega^n = (\omega')^n$. In fact the constant κ can be fixed by normalizing the volume $\int_M e^f \omega^n = \int_M \omega^n$. The simplification is that instead of requiring g' to lead to a prescribed $\mathcal{R}'$ one requires that it leads to a prescribed volume form and the statement about $\mathcal{R}$ and $\mathcal{R}'$ can be replaced by a statement about f. Similarly one can formulate the $[\omega'] = [\omega]$ condition in (C.b) as a search for a real function ϕ as in (339). ϕ can be made unique by requiring $\int_M \phi \mathrm{vol}_g = 0$. So the simplified version of (C.a) and (C.b) is

- (C'.a) that for every given Kähler metric g, Kähler form ω and a real smooth function f on M with $\int_M e^f \omega^n = \int_M \omega^n$

one can construct

- (C'.b) a unique smooth real function ϕ on M such that (i) $\omega + i\partial\bar{\partial}\phi$ is a positive $(1,1)$ form ω', (ii) $\int_M \phi \mathrm{vol}_g = 0$ and (iii) $(\omega + i\partial\bar{\partial}\phi)^n = e^f \omega^n$.

Yau proved that the non-linear p.d.e (iii) on ϕ admits a unique solution which fulfills (i) and (ii). This is an existence proof and up to date no explicit solutions for ϕ and[53] e.g. the Ricci-flat metric on any compact Calabi-Yau manifold has been given.

c.) $\rightarrow$ e.) at the end of Sec. 9.7 we argued that the holonomy group of a Kähler manifold is generically $U(n)$. Moreover wee saw that the Ricci-tensor is generating the $U(1)$ part of $U(n) \cong SU(n) \times U(1)$. On a Ricci-flat manifold this part is not generated and the holonomy is reduced to $SU(n)$.

e.) $\rightarrow$ d.) An $(n,0)$-form can always locally written as $\Omega_{i_1,\ldots,i_n} = f(x)\epsilon_{i_1,\ldots i_n}$. It is therefore in the total antisymmetric representation of the holonomy group $SU(n)$, i.e. a singlet invariant under Hol. By the fact quoted in the last paragraph of Sec. 9.7 one has that $\nabla\Omega = 0$. Since Γ has no mixed indices $\bar{\partial}_{\bar{i}}\Omega = \nabla_{\bar{i}}\Omega = 0$ and Ω is holomorphic. This implies that $f(x)$ has to be a globally defined holomorphic holomorphic function over the compact manifold M and hence a constant. Note that ω, locally written as $\omega = \frac{i}{2}(\mathrm{d}x^1 \wedge \mathrm{d}x^{\bar{1}} \wedge \ldots \wedge \mathrm{d}x^n \wedge \mathrm{d}x^{\bar{n}}$, and g, locally written $g = \sum_{i=1}^n |\mathrm{d}x^i|^2$, are also covariantly constant. The normalization (391) established at a point requires $|f| = 1$, but is since all quantities are covariantly constant (391) will hold at any point.

Ω is also harmonic $\Delta_{\bar{\partial}}\Omega = 0$ as beside $\bar{\partial}\Omega = 0$ also $\bar{\partial}^*\Omega = - * \partial * \Omega = 0$, because $* : A^{n,0} \rightarrow A^{n,0}$ and $\partial : A^{n,0} \rightarrow A^{n+1,0} = \{0\}$.

d.) $\rightarrow$ a.) We just constructed with Ω a trivial constant section of the canonical bundle $\wedge^n T^{*1,0} M$.

[53] It is not that diffi cult to fi nd a K¨ahler metric on a Calabi-Yau manifold, e.g. by constructing the induced metric of the Fubini-Study metric on the quintic in $\mathbf{P}^4$, see [190].

d) $\to$ b): Assume a nowhere vanishing holomorphic $(n,0)$ exists. We get then a globally well defined scalar function

$$|\Omega|^2 = \frac{1}{n!}\Omega_{i_1\ldots i_n}\bar\Omega^{i_1\cdots i_n} \ , \tag{392}$$

where the indices are raised by the hermitian metric $g^{i\bar\jmath}$. Locally Ω is given by $\Omega_{i_1,\ldots,i_n} = f(x)\epsilon_{i_1,\ldots i_n}$, where $f(x)$ is a non-vanishing holomorphic function in each patch. We can obtain $\bar\Omega^{i_1,\cdots i_n} = \frac{\bar f}{g}\epsilon^{i_1\cdots i_n}$ and it follows that $g = \det(g_{i\bar\jmath}) = \frac{|f|^2}{||\Omega||^2}$. Inserting in (360) we get $c_1(TM) = -\frac{i}{2\pi}\partial\bar\partial\log|\Omega|^2$ which is exact since $\log|\Omega|^2$ is a scalar, hence $c_1(TM)=0$ in cohomology.

f.) $\leftrightarrow$ d.) is proven in generality in [198]. This is done using representation theory. Let us just give a simple relevant example namely the threefold case, $n=3$. We must figure out how many spinors transforming as singlets under the holonomy $SU(3)$. Under generic rotations in the internal 6d space vectors transform by $SO(6)$ and the associated spin group with the same Lie algebra is isomorphic to $SU(4)$. The spinor representation in $6d$ is $2^{\frac{6}{2}} = 8$ dimensional and splits according to the chirality into representations $(\mathbf{4},\bar{\mathbf{4}})$ of this $SU(4)$. Now the holonomy is reduced to $SU(3)$ and embedding the $SU(3)$ in $SU(4)$ singles out an $U(1)$, i.e. one has $SU(3)\otimes U(1)\in SU(4)$. The decomposition of the $(\mathbf{4},\bar{\mathbf{4}})$ into the representations of this $U(1)$ and $SU(3)$ is unique $(\mathbf{4},\bar{\mathbf{4}}) = (\mathbf{3}^1\otimes\mathbf{1}^{-3},\bar{\mathbf{3}}^{-1}\otimes\mathbf{1}^3)$, where the superscripts are the $U(1)$-charges. Hence we can conclude that there are indeed one invariant and therefore covariantly constant spinor of each helicity. Bilinears of the covariantly constant spinors can be used to build the covariantly constant tensors discussed above. In particular the almost complex structure as $J^a_b = -i\xi^\dagger\Gamma^a_b\xi$, the metric as $g_{\mu\bar\nu} = i\xi^\dagger\Gamma_{\mu,\bar\nu}\xi$ and the $(3,0)$ form as $\Omega_{ijk} = e^{-i\alpha}\xi^T\Gamma_{ijk}\xi$. In this way one can show f.) $\to$ d.) see [32] for details. Furthermore it is easy to see that the eight spinors can be generated from $\xi\in\mathbf{1}^{-3}$ as $\Gamma_i\xi\in\bar{\mathbf{3}}^{-1}$, $\Gamma_{ij}\xi\in\mathbf{3}^1$, $\Gamma_{ijk}\xi\in\mathbf{1}^3$ and decomposed as

$$\eta = \Omega^{0,0}\xi + \Omega^{0,1}_{\bar\imath}\Gamma^{\bar\imath}\xi + \Omega^{0,2}_{\bar\imath\bar\jmath}\Gamma^{\bar\imath\bar\jmath}\xi + \Omega^{0,3}_{\bar\imath\bar\jmath\bar k}\Gamma^{\bar\imath\bar\jmath\bar k}\xi, \quad \text{where } \Omega^{0,n}_{\bar\imath_1\ldots\bar\imath_r}\mathrm{d}x^{\bar\imath_1}\wedge\mathrm{d}^{\bar\imath_r}\in H^{0,r}_{\bar\partial}(M) \ . \tag{393}$$

On $T^3_{\mathbf{C}}$ one has therefore eight covariant constant spinors and on $T^1_{\mathbf{C}}\times K3$ four.

A very general tool in Čech cohomology is Serre duality which states for any sheaf E on M that

$$H^k(E)^* \cong H^{n-k}(E^*\otimes K_M) \ . \tag{394}$$

Using the Čech-Dolbeault isomorphism $H^k(E)\cong H^k_{\bar\partial}(M,E)$, $H^r(M,\wedge^sT^*M) = H^{s,r}(M)$ and $K_M = \mathcal{O}_M$ we relate on a Calabi-Yau manifold the cohomology groups $H^{0,r}(M)\cong H^{0,n-r}(M)$ by taking $E=\mathcal{O}(M)$ or by complex conjugation the cohomology goups $H^{r,0}(M)\cong H^{n-r,0}(M)$. This particular result can be seen also in a more direct way by contracting a $(p,0)$ form $\omega_{i_1\ldots i_p}\mathrm{d}x^{i_1}\wedge\ldots\wedge\mathrm{d}x^{i_p}$ with the unique $(0,n)$ form to define a $(0,n-p)$-form $\hat\omega_{\bar\jmath_{p+1}\cdots\bar\jmath_n} = \frac{1}{p!}\bar\Omega_{\bar\jmath_1\cdots\bar\jmath_n}\omega^{\bar\jmath_1\cdots\bar\jmath_p}$. One shows easily that this is an invertible map that commutes with Δ, i.e. $H^{p,0}(M)\cong H^{0,n-p}(M)\cong H^{n-p,0}(M)$. As an exercise use the index theorem (365) to argue that $h^{1,0}=h^{2,0}$ on a Calabi-Yau 3-fold.

With $h^{n,0}(M) = h^{0,0} = 1$ e.q. (393) implies that one has at least two covariantly constant spinors on a Ricci-flat manifold. In order to show that one has only this two on a manifold with $\mathrm{Hol} = SU(n)$ we shall show that $h^{p,0}=0$ for $0<p<n$. On a compact Kähler manifold harmonicity of $(p,0)$-form implies holomorphicity as argued after (348) by consideration of type. Specializing (352) to $R_{ij\bar k\bar l}=0$ for Kähler- and $R_{i\bar\jmath}=0$ for Ricci-flat manifolds harmonicity means $\nabla^\nu\nabla_\nu\omega_{i_1\ldots i_p}=0$. On a compact manifold one can use pairing and partial integration to see that this requires $\nabla_j\omega_{i_1\ldots i_p}=0$ (and also $\bar\partial\omega=0$). From these equations we conclude that all harmonic $(p,0)$ forms are covariantly constant. However that would mean that they are invariant under $SU(n)$, which is impossible for $0<p<n$ as only the trivial and the total antisymmetric representation are invariant.

9.9 Bergers List

Let us finally show here Bergers list of the possible holonomy groups on simply connected irreducible and non-symmetric manifolds of real dimension m with some additional information about the properties of the metric and the number N_+, N_- of complex covariant constant spinors with positive and negative chirality [198] respectively. If m is odd the spinor representation is irreducible and we have just one type of spinor. The last part comments on the special forms that exist on this manifold. See [120, 104, 198] for more background.

- (i) $\mathrm{Hol}(g) = SO(m)$,*generic oriented manifold*, not nec. spin.
- (ii) $m = 2n$ with $m \geq 2$: $\mathrm{Hol}(g) = U(n)$, *K¨ahler manifold, K¨ahler*, not nec. spin; ω $(1,1)$ Kähler form.
- (iii) $m = 2n$, $n \geq 2$: $\mathrm{Hol}(g) = SU(n)$,*Calabi-Yau manifold, Ricci-flat, K¨ahler*, $N_\pm = 1$ for n odd, $N_+ = 2$ for n even; ω $(1,1)$ Kähler form and Ω $(n,0)$ holomorphic form.
- (iv) $m = 4n$, $n \geq 2$: $\mathrm{Hol}(g) = Sp(n)$, *Hyperk¨ahler manifold, Ricci-flat, K¨ahler*, $N_+ = m+1$; H, I, J $SU(2)$ triplet of $(1,1)$ forms.
- (v) $m = 4n$, $n \geq 2$: $\mathrm{Hol}(g) = Sp(n)Sp(1)$, *Quaternionic K¨ahler manifold, Einstein, not Ricci-flat, not K¨ahler* .
- (vi) $m = 7$: $\mathrm{Hol}(g) = G_2$, *G2-manifold, Ricci-flat*, $N = 1$; Φ associative 3-form, $*\Phi$ coassociative 4-form.
- (vii) $m = 8$: $\mathrm{Hol}(g) = Spin(7)$,*Spin(7) manifold, Ricci-flat*, $N_- = 1$; Ψ Cayley 4-form.

9.10 Examples of Calabi-Yau spaces

The tool that makes constructing of Calabi-Yau spaces easy is the perfect control over the first Chern class in algebraic geometry. As an application of some statements in Sec. 9.3 we want to calculate the first Chern class of $\mathbf{P}^n$, following [26]. As every projective space $\mathbf{P}^n$ has a tautological sequence

$$0 \to H^* \to \mathbf{P}^n \times \mathbf{C}^{n+1} \to Q \to 0 \,. \tag{395}$$

$H^* = \{(l,x) \in \mathbf{P}^n \times \mathbf{C}^{n+1} | x \in \hat{l}\}$, where $\hat{l}$ is the line in $\mathbf{C}^{n+1}$, which defines l as point in $\mathbf{P}^n$, and the quotient space Q is defined by (395). H^* is parametrized by the homogeneous variables $[x_1 : \ldots : x_{n+1}]$, which, as maps to $\mathbf{C}$, are section of the dual space H, called the hyperplane bundle. We can write tangent vectors in $T\mathbf{P}^n$ as linear combinations of $(\sum_{k=1}^{n+1} a_k^i x_k)\frac{\partial}{\partial x_i}$, which is scaling invariant under the $\mathbf{C}^*$ action and maps $H^{\oplus(n+1)}$ to $T\mathbf{P}^n$. There is a kernel $\mathbf{C}$ of that map, namely we have $\sum x_i \frac{\partial}{\partial x_i} = 0 \in T\mathbf{P}^n$ as it just generates the scaling action. These facts are expressed in the Euler sequence

$$0 \to \mathbf{C} \to H^{\oplus(n+1)} \to T\mathbf{P}^n \to 0 \,. \tag{396}$$

The Chern class of $\mathbf{C}$ is 1 and the Whitney formula and (trivial) splitting principle gives

$$c(T\mathbf{P}^n) = (1+x)^{n+1} \,, \tag{397}$$

where we denoted $x = c_1(H)$.

A weighted projective space WCP^n is defined similarly as $\mathbf{P}^n$ cff. (326), only that $\mathbf{C}^*$ acts now by

$$(x_1, \ldots, x_{n+1}) \sim (\lambda^{w_1} x_1, \ldots, \lambda^{w_{n+1}} x_{n+1}) \,, \tag{398}$$

where the integral weights w_i contain no common factor. Common factors k in subsets of the weights lead to Z_k quotient singularities of WCP^n. A similar argument as before shows that [63]

$$c(TWCP^n) = \prod_{i=1}^{n+1} (1 + w_i x) \ , \tag{399}$$

All weights are in $\mathbf{Z}$ and order to be compact $w_i > 0$. This prevents us to define compact WCP with $c_1(TWCP^n) = 0$, but $WCP(-2, 1, 1)$ is a well known example of a non-compact Calabi-Yau two manifold, better know as $\mathcal{O}(-2)$ line bundle over $\mathbf{P}^1$ called $\mathcal{O}(-2) \to \mathbf{P}^1$. The notation $\mathcal{O}(n) \to \mathbf{P}^1$ means the following. If we introduce local coordinates on $\mathbf{P}^1$, i.e. according to (327) $x^{(1)} = x_2/x_1$ in $\mathcal{U}^{(1)}$ and $x^{(2)} = x_1/x_2 = 1/x^{(1)}$ in $\mathcal{U}^{(2)}$, we have local coordinates $(l^{(i)}, x^{(i)})$ on $\mathcal{O}(n) \to \mathbf{P}^1$ with the transition function

$$(l^{(2)}, x^{(2)}) = \left(\frac{l^{(1)}}{(x^{(1)})^n}, \frac{1}{x^{(1)}} \right) \ . \tag{400}$$

$\mathcal{O}(-2)$ can be viewed as the cotangent bundle over $\mathbf{P}^1$ parametrized by $l\mathrm{d}x$ and $\Omega = \mathrm{d}l \wedge \mathrm{d}x$ is a non-vanishing $(2, 0)$ form. Note that $c_1(\mathcal{O}(n) = nH$.

Compact examples as easily obtained, e.g. as hypersurfaces in the projective spaces above. Let us consider a smooth degree d hypersurface M in $\mathbf{P}^n$. M is defined as zero locus of a degree d polynomial P, which is sufficiently general so that $P = 0$ and $\mathrm{d}P = 0$ has no common solution. It is a section of $H^d = \mathcal{O}_{\mathbf{P}^n}(d)$. Since P is smooth we have a splitting of the tangent bundle $T\mathbf{P}^n$ as follows

$$0 \to TM \to T\mathbf{P}^n|_M \to N_M \to 0 \ , \tag{401}$$

where N_M is the normal bundle to M, which is identified with $\mathcal{O}(d)|_M$ because P is a coordinate of N near M. $\mathrm{Ch}(H^d) = e^{dx} = 1 + c_1(H^d) = 1 + dx$, i.e. $c_1(H^d) = dx$ and the adjunction formula gives

$$c(M) = \frac{(1 + x)^{n+1}}{(1 + dx)} = 1 + (n + 1 - d)x + \ldots, \tag{402}$$

i.e. a Calabi-Yau hypersurface in $\mathbf{P}^n$ has to have degree $d = n + 1$. In this case P is a section $\mathcal{O}(K_{\mathbf{P}^n})$ of the canonical line bundle $K = -[c_1(\mathbf{P}^n)]$. This gives in for dimension three one case, the quintic in $\mathbf{P}^4$. For weighted projective spaces one has

$$c(M) = \frac{\prod_{i=1}^{n+1}(1 + w_i x)}{(1 + dx)} = 1 - (d - \sum_i w_i)x + \ldots, \tag{403}$$

where the degree d of a quasihomogeneous polynomial P is defined by the scaling $P(\lambda_1^w x_1, \ldots, \lambda^{w_{n+1}} x_{n+1}) = \lambda^d P(x_1, \ldots, x_{n+1})$. Together with the transversality condition $\mathrm{d}P = 0$ at $P = 0$ it leads 7555 examples of Calabi-Yau threefolds [136]. This sample contains many mirror pairs.

This in turn has a fairly obvious generalization to hypersurfaces (and complete intersections), which live over coordinate ring of a general toric variety defined by (157,158). In this context Batyrev provided a systematic construction of mirror pairs, as sections $M = \mathcal{O}(K_{(\mathbf{P}_\Delta)})$ and $W = \mathcal{O}(K_{\mathbf{P}(\Delta*)})$ respectively[15]. Here $\mathbf{P}_\Delta$ is the projective space associated to the integral polyhedron Δ [81]. Batyrev showed that if the Δ polyhedron is reflexive then a smooth sections of $\mathcal{O}(K_{\mathbf{P}(\Delta)})$ exists, the dual reflexive polyhedron Δ^* exists and the generically smooth section of $\mathcal{O}(K_{\mathbf{P}(\Delta)})$ has mirror Hodge numbers $h^{p,q}(M) = h^{3-p,q}(W)$. Reflexive polyhedra in four dimensions relevant for the CY threefold case have been classified [143]. This class of Calabi-Yau manifolds exihibits about 30.000 different Hogde numbers. As explained previously h^{11} and h^{21} are the only independent ones and the corresponding distribution for the sample is shown[54] in Fig. 38.

[54] Special thanks to Maximillian Kreuzer for sending me this fi gure

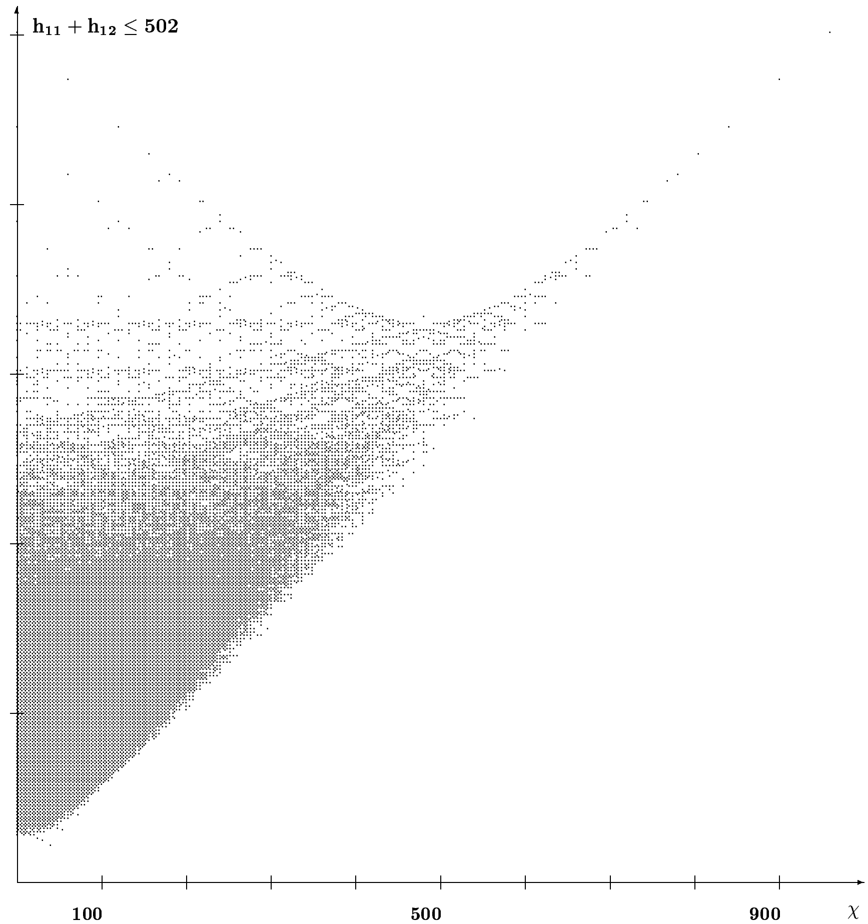

Fig. 38 Hodge Numbers of Hypersurface in Toric Varieties.

These and generalized constructions like complete intersections and orbifolds of tori and the afore mentioned manifolds are the bulk of the systematically explored examples of Calabi-Yau mirror pairs, see [144] for computer generated lists with about $10^4 - 10^8$ topological inequivalent examples[55], though slightly more exotic cases, e.g. hypersurfaces and complete intersections in Grassmannians and flag manifolds do exist in unknown numbers.

An encouraging observation in view of this enormous numbers is that at least in Type II string theory there is in some sense only one connected component of the Calabi-Yau moduli space. In fact a conjecture formulated by Miles Reid that all Calabi-Yau spaces are in the same moduli space connected by singular transitions [177] finds a physical application in that [184] shows that the singularity in physical quantities as calculated in conformal field theory at the conifold transition between topological different Calabi-Yau spaces is merely a breakdown of the perturbative low energy description due to a non-perturbative black hole becoming massless at the transition point. The full non-perturbative theory at low energy exhibits spontaneous breaking by acquiring an Higgs vacuum expectation value. Also it has been shown that all hypersurfaces in toric Calabi-Yau can be connected by such physically innocuous transitions.

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[55] The lower number is the number of inequivalent Hodge numbers the higher is an estimate of all topological different phases in the K¨ahlercone, which have not been systematically constructed.

References

[1] M. Aganagic, A. Klemm M. Marino and C. Vafa, *The topological vertex*, arXiv:hep-th/0305132.

[2] M. Aganagic, R. Dijkgraaf, A. Klemm, M. Marino and C. Vafa, *Topological strings and integrable hierarchies*, arXiv:hep-th/0312085.

[3] M. Aganagic, A. Klemm and C. Vafa, *Mirror Symmetry, D-branes and the duality web,* Z. Naturforsch. **A** 57, 1 (2002), arXiv:hep-th/0105045 .

[4] M. Aganagic and C. Vafa, *Mirror Symmetry, D-Branes and counting holomorphic Disk,* hep-th/0012041 .

[5] L. Alvarez-Gaume , Commun. Math. Phys. **90**, (1983) 161.

[6] O. Alvarez, *Topological Quantization And Cohomology,* Commun. Math. Phys. **100**, 279 (1985).

[7] L. Alvarez-Gaume, C. Gomez, G. W. Moore and C. Vafa, *Strings In The Operator Formalism*, Nucl. Phys. B **303**, 455 (1988).

[8] I. Antoniadis, E. Gava, K. S. Narain and T. R. Taylor, *Topological amplitudes in string theory,* Nucl. Phys. B **413**, 162 (1994) [arXiv:hep-th/9307158].

[9] I. Antoniadis, E. Gava, K. S. Narain and T. R. Taylor, *N=2 type II heterotic duality and higher derivative F terms*, Nucl. Phys. B **455** (1995) 109 [arXiv:hep-th/9507115].

[10] P. S. Aspinwall, B. R. Greene and D. R. Morrison, *Space-time topology change and stringy geometry,* J. Math. Phys. **35** (1994) 5321.

[11] P. S. Aspinwall, *D-branes on Calabi-Yau manifolds,* arXiv:hep-th/0403166.

[12] M.F. Atiyah and I.M Singer, *The index of an elliptic operator I& III*, Ann. Math. **87** (1968) 484.

[13] M. F. Atiyah and R. Bott, *The moment map and equivariant cohomology,* Topology **23** (1984) 1 .

[14] N. Barth, C. Peters, W. van de Ven "Compact Complex Surfaces," Springer Berlin (1984).

[15] V. Batyrev,*Dual Polyhedra and Mirror Symmetry for Calabi-Yau Hypersurfaces in Toric Varieties*, J. Algebraic Geom. 3 (1994), no. 3, 493–535

[16] S. Barannikov and M. Kontsevich, *Frobenius Manifolds and Formality of Lie Algebras of Polyvector Fields,* Internat. Math. Res. Notices **4** (1998) 201 – 215 [alg-geom/9710032].

[17] C. Beasley and E. Witten, *Residues and Worldsheet Instantons*, [arXiv:hep-th/0304115 .

[18] D.W Berry, *Quantal Phase Factors Accompanying Adiabatic Changes*, Proc. Royal Soc. London, Series A Mathematical and Phys. Sciences, **392** (1984) 45.

[19] M. Bershadsky, S. Cecotti, H. Ooguri and C. Vafa, *Holomorphic anomalies in topological field theories,* Nucl. Phys. B **405** (1993) 279 [arXiv:hep-th/9302103].

[20] M. Bershadsky, S. Cecotti, H. Ooguri and C. Vafa, *Kodaira-Spencer theory of gravity and exact results for quantum string amplitudes,* Commun. Math. Phys. **165** (1994) 311 [arXiv:hep-th/9309140].

[21] M. Bershadsky, C. Vafa and V. Sadov, "D-Branes and Topological Field Theories," Nucl. Phys. B **463**, 420 (1996) [arXiv:hep-th/9511222].

[22] A. L. Besse, *Einstin Manifolds*, Springer, Berlin (1987).

[23] D. Bessis, C. Itzykson and J. B. Zuber, *Quantum Field Theory Techniques In Graphical Enumeration,* Adv. Appl. Math. **1**, 109 (1980).

[24] J.-M. Bismut, H. Gillet and C. Soul´e, *Analystic Torsion and Holomorphic Determinant Line Bundles III, Quillen Metrics on Holomorphic Determinants*, Comm. Math. Phys. **115** (1998) 49.

[25] R. Borcherds, *Automorphic forms with singularities on Grassmannians,* alg-geom/9609022.

[26] R. Bott and L.W. Tu *Differential forms in algebraic Topology,* Springer Berlin (1982).

[27] G.E. Bredon, *Introduction to compact transformation Groups,* Academic Press (1972) New York .

[28] I. Brunner, M. R. Douglas, A. E. Lawrence and C. Romelsberger, JHEP **0008**, 015 (2000) [arXiv:hep-th/9906200].

[29] J. Bryan and R. Pandharipande, *The local Gromov-Witten theory of curves*, math.AG/0411037.

[30] R. Bryant and P. Griffi th, *Some observations on the Infi nitessimal Period Relations for Regular Threefolds with Trivial Canonical Bundle,* in Prgress in Mathematics 36, M. Artin and J. Tate, Eds.Birkh¨auser, (1983).

[31] A. Capelli, C. Itzykson, and J.B. Zuber, *Modular Invariant Partition functions in Two Dimensions,* Nucl. Phys. **B280** [FS 18] (1987) 445, A. Capelli, *Modular Invariant Partition Functions of Superconformal Theories,* Phys. Lett. **B215** (1987) 82.

[32] P. Candelas and X. del la Ossa, *Lectures on Complex Manifolds*, in Superstrings '87, "Proceedings of the Trieste Spring School," Ed. L. Alvarez-Gaume et. al., World Scientifi c, Singapore (1987).

[33] P. Candelas and X. de la Ossa, *Moduli Space Of Calabi-Yau Manifolds,* Nucl. Phys. B **355**, 455 (1991).

[34] P. Candelas, P. S. Green and T. Hubsch, *Rolling Among Calabi-Yau Vacua,"* Nucl. Phys. B **330**, 49 (1990).

[35] P. Candelas, G. Horowitz, A. Strominger and E. Witten, *Vaccum Confi gurations for Superstrings,* Nucl. Phys. **B**256 (1985) 46.

[36] P. Candelas and X. de la Ossa, *Comments on Conifolds*, Nucl. Phys. **B**341 (1990) 383.
[37] P. Candelas, X. C. De La Ossa, P. S. Green and L. Parkes, *A Pair Of Calabi-Yau Manifolds As An Exactly Soluble Superconformal Theory,* Nucl. Phys. B **359**, (1991) 21.
[38] P. Candelas, X. De La Ossa, A. Font, S. Katz and D. R. Morrison, *Mirror symmetry for two parameter models. I,* Nucl. Phys. B **416** (1994) 481 [arXiv:hep-th/9308083].
[39] P. Candelas and A. Font, *Duality between the webs of heterotic and type II vacua,* Nucl. Phys. B **511** (1998) 295 [arXiv:hep-th/9603170].
[40] P. Candelas, X. de la Ossa and F. Rodriguez Villegas, *Calabi-Yau manifolds over fi nite fi elds.I& II,* arXiv:hep-th/0012233 & arXiv:hep-th/0402133.
[41] S. Cecotti, S. Ferrara and L. Giradello, *Goemetry of Type II superstrings and the moduli of superconformal fi eld theories*, Int. J. Mod. Phys. **A4** (1989) 2475.
[42] S. Cecotti Nucl. Phys. B **355** (1991) 755.
[43] S. Cecotti and C. Vafa, *Ising model and N=2 supersymmetric theories,* Commun. Math. Phys. **157**, 139 (1993) [arXiv:hep-th/9209085].
[44] S. Cecotti and C. Vafa, *Topological Anti-topological Fusion,* Nucl. Phys. B **367** (1991) 369.
[45] T. M. Chiang, A. Klemm, S. T. Yau and E. Zaslow, *Local mirror symmetry: Calculations and interpretations,* Adv. Theor. Math. Phys. **3**, 495 (1999) [arXiv:hep-th/9903053].
[46] D. A. Cox and S. Katz, *Mirror Symmetry And Algebraic Geometry,* Mathematical Surveys and Monographs, **68** (1999), American Mathematical Society, Providence, RI
[47] D. Cox, *The Homogeneous coordinate ring of a toric variety,* J. Alg. Geom. **4** (1995) 17 [alg-geom/9210008].
[48] B. Craps, F. Roose, W. Troost and A. Van Proyen, Nucl. Phys. **B503**, (1997) 565, hep-th/9703082.
[49] E. Cremmer et. al, *Vector Multiplets Coupled to N=2 Supergravity: Superhiggs Effect, Flat Potentials and Goemtric Structures,* Nucl. Phys. **250** (1985) 385.
[50] V. I .Danilov, *The geometry of toric varieties*, Russian Math. Survey **33**, n.2 (1978) 97
[51] B. de Wit and A. van Proyen, *Potentials and symmetries of General Gauged N=2 Supergravity-Yang-Mills Models,* Nucl. Phys. **245** (1984) 89
[52] B. de Wit, P.G. Lauwers and A. Van Proyen, *Lagrangians of N=2 Supergravity Matter Systems,* Nucl. Phys. **B255** (1980) 569,
[53] Tammo tom Dieck, *Transformation Groups*, Walter de Gruyter Berlin (1987).
[54] P. Di Francesco, P. H. Ginsparg and J. Zinn-Justin, *2-D Gravity and random matrices,* Phys. Rept. **254**, 1 (1995) [arXiv:hep-th/9306153].
[55] E.D'Hoker and P.H. Phong, *The geometry of string perturbation theory*, Rev. Mod. Phys. **60** (1988) 917.
[56] R. Dijkgraaf, *Intersection theory, integrable hierarchies and topological fi eld theory,* arXiv:hep-th/9201003.
[57] R. Dijkgraaf and C. Vafa, *Matrix Model, topological strings and supersymmetric gauge theories*, Nucl. Phys. **B** 644, 3 (2002), arXiv:hep-th/0206255 .
[58] R. Dijkgraaf and C. Vafa, *On Geometry and Matrix Models*, Nucl. Phys. **B** 644, 21 (2002), arXiv:hep-th/0207106 .
[59] R. Dijkgraaf, E. Verlinde and H. Verlinde *Loop Equations and the Virasoro constraints in non-perturbative quantum gravity,* Nucl. Phys. B **384** (1991) 435.
[60] R. Dijkgraaf, H. Verlinde and E. Verlinde, *Topological strings in d¡1,* Nucl. Phys. B **352** (1991) 59.
[61] R. Dijkgraaf, *Fields, strings, matrices and symmetric products,* arXiv:hep-th/9912104.
[62] L. J. Dixon *Some world-sheet properties of superstring compactifi cations, on orbifolds and otherwise*, In "Superstrings, Unifi ed Theories, and Cosmology," (1987) Ed. G. Furla et al., World Scientifi c, Singapore.
[63] V. Dolgachev, *Weighted projective spaces*, Lectures Notes in Mathematics, Springer .
[64] S. K. Donaldson, R. P. Thomas, *Gauge theory in higher dimensions,* in *The geometric universe,* 31–47, (1998) Oxford Univ. Press.
[65] T Eguchi, P.B. Gilkey and A. Hanson, *Gravitation, Gauge Theory and Differential Geometry*, Phys. Rep. **66** (1980) 213 .
[66] T. Eguchi and S.-K. Yang, *N=2 superconformal models as topological fi eld theories,* Mod. Phys. Lett. **A5** 1693 (1990).
[67] J. Erler and A. Klemm, *Comment on the generation number in orbifold compactifi cations,* Commun. Math. Phys. **153**, 579 (1993) [arXiv:hep-th/9207111].
[68] C. Faber and R. Pandharipande, *Hodge integrals and Gromow-Witten theory,* Invent. Math **139** 173, math.AG./9810173.
[69] C. Faber, *Alogarithms for computing intersection numbers on the moduli space of curves, with a application to the locus of the Jacobian*, math.AG/9711218.

[70] S. Ferrara, J. A. Harvey, A. Strominger and C. Vafa, "Second quantized mirror symmetry," Phys. Lett. B **361**, 59 (1995) [arXiv:hep-th/9505162].
[71] S. Ferrara, R. Kallosh and A. Strominger, Phys. Rev. **D52** (1995) 5412, hep-th/9508072.
[72] A. Font and S. Theisen, *Introduction to String Compactification*, Lectures (2003) available with excercises at `http://www.aei.mpg.de/~theisen`.
[73] D. S. Freed, *Special Kaehler manifolds,* Commun. Math. Phys. **203**, 31 (1999) [arXiv:hep-th/9712042].
[74] D. Friedan, "Introduction to Polykov's String Theory," Les Houches (1882)
[75] D. Friedan, P. Windey, Nucl. Phys. **B235** (1984) 395.
[76] J. Fuchs, A. Klemm, C. Scheich and M. G. Schmidt, *Spectra And Symmetries Of Gepner Models Compared To Calabi-Yau Compactifications,* Annals Phys. **204**, 1 (1990).
[77] K. Fujikawa, *Path integral for gauge theories with fermions*, Phys. Rev. **D21** (80) 2848.
[78] K. Fujikawa, *Path integral for relativistic strings*, Phys. Rev. **D25** (81) 2584.
[79] K. Fujikawa and H. Suzuki, *Path integrals and quantum anomalies,* Clarendon Press, Oxford (2004).
[80] W. Fulton, *Intersection theory*, Springer, Berlin (1998).
[81] W. Fulton, *Introduction to toric varieties* (Princeton Univ. Press, Princeton 1993)
[82] D. Gaiotto and L. Rastelli, *A paradigm of open/closed duality: Liouville D-branes and the Kontsevich model,* arXiv:hep-th/0312196.
[83] S. J. . Gates, C. M. Hull and M. Rocek, *Twisted Multiplets And New Supersymmetric Nonlinear Sigma Models,* Nucl. Phys. B **248**, 157 (1984).
[84] D. Gepner, *Spacetime supersymmetries in compactified string theory and superconformal models*, Nucl. Phys. **B 296**, 757.
[85] D. Gepner, *Lectures On N=2 String Theory, Lectures at Spring School on Superstrings, Trieste, Italy, Apr 3-14, 1989*
[86] A. A. Gerasimov and S. Shatashvili, *Towards Integrability of Topological Strings I: Three-forms on Calabi-Yau manifolds*, arXiv:hep-th/0409238.
[87] P. H. Ginsparg and G. W. Moore, "Lectures on 2-D gravity and 2-D string theory," arXiv:hep-th/9304011.
[88] A. B. Givental, *Elliptic Gromov - Witten invariants and the generalized mirror conjecture*, Integrable systems and algebraic geometry (Kobe/Kyoto, 1997), 107, World Sci. Publishing, River Edge, NJ, 1998, math.AG/9803053 .
[89] D. Ghoshal and C. Vafa, *C = 1 string as the topological theory of the conifold," Nucl. Phys. B* **453**, 121 (1995) [arXiv:hep-th/9506122].
[90] R. Gopakumar and C. Vafa, *M-theory and topological strings. I,* [arXiv:hep-th/9809187].
[91] R. Gopakumar and C. Vafa, *M-theory and topological strings. II,* [arXiv:hep-th/9812127].
[92] M. B. Green, J.H. Schwarz and E. Witten, *Superstring Theory*, I & II, Cambridge University Press (1987).
[93] T. Graber and E. Zaslow, *Localization for open topological strings, and a mirror theorem*, hep-th/0109075 .
[94] T. Graber and R. Pandharipande, *Localization of the Virtual Class,* arXiv:alg-geom/9708001 .
[95] M. Green and N. Seiberg, Nucl. Phys. **B**299 (1988) 559.
[96] B. R. Greene and M. R. Plesser, *Duality In Calabi-Yau Moduli Space,* Nucl. Phys. B **338**, 15 (1990).
[97] B. R. Greene, *String theory on Calabi-Yau manifolds,* arXiv:hep-th/9702155.
[98] P. Griffith, *On the periods of certain rational integrals I and II,* Ann. of Math. **90** 460 and 498 (1969).
[99] P. Griffith, *Periods of integrals on algebraic manifolds I and II,* Amer. J. Math **90** 568 and 805 (1970).
[100] P. Griffith and J. Harris, *Principles of Algebraic Geometry,* J. Wiley & Sons New York (1978).
[101] P. Griffith (Ed.) *Topopics in Transcendental Algebraic Geometry,* "Annals of Mathematical Studies," Princeton University Press (1984).
[102] M. T. Grisaru, A. E. M. van de Ven and D. Zanon, Nucl. Phys. B **277**, 409 (1986).
[103] D. J. Gross and I. R. Klebanov, *One-Dimensional String Theory On A Circle,* Nucl. Phys. B **344**, 475 (1990).
[104] M. Gross, D. Huybrechts and D. Joyce, *Calabi-Yau manifolds and related Geometries,* Springer Berlin (2003).
[105] K. Hori, S. Katz, A. Klemm, R. Pandharipande, R. Thomas, C. Vafa, R. Vakil and E. Zaslow, *Mirror Symmetry*, American Mathemtical Society (2003).
[106] J. Harris and I. Morrison, *Moduli of curves,* Vol. 187 of Graduate Texts in Mathematics, Springer-Verlag, Berlin 1998.
[107] R. Harvey and H. B. Lawson, *Calibrated Geometries*, Acta Mathematica **148**, 47.
[108] J. A. Harvey and G. W. Moore, *Algebras, BPS States, and Strings,* Nucl. Phys. B **463**, 315 (1996) [arXiv:hep-th/9510182].
[109] A. Hashimoto, M. x. Huang, A. Klemm and D. Shih, *Open / closed string duality for topological gravity with matter,* arXiv:hep-th/0501141.

[110] , H. Hironaka *Resolutions of singularities of an algebraic variety over a field of characteristic zero I, II*, Ann. Math. **79** (1964) 109.

[111] F. Hirzebruch, *Topological Methods in Algebraic Topology,* Springer-Verlg, Berlin (1995) 3rd ed.

[112] N. Hitchin, *The Geometry of three-forms in six and seven dimensions,* arXiv:math.DG/0010054 .

[113] N. Hitchin, *Generalized Calabi-Yau manifolds,* Q.J. Math. **54** (2003) 281.

[114] S. Hosono, A. Klemm, S. Theisen and S. T. Yau, *Mirror symmetry, mirror map and applications to Calabi-Yau hypersurfaces*, Commun. Math. Phys. **167** (1995) 301 [arXiv:hep-th/9308122].

[115] S. Hosono, A. Klemm, S. Theisen and S. T. Yau, *Mirror symmetry, mirror map and applications to complete intersection Calabi-Yau spaces,* Nucl. Phys. B **433** (1995) 501 [arXiv:hep-th/9406055].

[116] S. Hosono, M. H. Saito and A. Takahashi, *Holomorphic anomaly equation and BPS state counting of rational elliptic surface,* Adv. Theor. Math. Phys. **3**, 177 (1999) [arXiv:hep-th/9901151].

[117] G. Horowitz, *What is a Calabi-Yau space?*, in Proceedings of the Workshop on Unified String theories, Santa Barbara 1985, Eds. M. Green and D. Gross.

[118] P. S. Howe, G. Papadopoulos and K. S. Stelle, Phys. Lett. B **174**, 405 (1986).

[119] S. Albeverio, J. Jost, S. Paycha and S. Scarlatti, A Mathematical Introduction to String Theory, London. Math. Soc. Lecture Notes Ser. 225, Cambridge Univ Press (1997).

[120] D. D. Joyce, *Compact Manifolds with special Holonomy,* Oxford University Press (2000).

[121] D. D. Joyce, *Compact Riemannian J-manofolds with holonomy G_2 I II ,* J. Diff Geometry, **43** (1996) 291, 329 .

[122] S. Kachru and C. Vafa, *Exact results for N=2 compactifications of heterotic strings,* Nucl. Phys. B **450**, 69 (1995) [arXiv:hep-th/9505105].

[123] S. Kachru, A. Klemm, W. Lerche, P. Mayr and C. Vafa, *Nonperturbative results on the point particle limit of N=2 heterotic string compactifications,* Nucl. Phys. B **459**, 537 (1996) [arXiv:hep-th/9508155].

[124] V. Kaplunovsky and J. Louis, *On Gauge couplings in string theory,* Nucl. Phys. B **444**, 191 (1995) [arXiv:hep-th/9502077].

[125] A. Kapustin and D. Orlov, *Lectures on mirror symmetry, derived categories, and D-branes,* arXiv:math.ag/0308173.

[126] A. Kapustin and Y. Li, *Topological sigma-models with H-flux and twisted generalized complex manifolds,* arXiv:hep-th/0407249.

[127] S. Katz, D. R. Morrison and M. Ronen Plesser, *Enhanced Gauge Symmetry in Type II String Theory,* Nucl. Phys. B **477**, 105 (1996) [arXiv:hep-th/9601108].

[128] S. Katz, A. Klemm and C. Vafa, *M-theory, topological strings and spinning black holes*, Adv. Theor. Math. Phys. **3** (1999) 1445 [arXiv:hep-th/9910181].

[129] T. Kawai, *String duality and modular forms,* Phys. Lett. B **397** (1997) 51 [arXiv:hep-th/9607078].

[130] T. Kawai and K. Yoshioka, *String Partition function and Infinite Products* [arXiv:hep-th/0002169].

[131] Y. Kazama and H. Suzuki, *Characterization Of N=2 Superconformal Models Generated By Coset Space Method,* Phys. Lett. B **216**, 112 (1989) and *New N=2 Superconformal Field Theories And Superstring Compactification,* Nucl. Phys. B **321**, 232 (1989).

[132] Phys. Usp. **37**, 1 (1994) [arXiv:hep-th/9303139].

[133] A. Klemm, W. Lerche and P. Mayr, *K3 Fibrations and heterotic type II string duality,* Phys. Lett. B **357**, 313 (1995) [arXiv:hep-th/9506112].

[134] A. Klemm, *On the geometry behind N = 2 supersymmetric effective actions in four dimensions,* arXiv:hep-th/9705131.

[135] A. Klemm and E. Zaslow, *Local mirror symmetry at higher genus,* in *Winter School on Mirror Symmetry, Vector Bundles and Lagrangian Submanifolds (Cambridge, MA, 1999)*, AMS/IP Stud. Adv. Math. **23** (2001) 183 – 207 [arXiv:hep-th/9906046].

[136] A. Klemm and R. Schimmrigk, *Landau-Ginzburg string vacua,* Nucl. Phys. B **411**, 559 (1994) [arXiv:hep-th/9204060], M. Kreuzer and H. Skarke, *On the classification of quasihomogeneous functions,* Commun. Math. Phys. **150**, 137 (1992) [arXiv:hep-th/9202039].

[137] A. Klemm and P. Mayr, *Strong Coupling Singularities and Non-abelian Gauge Symmetries in $N = 2$ String Theory,* Nucl. Phys. B **469** (1996) 37 [arXiv:hep-th/9601014].

[138] A. Klemm, M. Kreuzer, E. Riegler and E. Scheidegger, *Topological String Amplitudes, Complete intersection Calabi-Yau Spaces and Threshold Corrections*, hep-th/0410018.

[139] K. Kodaira, *Complex manifolds and Deformations of Complex Structures,* Springer-Verlag, Berlin (1985) .

[140] M. Kontsevich, *Intersection theory on the moduli space of curves and the matrix Airy function,* Commun. Math. Phys. **147**, 1 (1992).

[141] M. Kontsevich, "Enumeration of rational curves via Torus actions," arXiv:hep-th/9405035.

[142] M. Kontsevich, *Product formulas for modular forms on O(2,n) (after R.Borcherds)* [alg-geom/9709006].
[143] M. Kreuzer, H. Skarke, *Complete classification of reflexive polyhedra in four dimensions,* Adv. Theor. Math. Phys. **4** (2000) 6 [hep-th/0002240]
[144] `http://hep.itp.tuwien.ac.at/~kreuzer/CY/hep-th/yymmnnn.html`
[145] D. Kutasov, Phys. Lett. **B** 220 (1989) 153.
[146] J.M.F. Labastida, P.M. Llatas, *Topological Matter in Two Dimensions*, Nucl.Phys. **B379** (1992) 220, arXiv/hep-th/9112051.
[147] J.M.F. Labastida, M. Marino, *Type B topological matter, Kodaira Spencer Theory and mirror symmetry*, Phys.Lett. **B333** (1994) 386, arXiv/hep-th/9405151.
[148] A. Landmann, *On the Picard-Lefshetz transformations,* Trans. A.M.S. 181 (1973) 89.
[149] H.B. Lawson and M-L. Michelson, *Spin Geometry*, Princeton Univ. Press (1989).
[150] C. I. Lazaroiu, *D-brane categories,* Int. J. Mod. Phys. A **18**, 5299 (2003) [arXiv:hep-th/0305095].
[151] W. Lerche, C. Vafa and N.P. Warner, *Chiral Rings in N=2 superconformal Theories*, Nucl. Phys. **B324** (1989) 371.
[152] , *Branes and Toric Geometry*, Adv. Theor. Math. Phys. **2** (1998) 91, [arXiv:hep-th/9711013].
[153] A. S. Losev, A. Marshakov and N. A. Nekrasov, "Small instantons, little strings and free fermions," arXiv:hep-th/0302191.
[154] M. Marino and G. W. Moore, Nucl. Phys. B **543**, 592 (1999) [arXiv:hep-th/9808131].
[155] J. Maldacena, G. W. Moore, N. Seiberg and D. Shih, JHEP **0410**, 020 (2004) [arXiv:hep-th/0408039].
[156] Y. I. Manin, *Frobenius Manifolds, Quantum Cohomology, and Moduli Spaces,* Colloquium Publications AMS Vol 47, (1999).
[157] D. Maulik, N. Nekrasov, A. Okounkov and R. Pandharipande, *Gromow-Witten theory and Donaldson-Thomas theory I,* [arXiv:math.AG/0312059].
[158] M. Marino, *Les Houches lectures on matrix models and topological strings,* arXiv:hep-th/0410165.
[159] M. Marino, *Chern-Simons theory and topological strings,* arXiv:hep-th/0406005.
[160] M. Mirzakhani, *Weil-Peterson volumes and the Witten-Kontsevich formula*, Preprint (2003), Homepage Maryam Mirzakhani.
[161] D. R. Morrison, *Where is the large radius limit?,* arXiv:hep-th/9311049. .
[162] M. Nakahara, *Geometry, Toplogy and Physics*, Adam Hilger (1990) Bristol.
[163] A. Neitzke and C. Vafa, *Topological strings and their physical applications,* arXiv:hep-th/0410178.
[164] D. Nemeschansky and A. Sen, "Conformal Invariance Of Supersymmetric Sigma Models On Calabi-Yau Manifolds," Phys. Lett. B **178**, 365 (1986).
[165] N. Nekrasov, H. Ooguri and C. Vafa, *S–duality and topological strings*, arXiv:hep-th/0403167.
[166] A. Newlander and L. Nirenberg, *Complex analytic coordiantes in almost complex manifolds*, Ann. Math. **65** (1957) 351–404
[167] T. Oda, *Convex bodies and algebraic geometry* (Springer, Berlin Heidelberg 1988).
[168] A. Okounkov and R. Pandharipande *Gromow-Witten theory, Hurwitz numbers and Matrix models I,* math/AG/0101147.
[169] H. Ooguri and C. Vafa, *Knot invariants and topological strings,* Nucl. Phys. B **577**, 419 (2000) [arXiv:hep-th/9912123].
[170] H. Ooguri and C. Vafa, *Gravity Induced C-Deformation,* [arXiv:hep-th/0303063].
[171] M. Peskin and D. Schroeder, *An Introduction to Quantum fi eld theory*, Westview Press (1995).
[172] V. Pestun and E. Witten, *The Hitchin functionals and the topological B-model at one loop,* arXiv:hep-th/0503083.
[173] J. Polchinski, *String Theory*, I & II, Cambridge University Press (1998).
[174] J. Polchinski and A. Strominger, *New Vacua for Type II String Theory,* Phys. Lett. B **388**, 736 (1996) [arXiv:hep-th/9510227].
[175] D. B. Ray and I. M. Singer, "Analytic Torsion for complex manifolds," Ann. of. Math. **98**, 154, (1973).
[176] A. Recknagel and V. Schomerus, Nucl. Phys. B **531**, 185 (1998) [arXiv:hep-th/9712186].
[177] M. Reid, *The moduli space of 3-folds with* $K = 0$ *may nevertheless be irreducible,* Math. Ann. **278** (1987) 329.
[178] A. Schwimmer and N. Seiberg, *Comments on the N=2,3,4 superconformal Algebras in two dimensions,* Phys. Lett. **184B** (1987) 191.
[179] W. Schmidt, *Variation of Hodge structure: The singularities of the period map,* Inventiones Math. 22 (1973) 211.
[180] E.H. Spanier, *Algebraic Topology* Springer (1966).
[181] E. Sharpe, *Lectures on D-branes and sheaves,* arXiv:hep-th/0307245.
[182] K. Strebel, *Quadratic Differentials*, Springer 1984.

[183] A. Strominger and E. Witten, Comm. Math. Phys. **101** (1985) 341.

[184] A. Strominger *Massless black holes and conifolds in string theory,* Nuc. Phys. **B245** 96.

[185] A. Strominger, *Special Geometry,* Comm. Math. Phys. 133 (1990) 163.

[186] A. Strominger, *Macroscopic Entropy of N=2 Extremal Black Holes,* Phys. Lett. **B238** (1996) 39, hep-th/9602111.

[187] A. Strominger, *Superstrings With Torsion,* Nucl. Phys. B **274**, 253 (1986).

[188] R. P. Thomas, *A holomorphic Casson invariant for Calabi–Yau 3-folds, and bundles on K3 fibrations*, J. Differ. Geom. **54** (2000) 367–438, [arXiv:math.ag/9806111].

[189] G. Tian, *Smoothness of the Universal Deformation Space of Compact Calabi-Yau manifolds and its Weil-Peterson metric*, in "Mathematical Aspects of String Theory" Ed. S.T. Yau, World Scientific Singapore (1987).

[190] G. Tian, *Canonical Metrics in Kähler Geoemtry,* Birkhäuser (2002).

[191] G. 't Hooft, *Perturbative Quantum Gravity*, in Proceedings of the International School of Subnuclear Physics, Erice 2002, From Quarks and Gluons to Quantum Gravity, Subnuclear Series Vol. 40, ed. A. Zichichi, World Scientific, 249, `http://www.phys.uu.nl/~thooft/`

[192] A. Todorov, *The Weyl-Peterson Geometry of the Moduli-Space of* $SU(n \geq 3)$ *(Calabi-Yau) Manifolds I,* Commun. Math. Phys. 126 (1989) 325.

[193] C. Vafa, *Topological Landau-Ginzburg models,* Mod. Phys. Lett. A **6** (1991) 337.

[194] C. Vafa, *Topological Landau Ginzburg models*, Mod. Phys. Lett. **A6** (1991) 337.

[195] E. Verlinde and H. Verlinde, *A solution of two-Dimensional Topological Quantum Gravity*, Nucl. Phys. **B348** (1991) 457.

[196] E. Viehweg, *Quasi-Projective Moduli for Polarized Manifolds*, Ergeb. der Mathematik und ihrer Grenzgebiete 3 Bd 30, Springer Berlin 1995.

[197] M. Vonk, *A mini-course on topological strings,* arXiv:hep-th/0504147.

[198] McKenzie Y. Wang, *Parallel Spinors and Parallel Forms,* Ann. Global Anal. Geom. **7** (1989) 59-68.

[199] E. Witten, *Chern-Simons gauge theory as a string theory,* Prog. Math. **133**, 637 (1995) [arXiv:hep-th/9207094].

[200] E. Witten, *Two-Dimensional Gravity and intersection theory on moduli spaces,* Surveys in Diff. Geometry **1** (1991) 243.

[201] E. Witten, *Quantum Field theory and the Jones Polynomial,* Commun. Math. Phys. **121**, 351 (1989) [arXiv:hep-th/9207094].

[202] E. Witten, *On the Structure of the Topological Phase of Two-Dimensional Gravity,* Nucl. Phys. B **340**, 281 (1990).

[203] E. Witten, *Two-Dimensional Gravity And Intersection Theory On Moduli Space,* Surveys Diff. Geom. **1**, 243 (1991).

[204] E. Witten, *Algebraic geometry associated with matrix models of two dimensional gravity,* Topological Methods in Modern Mathematics, (Stony Brook, NY, 1991), 235–269, Publish or Perish, Houston, TX, 1993.

[205] E. Witten, *The N matrix model and gauged WZW models,* Nucl. Phys. B **371**, 191 (1992).

[206] E. Witten, *Topological Sigma Model*, Commun. Math. Phys. **118** (1988) 411.

[207] E. Witten, *Mirror Manifolds and Topological Field Theory,* arXiv:hep-th/9112056 .

[208] E. Witten, *Constraints on supersymmetry breaking,* Nucl. Phys. **B202** (1982) 253.

[209] E. Witten, *Elliptic Genera and Quantum Field Theory,* Commun. Math. Phys. **109** (1987) 525.

[210] E. Witten, *The index of the Dirac operator in Loop-space,* in "Elliptic Curves and Modular Forms in Algebraic Topology", P.S. Landweber Ed., Lecture Notes in Mathematics 1326, Springer Berlin (1980).

[211] E. Witten, *On the Landau description of* $N = 2$ *minimal models,* Int. Mod. Phys. **A9** (1994) 4783, hep-th/9304026.

[212] E. Witten, *Two-Dimensional Models With (0,2) Supersymmetry: Perturbative Aspects*, hep-th/0504078.

[213] E. Witten, *Phases of N=2 theories in two dimensions,* Nucl. Phys. **B403** (1993) 159 [hep-th/9301042]

[214] S.-T. Yau, *On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equations I,* Comm. on pure and applied math.**31** (1978) 339.

[215] S. T. Yau and E. Zaslow, *BPS States, String Duality, and Nodal Curves on K3,* Nucl. Phys. B **471**, 503 (1996) [arXiv:hep-th/9512121].

[216] B. Zumino, *Supersymmetry And Kähler Manifolds,* Phys. Lett. B **87**, 203 (1979).

[217] D. Zvonkine, *Strebel differentials on stable curves and Kontsevichs's proof of Witten's conjecture,* [arXiv:math.AG/0209071].