Generalized Givental's theorem and classification of Fano threefolds

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1. GOLYSHEV CONJECTURE

Let V be a smooth Fano threefold with $\operatorname{Pic} V \cong \mathbb{Z}H$. We will consider only Gromov–Witten invariants of genus 0.

Definition 1. Put $K = -K_V$. Consider a matrix of normalized two-pointed invariants

$$A = \begin{bmatrix} 0 & a_{01} & a_{02} & a_{03} \\ 1 & a_{11} & a_{12} & a_{13} \\ 0 & 1 & a_{22} & a_{23} \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

where $a_{ij} = \frac{1}{\deg V}\langle K^{3-i}, K^j, K\rangle_{j-i+1} = \frac{j-i+1}{\deg V}\langle K^{3-i}, K^j\rangle_{j-i+1}$, $\deg V = (-K_V)^3$. This matrix is called **the counting matrix** of V.

Definition 2. Given the counting matrix A, define the matrix M in the following way. Consider $\mathbb{C}[t]$, $D = t\frac{\partial}{\partial t}$ and the matrix M

$$M_{i,j} = \begin{cases} A_{i,j} \cdot (Dt)^{j-i+1} & \text{if } j-i+1 \geq 0, \\ A_{i,j} & \text{otherwise.} \end{cases}$$

Put $\widetilde{L}(\alpha) = \det_{\text{right}}(D(1-\alpha t)E - M)$, $\alpha \in \mathbb{C}$. Divide $\widetilde{L}(\alpha)$ by D from the left: $\widetilde{L}(\alpha) = DL(\alpha)$. The equation of type $L(\alpha)\Phi = 0$, where $\widetilde{L} = DL$, is called $D3$ **equation**.

This procedure means the following. We consider the quantum D-module Q_V on $\mathbb{C}[t,t^{-1}]$. I. e. let $H(V)$ be a algebraic cohomology ring with basis $\{H_i\}$. Consider a trivial vector bundle over $\mathbb{C}[t,t^{-1}]$ with fiber $H(V)$. Denote the global sections given in the fibers by $\{H_i\} \otimes 1 \in H(V) \otimes \mathbb{C}[t,t^{-1}]$ by $\{h_i\}$. This, the space of the sections is $H(V) \otimes \mathbb{C}[t,t^{-1}]$ and generated (over $\mathbb{C}[t,t^{-1}]$) by $\{h_i\}$. Consider and a (flat) connection ∇ defined on sections h_i as

$$\langle \nabla h_i, t\frac{d}{dt}\rangle = h \star h_i,$$

where h corresponds to H and $\star$ is a quantum multiplication . Put $\mathcal{D} = \mathbb{C}[t,t^{-1},D]$. Then this module is represented by some operator $\widehat{L}_V$: $Q_V \simeq \mathcal{D}/\mathcal{D}\widehat{L}_V$. To state the mirror–type conjecture on the Fano threefolds we need to **regularize** it. This means that we need to convolute it with the canonical exponent, i. e. with the push-forward under the morphism $x \to \frac{1}{x}$ of $\mathcal{D}/(z\partial - z)\mathcal{D}$. Notice that the operator $\widehat{L}_V$ is divisible by t on the left. Divide. In fact the convolution means that we need (after naively extension to $\mathbb{C}[t]$ as $\mathcal{D}/\mathcal{D}t^{-1}\widehat{L}_V$) to do the Fourier transform and pull back with respect to the inversion-of-coordinate morphism. After changing variables we obtain the counting operator.

Now state Golyshev's mirror–type conjecture.

Put $d = \operatorname{ind}(V)$ (i. e. $-K_V = dH$), $n = (-K_V)^3$, $N = \frac{n}{2d^2}$. Let $X_0(N)^W$ be the quotient of the modular curve $X_0(N)$ by the Atkin–Lehner involution (given by $z \to -\frac{1}{Nz}$, where z is the coordinate on the upper half-plane). Consider the local coordinate $q = e^{2\pi i z}$ on $X_0(N)^W$ around the image of the cusp $(i\infty)$. Notice that for N's that correspond to the considering Fano threefolds $X_0(N)^W$ are rational curves. Consider a (global) coordinate

T (the inverse of **a Conway–Norton uniformizer**) with center in the image of the cusp $(i\infty)$ which behaves locally as q, i. e. at this point $T(q) = q + q^2 \cdot F(q)$, where F is a series on q.

Conjecture (Golyshev). For each smooth Fano threefold V with Picard group $\mathbb{Z}$ there exist a particular α_V such that the function

$$\Phi = (q^{\frac{1}{24}} \prod (1-q^n) q^{\frac{N}{24}} \prod (1-q^{Nn}))^2 \cdot T^{-\frac{N+1}{12}}$$

is a solution of the equation $L(\alpha_V)\Phi = 0$ with respect to $t = T^{\frac{1}{d}}$.

So, we have the predictions for counting $D3$ (and the numbers $a_{i,j}$) for all of 17 varieties of the Iskovskikh list.

This conjecture has been checked recently for all varieties. Check it for three of them using the theorem for complete intersections in toric varieties.

2. COMPLETE INTERSECTIONS IN THE TORIC VARIETIES

Consider those Fano threefolds from the Iskovskikh list that can be represented as complete intersections in toric varieties (except for the complete intersections in projective spaces, whose Gromov–Witten invariants are well-known). That is,

- V_1**:** a smooth hypersurface of degree 6 in $\mathbb{P}(1,1,1,2,3)$.
- V_2**:** a smooth hypersurface of degree 4 in $\mathbb{P}(1,1,1,1,2)$.
- V_2'**:** a smooth hypersurface of degree 6 in $\mathbb{P}(1,1,1,1,3)$.

We need to obtain the following theorem.

Theorem 1. Counting matrices for V_1, V_2 and V_2' are:

0	240	0	576000
1	0	1248	0
0	1	0	240
0	0	1	0

0	48	0	2304
1	0	160	0
0	1	0	48
0	0	1	0

0	137520	119681240	21690374400
1	624	650016	119681240
0	1	624	137520
0	0	1	0

To prove it, we find their one-pointed invariants (with descendants) and then find prime two-pointed ones in their terms.

Combine one-pointed invariants in the following generating series.

Definition 3. Let γ_i and $\check{\gamma}_i$ be the dual bases of $H^*(V)$, $\beta \in H_2(V)$, $\deg \beta = (-K_V)\cdot\beta = d$. Then

$$I_d^V = I_\beta^V = ev^*(\frac{1}{1-\psi} \cdot [\bar{M}_1(V,\beta)]^{\mathrm{virt}}) = \sum_{i,j} \langle \psi^i \gamma_j \rangle_\beta \check{\gamma}_j,$$

$$I^V = \sum_{d \geqslant 0} I_d^V \cdot q^d.$$

Givental's theorem for complete intersections with non-negative canonical class in **smooth** toric varieties enables one to find the I-series for them. To find the I-series in our case, i. e. in the case of smooth Fano complete intersections in the **singular** toric varieties we should generalize this theorem.

<u>Theorem</u> 2. Let Y be a $\mathbb{Q}$-factorial toric variety and $Y_1, \ldots, Y_k$ be the divisors that correspond to the edges of the fan of Y. Consider a smooth complete intersection V of hypersurfaces $V_1, \ldots, V_l$ that does not intersect the singular locus of Y. Assume that $-K_V > 0$ and $\operatorname{Pic} V = \mathbb{Z}$. Let $i\colon V \to Y$ be the natural embedding. Let ℓ be a nef generator of $H_2(Y)$. For $\beta = d\ell$ put $q^\beta = q^d$. Let $\Lambda \subset H_2(V)$ be the semigroup of algebraic curves as cycles on V.

Then I-series of V is the following.

$$I^V = e^{-\alpha_V q} \sum_{\beta \in \Lambda} q^\beta \cdot i^* \left(\frac{\prod_{a=1}^{l}((V_a+1)\cdot \ldots \cdot (V_a + \beta \cdot V_a))}{\prod_{a=1}^{k}((Y_a+1)\cdot \ldots \cdot (Y_a + \beta \cdot Y_a))} \right),$$

where $\alpha_V = 0$ if the index of V is greater than 1, and $\alpha_V = \prod_{a=1}^{r}(\ell \cdot V_a)!/\prod_{a=1}^{k}(\ell \cdot Y_k)!$ if the index is 1.

(Remark that the correction term here is exactly one from Golyshev's conjecture.)

The idea of the proof of this theorem is the following. Blow up the singularities of Y.

$$\begin{array}{ccc} \widetilde{V} & \xrightarrow{\;j\;} & \widetilde{Y} \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ V & \xrightarrow{\;i\;} & Y \end{array}$$

They are away from V, so in the neighbourhood of V the map g is isomorphism. Then apply Givental's theorem for complete intersections in the smooth toric varieties for $\widetilde{Y}$ and $\widetilde{V}$ (find the correction term by the dimensional reasons). The terms with the exceptional divisors vanish, and we get the expressions for the I-series of $\widetilde{V}$, which is the same (because of the isomorphism) as for V.

Remark that we suppose that V is Fano with Picard number 1 just for simplicity and for our case. We can proof the analogous theorems for the cases of greater Picard number and Calabi–Yau varieties. Such theorems will differ only in the correction term.

This theorem enables us to find one-pointed invariants $\langle \tau_i H^j \rangle_d$ of our Fano threefolds. Now we have to find the two-pointed ones.

Applying twice the divisor axiom in the form

$$\langle \tau_{d_1}\gamma_1, \ldots, \tau_{d_n}\gamma_n \rangle_\beta = \frac{1}{(\gamma_0 \cdot \beta)} \Big(\langle \gamma_0, \tau_{d_1}\gamma_1, \ldots, \tau_{d_n}\gamma_n \rangle_\beta - \sum_{k, d_k \geqslant 1} \langle \tau_{d_1}\gamma_1, \ldots, \tau_{d_k - 1}(\gamma_0 \cdot \gamma_k), \ldots, \tau_{d_n}\gamma_n \rangle_\beta \Big)$$

(where γ_0 is the divisor class) and by induction we get the expressions for one-pointed invariants in terms of three-pointed ones with descendants (such that at least one of cohomology class in them is of dimension 2). Now use the topological recursion

$$\langle \tau_{d_1}\gamma_1, \tau_{d_2}\gamma_2, \tau_{d_3}\gamma_3 \rangle_\beta = \sum_{a, \beta_1 + \beta_2 = \beta} \langle \tau_{d_1 - 1}\gamma_1, \Delta^a \rangle_{\beta_1} \langle \Delta_a, \tau_{d_2}\gamma_2, \tau_{d_3}\gamma_3 \rangle_{\beta_2}$$

(where Δ_i and Δ^i are dual bases of $H^*(V)$).

Thus we have the following expressions for one-pointed invariants in terms of two-pointed prime ones. Put $I^V = \sum_{d\geq 0} d_i q^i$. Then

$$d_2 = \frac{1}{4}a_{01},$$

$$d_3 = \frac{1}{18}a_{01}a_{11} + \frac{1}{27}a_{02},$$

$$d_4 = \frac{1}{64}a_{01}^2 + \frac{1}{96}a_{01}a_{11}^2 + \frac{1}{144}a_{02}a_{11} + \frac{1}{128}a_{01}a_{12} + \frac{1}{192}a_{02}a_{11} + \frac{1}{256}a_{03},$$

$$d_5 = \frac{17}{3600}a_{01}^2a_{11} + \frac{13}{2700}a_{01}a_{02} + \frac{1}{600}a_{01}a_{11}^3 + \frac{47}{18000}a_{02}a_{11}^2 + \frac{43}{12000}a_{01}a_{11}a_{12} + \frac{9}{8000}a_{03}a_{11} + \frac{1}{1125}a_{02}a_{12},$$

$$d_6 = \frac{191}{103680}a_{01}a_{02}a_{11} + \frac{13}{28800}a_{02}a_{11}a_{12} + \frac{19}{43200}a_{02}a_{11}^3 + \frac{25}{41472}a_{01}^2a_{12} + \frac{1}{13824}a_{03}a_{12} + \frac{29}{82944}a_{01}a_{03} + \frac{49}{51840}a_{01}^2a_{11}^2 + \frac{37}{172800}a_{03}a_{11}^2 + \frac{83}{86400}a_{01}a_{11}^2a_{12} + \frac{1}{2304}a_{01}^3 + \frac{1}{3888}a_{02}^2 + \frac{1}{4320}a_{01}a_{11}^4 + \frac{1}{6912}a_{01}a_{12}^2.$$

These expressions are **birational**, so we can inverse them.

$$a_{01} = 4d_2,$$

$$a_{11} = \frac{1}{2}\frac{3000d_4d_5 - 168d_2d_4d_3 - 1000d_2^2d_5 + 280d_2^3d_3 + 729d_3^3 - 3888d_6d_3}{-495d_3d_5 + 261d_2d_3^2 - 312d_4d_2^2 + 432d_4^2 + 56d_2^4},$$

$$a_{02} = 3(-4455d_3^2d_5 + 1620d_2d_3^3 - 2640d_3d_4d_2^2 + 3888d_3d_4^2 + 224d_3d_2^4 - 3000d_2d_4d_5 + 1000d_2^3d_5 + 3888d_2d_6d_3)/(-495d_3d_5 + 261d_2d_3^2 - 312d_4d_2^2 + 432d_4^2 + 56d_2^4),$$

$$a_{12} = -\frac{1}{4}(11648448d_2^3d_3^2d_6 + 64300500d_3^2d_5^2d_2 - 28921320d_2^2d_5d_3^3 - 16547328d_2^4d_6d_4 - 19740000d_4d_5^2d_2^2 + 10065024d_2^2d_4^2d_3^2 - 25660800d_2d_4d_3^2d_6 + 69517440d_4d_5d_6d_3 + 34223040d_4d_5d_2^3d_3 - 5387200d_2^5d_5d_3 + 44789760d_2^2d_6d_4^2 + 4811400d_2d_4d_3^4 - 13034520d_4d_5d_3^3 - 8755008d_2^4d_4d_3^2 + 2032128d_2^6d_6 - 40837500d_3d_5^3 - 1748992d_2^7d_4 + 8689152d_2^5d_4^2 - 40310784d_6d_4^3 + 14432256d_2d_4^4 - 18524160d_2^3d_4^3 - 15116544d_6^2d_3^2 + 5668704d_3^4d_6 + 3719736d_2^3d_3^4 + 1391936d_2^6d_3^2 + 3620000d_2^4d_5^2 + 26640000d_4^2d_5^2 + 7558272d_3^2d_4^3 - 531441d_3^6 + 125440d_2^9 - 52853760d_4^2d_5d_2d_3 - 25738560d_2^2d_5d_6d_3)/(-495d_3d_5 + 261d_2d_3^2 - 312d_4d_2^2 + 432d_4^2 + 56d_2^4)^2,$$

$$a_{03} = -2(448d_2^6 - 1600d_4d_2^4 + 36288d_2^3d_6 + 1584d_2^3d_3^2 - 20352d_2^2d_4^2 - 49560d_2^2d_3d_5 - 93312d_2d_6d_4 + 54432d_2d_4d_3^2 + 82500d_2d_5^2 - 15309d_3^4 - 126360d_4d_5d_3 + 55296d_4^3 + 81648d_3^2d_6)/(495d_3d_5 - 261d_2d_3^2 + 312d_4d_2^2 - 432d_4^2 - 56d_2^4).$$

(Remark that we use here algebraic minimality of our varieties, i. e. that the algebraic cohomologies of them are generated by the Picard group.)

Thus we obtain the counting matrices of V_1, V_2 and V_2' and prove theorem 1.

Thus, Golyshev's conjecture reproduces the Iskovskikh classification. What is the next step? There are three ways to develop:

- We can consider a smooth Fano threefolds with greater Picard number and reproduce Mukai's classification of them. The problem is: we need to consider a multi-dimensional version of all above, which is more technically difficult.
- We can go to the four- and more-dimensional land. The problem of classification of Fanos of dimension greater than 3 is opened. The difficulty we meet is: in this case we can't suppose that our varieties are almost minimal (i. e. whose cohomologies are generated by Picard group generator except maybe for the middle ones), which is used to state Golyshev's conjecture.
- We can try to classify the singular (say, terminal or canonical) Fano threefolds. This problem is also opened. The difficulty in this way is that we need to work with Gromov–Witten invariants of singular varieties.

References

[1] A. Givental, *A mirror theorem for toric complete intersections*, Topological field theory, primitive forms and related topics (Kyoto, 1996), 141-175, alg-geom/9701016.

[2] V. Golyshev, *The geometricity problem and modularity of some Riemann-Roch variations*, Dokl. Akad. Nauk 386 (2002) 583-588.

[3] V. Przyjalkowski, *Quantum cohomology of smooth complete intersections in weighted projective spaces and singular toric varieties*, math.AG/0507232.