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Modal Logic for Changing Systems

by

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To

the reader, for his patience.

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Abstract

In this thesis we study various extensions of classical modal semantics to changing relational structures. Extensions of classical modal languages are considered in order to reason about these structures. The thesis is divided in two parts each dedicated to a particular kind of change: change happening along a linear order, to which we refer, metonymically, as 'time'; and change caused by transitions while being dependent on the path previously covered (reactivity).

In the first part we solve some open problems regarding the (non-)finite axiomatisability of product logics where one of the components is linear. In particular, we give the first examples of recursively enumerable (even decidable) two-dimensional products of finitely axiomatisable modal logics that are not finitely axiomatisable. We show that any axiomatisation of some bimodal logics that are determined by classes of product frames with a linearly ordered component must be infinite in two senses: it should contain infinitely many propositional variables, and formulas of arbitrarily large modal nesting-depth. We also present some further results regarding the problem of finding an explicit axiomatisation for the considered logics.

In the second part we consider the concept of reactivity introduced by Dov Gabbay in his first approach to reactive modal semantics [29] using higher order arrowed structures representing the dependence of the state of a connection upon each transition. We start by generalising this semantics to reactive Kripke frames, in order to provide representation-free tools for analysing all possible changes. Then we use classical modal techniques to axiomatise some classes of reactive frames, and to obtain decidability/complexity results. We also show that the 'higher order arrows' formalism is powerful enough to generate all the relational reactive behaviours. Finally, a logic is introduced to reason directly about these structures, and its completeness is proved.

1

Introduction

The study of modal logic started with Aristotle's analysis of necessary and possible truths. However, even though modal logic has been around since the Greeks, its most popular period blossomed with Kripke's introduction of its relational semantics in the late 1950's [54, 55]. For a detailed account of its evolution see [39]. Since then, modal languages have established themselves as the simplest languages in which relational structures can be described, constrained and reasoned about. Furthermore, relational structures play a fundamental modelling role in various disciplines, from economics to philosophy, from knowledge representation to law, and probably most obviously in theoretical computer science, for example when labelled transition systems are used to model program execution. These two facts fuelled a boom in the number of studied modal systems and the continuous increase of its application range.

One can say that the relational semantics of modal logic already encompasses change. In fact, one can consider (or access to) different worlds, and the propositional truths, given by the propositional variables valuation, may change. Also the accessible worlds may change with these transitions. Yet, the truths at a given world are 'still'. In a Kripke model both propositional truths and the accessible worlds are fixed for each world. In this thesis we take it a step further and let these vary. In many situations it makes sense to consider semantics such that when certain operators are evaluated, the model, where the formula is being evaluated, changes. Therefore, the interpretation of a formula in the scope of a modal operator is given by a general condition of the type:

$$\mathfrak{M}, x \models \Diamond\varphi \text{ if } \mathfrak{M}', x' \models \varphi$$

where x' is a point in a new model $\mathfrak{M}'$. Indeed there are various examples of such approaches, e.g.:

- in dynamic epistemic logics with agent's public announcements [74];
- in sabotage logics edges can be deleted [73];
- in memory logics one may keep the information that a certain world was visited, adding it to the memory of the model [5];
- in Hyper-modalities the meaning of the modal operators depends on where in the formula they occur [27];
- in product logics one may think that while moving along one direction the valuation of the remaining hyper-plan is changing, for example modelling valuation change in time if that direction is a time flow [23];
- in reactive Kripke semantics the accessibility relation depends on the path crossed [29, 31].

In this thesis we concentrate on the two kinds of change captured by last two approaches. One happens along a linear order, to which we call, metonymically, 'time'. That is, the propositional variables' truth value depends not only on the world we are in, but also on which 'moment' we are. Whereas the other depends on where we have been before, that is, neither relation nor propositional variables values change with the clock ticking but they react when and because we move. Moreover, the changes are sensitive to the way we got to the current world. We call the latter reactivity. We do not claim that the classical Kripke frames cannot cope with these types of change, the matter is more about how to incorporate these meta-level notions into the models and which language to consider in order to reason about them.

The thesis is divided in two parts, each dedicated to one of the considered varieties of change. These two parts are of a quite different nature. While in the first part we consider open problems in the long studied subject of products of modal logics, in the second, we take a more logical engineering approach by embracing the development of a recent field of research. In the next two sections we give an overview of the ideas present in the thesis together with some modelling examples, and close the chapter with a section giving the detailed thesis outline.

1.1 Change in (linear) 'time'

In the first part we consider the problem of axiomatizing 2-products of unimodal logics where one of the components is linearly ordered. Here we introduce briefly these structures and give a hint about the interest of such structures from the point of view of the applications.

Thinking multidimensionally - Products Modal logics are used to reason about different entities like time, knowledge, beliefs, actions, space, etc. When one wants to reason about these in an integrated manner one needs to consider multimodal logics interpreted over appropriate structures. Some of the most common among these are products of Kripke frames, since they capture the interaction between the component needed in many situations. The product construction shows up in various disguises, and it is related to many other logical formalisms, such as algebras of relations in algebraic logic, finite variable fragments of classical, intuitionistic and modal predicate logics, temporal-epistemic logics, dynamic topological logics, modal and temporal description logic, see e.g. [1, 6, 7, 17, 18, 19, 23, 52, 66]. Ever since their introduction [70, 71, 25], products of modal logics — propositional multimodal logics determined by classes of product frames — have been extensively studied, see [23, 58] for comprehensive expositions and further references. The first part of this thesis targets some open problems in this area.

Adding a dimension A way of adding a dimension to a modal logic L is to extend the language of L with the chosen language to describe the new dimension, let us say T, and obtain a logic $T(L)$, where the models for $T(L)$ are sequences of snapshots of L-models changing along the new dimension. In Figure 1.1 we see such an example where the new dimension is a commonly used time flow: $(\omega, <)$. In the simplest case, when frames are allowed to change without restriction the resulting logic is called the fusion of L and T [21, 53, 20]. Many possibilities and restrictions have been studied; check [52] for a survey on the result of this process with various values of L as spatial logics (from metric and topological spaces) and choices of T.

Products with linear orders We concentrate on the case where these snapshots are Kripke models based on the same Kripke frame, and only their valuations are changing along a new linearly ordered dimension. In this case $T(L)$-models are Kripke models over products of

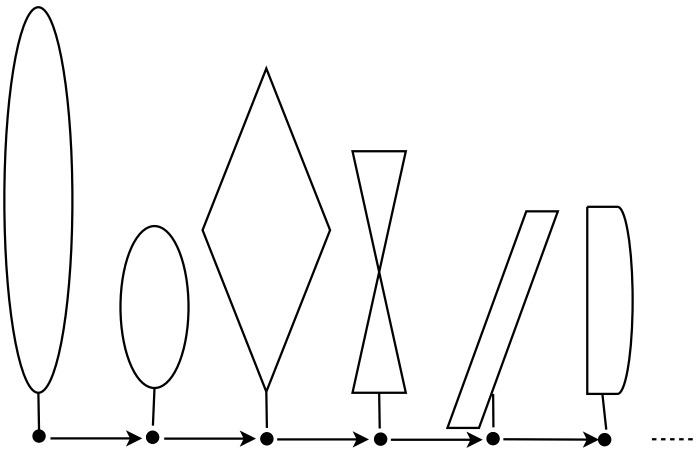

Figure 1.1: A structure changing along a time flow, $(\omega, <)$.

frames, where one of the component frames is a linear order (the 'time'[1]), see Figure 1.2. Moreover, we consider L and T to be unimodal logics over these components.

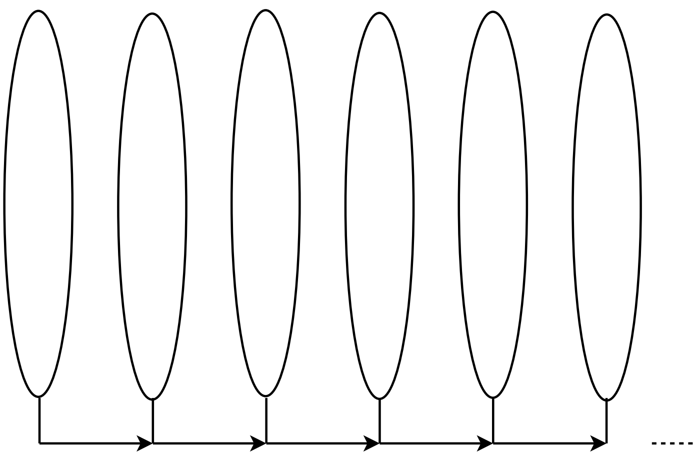

Figure 1.2: Product with $(\omega, <)$.

Our results In Section 3.2 we consider the problem of finding finite axiomatisations for products of two finitely axiomatisable modal logics and show the first examples of re-

[1]We use this term as a shorthand to designate the linear order component, and even though our our results apply to the usual linear flows of time, there are other non-linear models of time, having multiple (usually only towards the future) time lines [64], that are not considered. See [76] or [47] for an overview on temporal logics.

cursively enumerable (even decidable) two-dimensional products of finitely axiomatisable modal logics that are not finitely axiomatisable. In particular, we show that any axiomatisation of some bimodal logics that are determined by classes of product frames with a linearly ordered component must be infinite in two senses: it should contain infinitely many propositional variables, and formulas of arbitrarily large modal nesting-depth. Examples of such bimodal logics are given by $L \times \mathbf{K}$, where L is any of the logics of the most common models of the flow of time $\mathtt{Log}\{(\omega,<)\}$, $\mathtt{Log}\{(\mathbb{Q},<)\} = \mathtt{Log}\{(\mathbb{R},<)\}$, $\mathtt{Log}\{(\omega,\leq)\}$, $\mathbf{S4.3} = \mathtt{Log}\{(\mathbb{Q},\leq)\} = \mathtt{Log}\{(\mathbb{R},\leq)\}$, as well as the logic of all linear orders, **K4.3**. Finally, in Section 3.3 we present some results regarding the problem of finding an explicit axiomatisation of the considered logics.

Let us finalise with a modelling example.

Example 1.1.1. Let W be a set of cities. The fact that there is a flight from a city to another can be represented by a binary relation $R \subseteq W \times W$ (usually R should be symmetric). Clearly, $(\omega,<)$ can represent the successions of days (hours, or any unit of time) following a certain moment. Furthermore propositional symbols can be associated to predicates about various properties that depend both on the city and on the moment in question, e.g. temperature, pressure, sunshine, rain, U.V. degree, etc. If we consider that the flight connections are fixed, the product of $(\omega,<)$ and (W,R) is the right structure to model the evolution of such predicates together with the flight accessibility information. Let us exemplify what we can express in the associated bimodal logic. We have a modal operator associated with the time flow, $\Diamond_0$, and another, $\Diamond_1$, associated with the flight accessibility relation. So $\Diamond_0\varphi$ stands for φ is true somewhere in the future and $\Diamond_1\varphi$ stands for φ is true in some city reachable by flights in the considered airline. Furthermore let us give the following interpretation to some proposition symbols: s is true if it is sunny; $t_{\leq 15}$ if the temperature is inferior or equal to 15° celsius; r if it is raining; c if it is cloudy. See Table 1.1 for some possible statements regarding the situation in hand.

From the results proven in this thesis follows that the logic $\mathtt{Log}\{(\omega,<)\} \times \mathbf{T}$ is not finitely axiomatisable. That is the logic determined by all frames resulting of the product of frames for $\mathtt{Log}\{(\omega,<)\}$ by symmetric frames (modelling general symmetric flights routes). But, it is left open if we consider only flights that form more specific form if that is the case. For example if we consider only flights configurations forming an equivalence relation (stranger things have happened!), the obtained logic is $\mathtt{Log}\{(\omega,<)\} \times \mathbf{S5}$, to which our results do

Modal language	Natural language
$\Diamond_0 \Box_0 s$	There is a moment after which it will be always sunny (in the city we are).
$\Diamond_0 \Box_1 s$	There is a moment in which it will be sunny everywhere one can reach by flight.
$\Diamond_1 \Box_0 c$	There is a reachable city where it will be cloudy forever.
$\Diamond_1 \Diamond_0 (s \wedge r)$	There is a reachable city where one will find at a certain point sun and rain at a certain point (good for rainbows).
$(\Box_1 \Box_0 s \wedge \Box_0 s) \rightarrow \Diamond_0 \Box_0 \neg t_{\leq 15}$	If it is sunny forever here and everywhere we can reach then there will be a moment in time from which onwards the temperature will be superior to 15

Table 1.1: Possible statements in the considered modal semantics.

not apply. Interestingly, in this case if we have an extra operator corresponding to the next moment then we know that the resulting logic is finitely axiomatisable, see section's 3.4 last item.

1.2 Change by (re)action: reactivity

In computer science the word reactivity has been used to denote systems that react to their environment and are not meant to terminate, as coined by Pnueli and Harel in [42]. In this thesis the word has a different meaning, reactive systems are history-dependent relational structures, where the accessibility relation is determined not only by the point where one is, but also by the previous transitions. This concept was introduced by Dov Gabbay in 2004, see [28] and the extended version [29]. In the second part of this thesis we consider the concept of reactivity by presenting some structures that embody it and some logics to reason about them. Let us start by explaining how the concept of reactivity was born and outlining its short life-story.

New kind of arrows. In [28], Dov Gabbay introduced the idea of enriching graph-based structures with arrows of a new type, calling it the double arrows. Double arrows, instead of connecting points, connect arrows with arrows or other double arrows, see Figure 1.3. The idea is that this new kind of arrows can represent the dependence of the state of the targeted

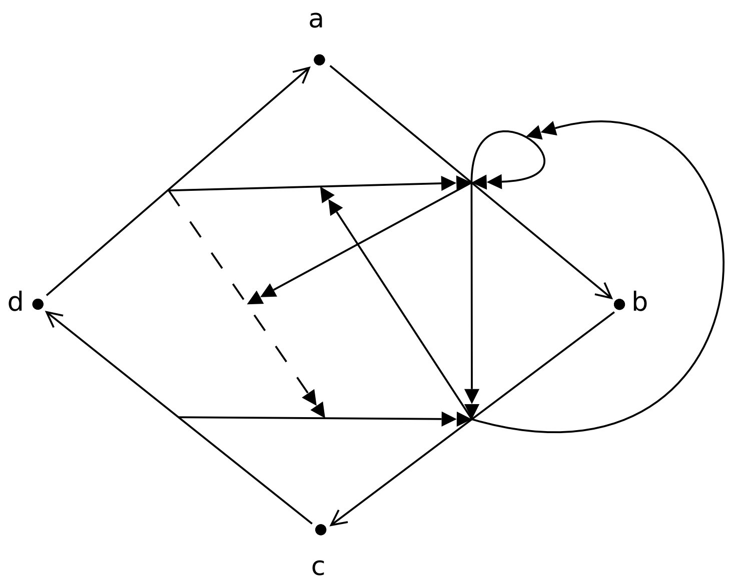

Figure 1.3: An enriched graph.

arrow (or double arrow) upon the crossing of the arrow in its origin. In this first presentation the double arrows would simply change the targeted arrow state. Let us see how it works by playing with the example in Figure 1.3. We represent the fact that an arrow is off by drawing its body as a dotted line. Let us see the effect that crossing some of its edges has. As shown

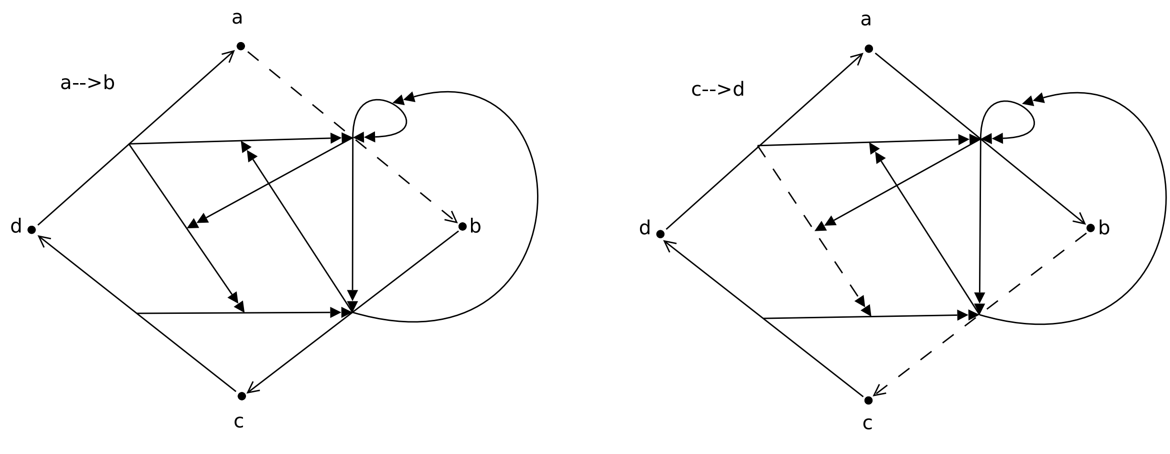

Figure 1.4: The effect of crossing edges.

in Figure 1.4, when we cross the edges (a,b) (left) or (b,c) (right), the arrows that are in the scope of the double arrows coming out of them, become off or on if they are on or off respectively (that is, their state changes). This process is cumulative, after crossing (a,b) we can also cross (b,c) and the effects are determined by the new state of double arrows, see Figure 1.5. These ideas were presented using suggestive motivational cases, for example in

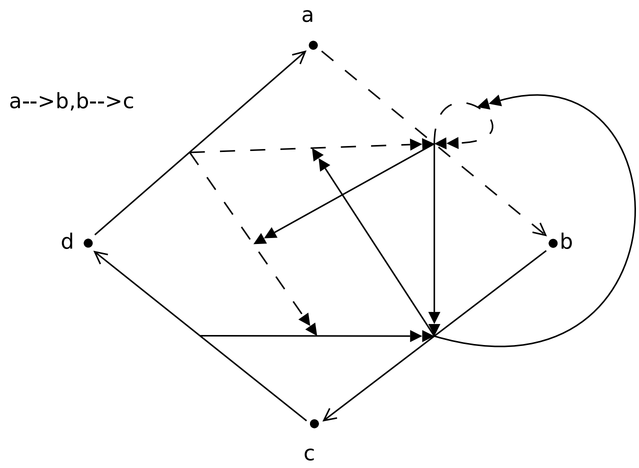

Figure 1.5: The effect of crossing edges.

Figure 1.6 we see how these new arrows can represent a classical inheritance networks case.

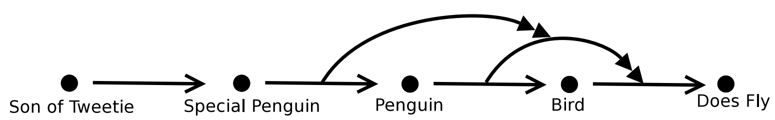

Figure 1.6: A classical inheritance networks example. We can represent simple exceptions: birds do fly, but although penguins are birds they do not fly. And we can also represent higher order exceptions (exceptions to exceptions): even though the son of Tweety is a special penguin (so also a penguin) he does fly.

The general idea that a relational structure may vary when one moves through it and the enriched kind of frames that came along with it fuelled publications in many areas. Indeed, there are applications of the reactive ideas in such diverse areas as modal logic, preferential non-monotonic logic, inheritance systems, context-free grammars, automata theory, deontic logic and contrary to duty, argumentation and other networks, see papers [8, 32, 16, 30, 33, 24, 34, 22, 62]. For example when one adds these kind of double arrows to the structure of an automata, one allows it to modify its transition relation while reading a sentence, that is, one makes it reactive. This alternative paradigm competes with non-determinism in the task of obtaining automata with minimal number of states accepting a language. Indeed, the following theorem (Proposition 6.1, [16]) is proven:

> If A is deterministic automaton with k^n states, it has an equivalent reactive automaton $R(A)$ with kn states.

Another interesting application of this kind of enriched graphs (though not using the dynamical counterpart) can be found in [62], where Dung's abstract argumentation theory is extended incorporating the meta-level argumentation-based reasoning, about possibly conflicting preferences between arguments.

Reactive logics: making Kripke reactive In [29] we can also find a semantics based on Kripke frames enriched with these double arrows, where the basic relation changes along the interpretation of a formula by the action of the double arrows. Subsequently it is proven that this semantics strictly generalises the Kripke semantics, in fact, we may have classes of these frames originating logics that are not closed under substitution.

In Chapter 4 we introduce a more abstract notion of reactive Kripke frames. Whereas in [29] the changes in the accessible relation are the ones produced by the action of the double arrows, in the abstract notion of reactive Kripke frames these changes are given. In reality, in the usual semantics of modal logic the only important information to the value of a modal formula is the set of successors at each moment, i.e the **local** accessibility relation. Therefore the notion of reactive Kripke frame boils down to a set of admissible sequences of points, that is, the set of admissible paths. One can picture the initial accessibility relation by considering the paths of size two, and its evolution is encoded in the one step prolongment relation on the bigger paths. The semantics presented in [29] is generalised over these abstract structures, allowing the valuation to vary along the paths, and the language is enriched with an extra operator relating paths that have the same endpoint. This is similar to what it is done in the branching-time logic with quantification over branches in [79].

Let us look to a concrete case and see some examples of what can be expressed in the considered language.

Example 1.2.1. Let us consider the situation of a traveller with a budget. The set of his possible moves depends on whether he has enough money to do them (to pay tolls, oil, train or flight tickets), furthermore his actual moves also determine his future possibilities. So the paths of the correspondent reactive frame are the sequences of cities he can visit with a certain budget. The formulas are interpreted over these paths. Let $\Diamond_R$ stand for the dynamics operator, that is, corresponding to the accessibility relation, and $\Diamond_P$ to the relation identifying the paths with the same endpoint. So, $\Diamond_R\varphi$ means that after the current path we can access to a city such that the resulting path satisfies φ. $\Diamond_R\varphi$ means that there is a path to the current city satisfying φ. Let us consider that the propositional symbols p_b and p_w correspond to the predicate of being able to buy bread and wine respectively, and m be true if there is still some money left. See Table 1.2 for examples of what can be said.

Modal language	**Natural language**
$\neg m \to \Box_R\bot$	If the traveller has no money left then he cannot move
$\Diamond_P(\Box_R p_b \wedge \Diamond_R\top)$	There is a path to the current city, after which the traveller is not blocked and he has enough money to buy bread every city he can access to
$(p_w \wedge \Box_R p_w) \to \Box_P p_b$	If the traveller can buy wine now, and at any immediate next stop, then he would always be able to buy bread in the current city regardless of how he got there

Table 1.2: Possible statements in the considered modal semantics.

The truth values of the sentences in Table 1.2 depend on the particular valuation we pick, but as we shall see, with this language we can capture interesting structural aspects of these frames. One obvious place to start is to consider the familiar relational notions of reflexivity, symmetry and transitivity. In this dynamical context, these properties have various possible generalizations. Symmetry may mean just that we can retrace our steps, that is, we can always return to a world where we have been, but it may mean something stronger, it may mean that when we get back we can access the same worlds as before. It may mean something less strong as well, it may mean that if we go from a to b then there is a path to b such that we can reach a from b. For instance in the case of the traveller above, it could make sense to say it is strongly reflexive, that is, if he stays in one city (supposing the staying cost is not supported by the travelling budget) then he can still reach the same worlds he could when he got there. In this chapter we study logics for classes of reactive frames satisfying this kind of properties, obtaining completeness and decidability results where we can.

Switch graphs While the notion of reactive frame contains exactly the necessary information to generalise the usual Kripke semantics to the reactive case, it ignores the state of the global accessibility relation. In order to model this global dependence we consider the concept of reactive graphs (Chapter 5). Intuitively a reactive graph consists in a graph that may change its configuration when a certain edge is crossed.

As in the case of reactive frames, we define a reactive graph to be a set of admissible sequences of edges. Clearly, from such a set one can extract the evolution of the whole relational structure while transversing the graph edges. For any admissible sequence λ, the relational state of the reactive graph after λ is given by

$$R_\lambda = \{(w, w') : \lambda(w, w') \text{ is an admissible sequence of edges}\}.$$

At this point a natural question arises: can all these relational behaviours be encoded by double arrows? In order to answer this question we introduce the concept of a switch graph. A switch graph is a graph enriched with two kinds of double arrows, the connecting and the disconnecting switches. As their names suggest, when the origin of a connecting/disconnecting switch is crossed its target is connected/disconnected.

Given a set W, a switch is either an edge $s \in W^2$ (switch of level 0, neither connecting or disconnecting) or a triple $s = (a, s', *)$ (of level $n > 0$), where

- $a \in W^2$ in the edge that triggers its action,
- s' is the targeted switch (of level $n - 1$)
- $* \in \{\bullet, \circ\}$ says if it is a connecting (black circle) or disconnecting (white circle) switch.

The type of the switches of level 0 is ϵ (the empty sequence) and of $s = (a, s', *)$ is $\sigma*$ where σ is the type of s'. We use the following notation to refer to switches in an easier fashion:

- $(ab, \epsilon) = (a, b)$,
- $(v_1 v_2, \ldots, v_{2n+1} v_{2n+2}, a, *_1 \ldots *_{n+1}) = ((v_1, v_2), (v_3 v_4, \ldots, v_{2n+1} v_{2n+2}, *_1 \ldots *_n), *_{n+1})$.

In graphical representations we use white headed arrows to represent the disconnecting switches and black headed arrow to represent the connecting ones. Let us see an example of a situation where the dynamical restrictions are easily represented by these structures.

Example 1.2.2. Switches can easily grasp the fact that certain resources are finite, that is, one can use them a finite number of times. Depending on the meaning of the accessibility relation (e.g. crossing a bridge, driving a road, taking a pill from a tablet, printing pages, ask a person for a cigarette, etc) the switch configuration presented in Figure 1.2.2 represents the fact that a particular action can be taken exactly $k > 1$ number of times. The set of switches is given by $\{(a, b), (ab, \ldots, ab, \circ\bullet^{k-1})\}$. For $k = 0$ we would have (a, b) and $(ab, ab, \circ)$ on, and would not need more switches.

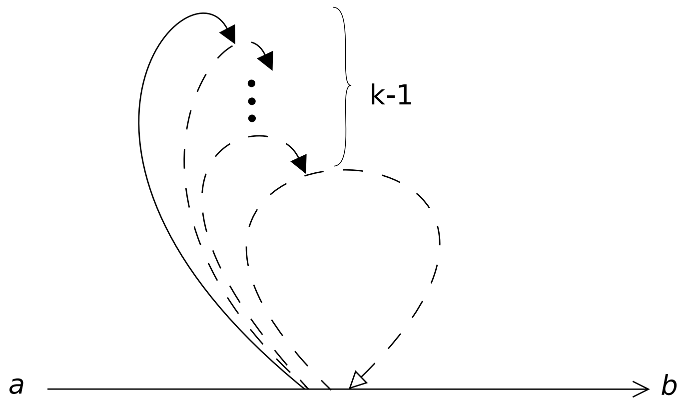

Figure 1.7: The edge (a, b) can be crossed exactly $k > 1$ times.

The main result of this section answers to the above question, we prove that any reactive graph can be generated by a switch graph. We also provide an example where switches are used to represent the dynamical restrictions required for the solution of the mutual exclusion problem.

Switch reactive hybrid logic We close this part by introducing an interpretation of a hybrid modal language over Kripke frames generated by switch graphs dynamics. Usually nominals are valid in exactly one point in each model. In the reactive setting they are valid in exactly one point for each of the components representing different reactive moments, with different relational states.

- Instead of having an operator relating the different relational states of a point, we use nominals to identify them.

- Also, using the fact that double arrows are relations (of different arities) over the carrier, we consider the correspondent modal operators. To each type of switches we consider the associated $2n + 2$ relation

  $$R^{\sigma} = \{(w_1, w_2, \ldots, w_{2n+1} w_{2n+2}) : (w_1 w_2, \ldots, w_{2n+1} w_{2n+1}, \sigma) \textbf{ is on}.\}$$

  for $|\sigma| = n$, and its correspondent modal operator $2n + 1$-ary modal operator $\Diamond_{\sigma}$.

- Moreover the use of the hybrid operator @ allows us to have a global view over the switch configuration at each moment.

- To the fragment of the language introduced above, that allows us to talk about the switches state in each moment, we add the modal operator $\diamondslash$ relating the different states of the switch graph, being the real dynamics operator (corresponding to $\Diamond_R$ in the reactive Kripke frames).

The task of this chapter is to give an axiomatisation of the switches evolution by capturing the interaction between these components.

Let us look at some examples of what we may express in this language.

Example 1.2.3. Let us consider a non-local version of Example 1.2.1. Instead of considering a single traveller, that can be only at one place at a time, let us consider the same problem but with a truck company (or group of travellers) with a common budget. Clearly each move affects all the subsequent possible moves. We know from the result proven in Chapter 5 that any such reactive dynamics can be expressed by switches. If we consider a switch graph generating this dynamics and the language described above we may express the local interdependencies explicitly, e.g.

$$@_a \wedge \diamondvert_{\circ}(b, c, d)$$

means that the fact that a truck in a goes to b implies that no truck in c can go to d. That is

$$@_a \diamondslash (b \to @_c \neg \diamondslash d).$$

In the case of Example 1.2.2 dealing with bounded resources, where we allow only certain kinds of switches:

$$@_a(\text{⟐}_\epsilon b \wedge \text{⟐}_\circ(b,a,b))$$

means that (a,b) can be crossed exactly once, $@_a \,\Diamond\!\!\!\!/\, (b \to @_a \neg \Diamond\!\!\!\!/\, b)$,

$$@_a(\text{⟐}_\epsilon b \wedge \neg\, \text{⟐}_\circ\, (b,a,b) \wedge \neg\, \text{⟐}_{\circ\bullet}\, (b,a,b,a,b) \wedge \text{⟐}_{\circ\bullet\bullet}(b,a,b,a,b,a,b))$$

means that (a,b) can be crossed exactly three times, $@_a \,\Diamond\!\!\!\!/\, (b \to @_a \,\Diamond\!\!\!\!/\, (b \to @_a \neg \Diamond\!\!\!\!/\, b))$, and so on.

We do not claim that this is the most appropriate language to reason about all cases of reactivity. For instance, in Examples 1.2.1 and 1.2.3 we can envisage a language that could explicitly reason about the cost of each move and the remaining budget after it. And in the case of Example 1.2.2, where only some shapes of switches are allowed, the language could be simplified. Still, the fact is that all reactive systems can be generated by switches and this language seems adequate to express the local dynamic dependencies on each move, imposed by the switches. We hope that it represents a kind of skeleton for the various possibilities. In the conclusion of Chapter 6 we discuss possible extensions of this language with operators of the kind we find in CTL and CTL^*. The main result of this chapter is the proof of the usual hybrid completeness result in this dynamic context. An important fact to retain is that in each moment the whole future relational dynamics is coded in the switches, so it may be that some properties (that depend only on the worlds - corresponding to pure formulas, having no propositional symbol that is not a nominal) can be derived by reasoning locally, using only the information contained in switches configuration.

1.3 Thesis Outline

The detailed structure of the thesis is as follows:

Chapter 2 The background notions and the main used results are presented.

Chapter 3 We consider the problem of axiomatising products of modal logics where the frames for one component are linear orders, covering the usual time-flow models.

We start by presenting the definitions and results of general 2-products of modal logics in Section 3.1. It is known that if the classes of frames for the product component logics are definable by recursive sets of first-order sentences, then their product is a recursively enumerable bimodal logic [25]. It was also known, as is shown in [23, Theorem .5.15], that no product logic of the form $\mathbf{K4.3} \times L$ is product-matching, whenever L is any Kripke complete modal logic containing **K4** and having the two-element reflexive chain among its frames. Being open if these logics have finite axiomatisations.

In Section 3.2 answers very generally to this question. We prove that for many of these logics there is no finite axiomatisation, and that this non finiteness comes in two forms. Let L be any bimodal logic:

- If L contains $\mathbf{K4.3} \times \mathbf{K}$ and the product of $(\omega, \leq)$ and an (irreflexive or reflexive) ω-fan is a frame for L, then L is not axiomatisable using finitely many propositional variables (Theorem 3.2.1).
- If $\mathbf{K4.3} \times \mathbf{K} \subseteq L \subseteq \mathtt{Log}\{(\omega, \leq)\} \times \mathbf{K}$ then every axiomatisation of L must contain formulas of arbitrarily large vertical depth (Theorem 3.2.3).

These results give negative answers to questions in [25], and to Questions 5.18 and 5.19 in [23].

In Section 3.3 we present some preliminary results towards finding an explicit axiomatisation for such (recursively enumerable) logics. We also prove that $\mathbf{K4.3} \times \mathbf{S5}$ is not product matching (Theorem 3.3.13). This chapter is joint work with Agi Kurucz.

Chapter 4 In Section 4.1 we extend the semantics over reactive Kripke structures introduced in [29] in two ways (Definition 4.1.2): we define a more general changing Kripke structure, called reactive Kripke frame, that allows us to consider all possible relational changes, not being limited by specific representations; and we add an extra operator ranging through all the reactive states of a point. In order to use classical tools in the study of the resulting logics we translate these structures into more familiar ones. We define a correspondence between reactive frames and a certain sub-class of bimodal frames for $(\mathbf{K}, \mathbf{S5})$ (called cs-frames and shattered frames respectively, Definition 4.1.7), such that there is a bijection between their models that preserves truthness (Remark 4.1.9 and Theorem 4.1.10), concluding that they define the same logic (Corollary 4.1.11).

In Section 4.2 we establish various completeness results using variations of the so-called 'blow up method' (also used in [40, 56]): general completeness (Theorem 4.2.4); for generalisations of reflexivity (Theorems 4.2.8 and 4.2.10); for generalisations of transitivity (Theorems 4.2.9 and 4.2.11); the static and quasi-static (Theorems 4.2.12 and 4.2.13); strong reflexivity plus strong symmetry (Theorem 4.2.15).

In Section 4.3 we establish some decidability results using variations of the filtration technique getting an upper bound for the general system, the generalised reflexivities and strong reflexivity of *co-NEXPTIME* (Corollary 4.3.2) and for the static and quasi-static of *co-2NEXPTIME* (Corollary 4.3.5).

For a more complete description of these results see Table 4.1. Part of this chapter is joint work with Dov Gabbay and has been published in [31].

Chapter 5 We start by introducing the concept of a reactive graph (Definition 5.1.1), corresponding to the abstract notion of a(n accessibility) relation that depends on the previously crossed edges. We pick the idea introduced in [29], of locally representing the dependency between crossing an edge and the (global) state of the relation by adding higher order arrows expressing it, and prove that such structures, the switch graphs (Definition 5.1.3), can express all the relational behaviours, that is, all reactive graphs are generated by switch graphs (Theorem 5.1.8). This result allows us to see the switches formalism as a possible representative of the paradigm of reactivity. They are expressive enough to cover all reactive relational dynamics and we believe they provide a useful tool for modelling real situations, by gathering in an homogeneous structure the various (meta-)levels of reactivity.

In Section 5.2 we define the dynamics of k elements over a reactive graph (or over a switch graph, that is in the reactive graph generated by it) (Definition 5.2.1), and give an example of the use of these structures in modelling the problem of mutual exclusion (Example 5.2.2).

Chapter 6 In Section 6.1 we introduce a hybrid logic to reason about the switch dynamics introduced in Chapter 5. We adapt hybrid semantics to this dynamical context (Definition 6.1.2). We use the power of names to identify worlds in different reactive moments and @ to have a global view on the current configuration of the switches (read here as usual relations over the set of worlds thus having the classical modal correspondents). In Section 6.2 we introduce an axiomatic system (Figure 6.3) that describes the interaction between these elements. We then prove the corresponding completeness theorem, with the usual immediate completeness for additional pure axioms (Theorem 6.2.2), confirming that the switch behaviour was captured.

We close each chapter with a section where we discuss the obtained results and present a list of open questions.

2

Preliminaries

We assume the reader is familiar with the basics of propositional, predicate and modal logics ([15, 11]), computability and complexity ([63]) notions. Let us here summarise the necessary notions and notation for the thesis.

2.1 Modal Logic

Here we introduce some of the fundamental concepts of modal logic that we use throughout the thesis. The presented definitions and results[1] can be found in every textbook presentation of the subject of modal logic, we used [38, 15, 11] as references and rely on them also regarding what we do not include here.

2.1.1 Syntax

Languages

Definition 2.1.1. *Modal similarity* type is a pair $\tau = (O, \rho)$ where O is a non-empty set, and ρ is a function $O \to \omega$. The elements of O are called modal operators; we use $\Diamond$'s ('diamonds' with different subscripts or/and drawing inside) to denote elements of O. The function ρ assigns to each operator a finite arity.

The *set of modal formulas*[2] $\mathcal{L}_\tau(\Pi)$ (or $\mathcal{L}_\tau$ for short) is built up using a modal similarity

[1]We do not give the original references but we say where one can find the results in [11].

[2]We have chosen a general notation to present these concepts, using polyadic modal operators, because they are crucial in part II. We realise this is not the usual choice but in an introduction but, exactly because of that, we feel it may be useful to make the reader used to the general case from the start.

type $\tau = (O, \rho)$ and a denumerable set of propositional letters (or variables) Π using the following rule:

$$\varphi \ ::= \ p \mid \neg\varphi \mid \varphi_1 \wedge \varphi_2 \mid \Diamond(\varphi_1, \ldots, \varphi_{\rho(\Diamond)}) \,.$$

The other connectives: $\top$, $\bot$, $\vee$, $\rightarrow$, $\leftrightarrow$ are introduced by the usual abbreviations. And dual operators of non-nullary operators are defined as $\Box(\varphi_1, \ldots, \varphi_{\rho(\Diamond)}) = \neg \Diamond (\neg\varphi_1, \ldots, \neg\varphi_{\rho(\Diamond)})$.

Modal nesting-depth

Definition 2.1.2. Given a formula $\varphi \in \mathcal{L}_\tau$ and a $\Diamond \in O$ we define the *$\Diamond$-modal nesting-depth* (or modal depth) of φ, $md_\Diamond$ inductively as follows:

$$\begin{aligned}
md_\Diamond(p) &= 0,\\
md_\Diamond(\neg\varphi) &= md_\Diamond(\varphi),\\
md_\Diamond(\varphi_1 \wedge \varphi_2) &= max\{md_\Diamond(\varphi_1), md_\Diamond(\varphi_2)\},\\
md_\Diamond(\Diamond(\varphi_1, \ldots, \varphi_{\rho(\Diamond)})) &= 1 + max\{md_\Diamond(\varphi_1), \ldots, md_\Diamond(\varphi_{\rho(\Diamond)}),\\
md_\Diamond(\Diamond'(\varphi_1, \ldots, \varphi_{\rho(\Diamond)}) &= max\{md_\Diamond(\varphi_1), \ldots, md_\Diamond(\varphi_{\rho(\Diamond)}), \text{ if } \Diamond \neq \Diamond'.
\end{aligned}$$

We can also refer to the modal depth of a formula without identifying a specific operator in the subscript, $md(\varphi)$ is defined as above by removing the last line.

Formulas and sets of formulas

Definition 2.1.3. A *substitution* is a map $\sigma : \Pi \rightarrow \mathcal{L}_\tau$ and it induces a map $(.)^\sigma : \mathcal{L}_\tau \rightarrow \mathcal{L}_\tau$ (the process of uniform substitution), which is recursively defined as follows:

$$\begin{aligned}
p^\sigma &= \sigma(p),\\
(\neg\varphi)^\sigma &= \neg\varphi^\sigma,\\
(\varphi_1 \wedge \varphi_2)^\sigma &= \varphi_1^\sigma \wedge \varphi_2^\sigma,\\
\Diamond(\varphi_1, \ldots, \varphi_{\rho(\Diamond)})^\sigma &= \Diamond (\varphi_1^\sigma, \ldots, \varphi_{\rho(\Diamond)}^\sigma).
\end{aligned}$$

Axiomatisations

We consider the question of axiomatisability in relation to logics (sets of formulas) that are defined syntactically in different ways, so we leave here just a general definition. An axiomatisation system is a set of base axioms Σ and a set of closure rules (rules of the form: if $a \in A$ then $R(a) \subseteq A$). We say that a set of formulas Σ' (in a particular axiomatisation system) axiomatises a logic L if L is the smallest set of formulas containing $\Sigma \cup \Sigma'$ and closed under the closure rules. All logics we consider are normal modal logics in the following sense:

Definition 2.1.4. We call modal logic to any set of formulas $L \subseteq \mathcal{L}_\tau$ that contains all propositional tautologies and is closed under modus ponens (that is $\varphi, \varphi \to \varphi' \in L$ then $\varphi' \in L$). We say that $L \subseteq \mathcal{L}_\tau$ is *uniform* it it is closed under uniform substitution and that it is *normal* if for all $\Diamond \in O$ and $\varphi, \psi, \varphi_1, \ldots, \varphi_{\rho(\Diamond)}$ we have:

$$\vdash \Box(\varphi_1, \ldots, \varphi \to \psi, \ldots, \varphi_{\rho(\Diamond)}) \to (\Box(\varphi_1, \ldots, \varphi, \ldots, \varphi_{\rho(\Diamond)}) \to \Box(\varphi_1, \ldots, \psi, \ldots, \varphi_{\rho(\Diamond)})),$$

where $\varphi \to \psi$, in the antecedent, and φ and ψ, in the consequent, occur in the *i*-th position for $1 \leq i \leq \rho(\Diamond)$. And if it is closed for the rule of necessity (or generalisation):

$$\vdash \varphi \text{ implies } \vdash \Box(\bot, \ldots, \bot, \varphi, \bot, \ldots, \bot),$$

where occurs in the *i*-th position for $1 \leq i \leq \rho(\Diamond)$.

In Part I the axiomatic system considered is formed by the rules of modus ponens, generalisation plus the rule of uniform substitution. But in Part II not all logics are closed under uniform substitution. Furthermore in Chapter 6 we use some non-orthodox rules including side conditions.

Definition 2.1.5. Let $L \cup L' \cup \{\varphi\} \subseteq \mathcal{L}_\tau$ and $X \subseteq \Pi$, we then define:

- (L, L') is the closure by the rules of modus ponens and necessity of the set

$$\{\, \psi : \psi \text{ is a propositional instance of } \varphi \in L \cup L' \,\}.$$

- $L \oplus \varphi = (L, \{\varphi\})$.
- $L +_X \varphi$ is the closure by the rules of modus ponens and necessity of the set

$$L \cup \{\, \psi : \psi \text{ is the result of substituting the variables of } \varphi \text{ by variables in } X \,\}.$$

Logics: Consistency and maximal consistent sets In order to define consistency we use the general notion of axiomatisation for propositional-based logics.

Definition 2.1.6. A *logic* $L \subseteq \mathcal{L}_\tau$ is a set of formulas that contains all propositional tautologies and is closed under modus ponens (that is $\varphi, \varphi \to \varphi' \in L$ then $\varphi' \in L$). Every element of a logic $\varphi \in L$ is called *theorem*, written $\vdash_L \varphi$. Given $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}_\tau$ we say that φ is *L-deducible* from Γ if there exist $\varphi_1, \ldots, \varphi_n \in \Gamma$ such that

$$\vdash_L \varphi_1 \to (\varphi_2 \to (\ldots \to (\varphi_n \to \varphi)\ldots)).$$

A set $\Gamma \subseteq \mathcal{L}_\tau$ is *L-consistent* if $\Gamma \nvdash_L \bot$, and it is a *L-maximal consistent* set (L-MCS) if it is L-consistent and for all $\varphi \in \mathcal{L}_\tau$ then either $\varphi \in \Gamma$ or $\neg\varphi \in \Gamma$.

Lemma 2.1.7 (Lindenbaum's lemma: Lemma 4.17, [11])**.** *Every L-consistent set of formulas is contained in a L-maximal consistent set.*

2.1.2 Semantics

A *relational structure* is a tuple $\mathfrak{F}$ whose first component is a non-empty set W called the universe (or domain) of $\mathfrak{F}$, and whose remaining components are relations on W. The elements of W can be named: points, states, nodes, worlds, times, instants, etc. When a relation is binary we say it is an accessibility relation.

Frames and models

Definition 2.1.8. A frame for the similarity type $\tau = (O, \rho)$ (τ-frame) is a tuple $\mathfrak{F}$ consisting of the following ingredients:

- a non-empty set W,
- for each $\diamond \in O$ an $\rho(\diamond) + 1$-ary relation $R_\diamond$, we say $\diamond$ is the modal correspondent of $R_\diamond$.

A τ-model is a pair $\mathfrak{M} = (\mathfrak{F}, V)$ where $\mathfrak{F}$ is a τ-frame and V is a valuation, that is a function from $\Pi \to 2^W$, $\mathfrak{M}$ is also called a model over $\mathfrak{F}$. The notion of a formula φ being satisfied (or true) at a state w in a model $\mathfrak{M}$ ($\mathfrak{M}, w \vDash \varphi$) is defined inductively:

- $\mathfrak{M}, w \vDash p$ iff $w \in V(p)$ for $p \in \Pi$,

- $\mathfrak{M}, w \vDash \neg\varphi$ iff $\mathfrak{M}, w \nvDash \varphi$,
- $\mathfrak{M}, w \vDash \varphi_1 \wedge \varphi_2$ iff $\mathfrak{M}, w \vDash \varphi_1$ and $\mathfrak{M}, w \vDash \varphi_2$,
- $\mathfrak{M}, w \vDash \Diamond(\varphi_1, \ldots, \varphi_{\rho(\Diamond)})$ iff there are $v_1, \ldots, v_{\rho(\Diamond)} \in W$ with $(w, v_1, \ldots, v_{\rho(\Diamond)}) \in R_\Diamond$ and for each i, $\mathfrak{M}, v_i \vDash \varphi_i$.

Definition 2.1.9. We say a formula φ is satisfiable in a model if it is true at some point in that model. A formula is satisfiable in a frame if it is satisfiable in a model over that frame. We say a formula φ is valid at a state w in a frame $\mathfrak{F}$, $\mathfrak{F}, w \vDash \varphi$ if φ is true at w in every model based on $\mathfrak{F}$; φ is valid in $\mathfrak{F}$, $\mathfrak{F} \vDash \varphi$, if it is valid at every state in $\mathfrak{F}$. The notion of validity is easily extended to sets of formulas, Γ, $\vDash \Gamma$. Given a class of frames, F, the logic of F is the set of all formulas that are valid in all frames in F:

$$\mathsf{Log}\{\mathsf{F}\} = \{\varphi : \mathsf{F} \vDash \varphi\}.$$

It is easy to see that the logic of a class of frames is normal.

Frame and model constructions

Definition 2.1.10. Given two frames $\mathfrak{F} = (W, R_\Diamond)_{\Diamond \in O}$ and $\mathfrak{F}' = (W', R'_\Diamond)_{\Diamond \in O}$, we say that $\mathfrak{F}'$ is a *subframe* of $\mathfrak{F}$ if $W' \subseteq W$ and for all $\Diamond \in O$ we have $R'_\Diamond = R_\Diamond \cap W'^{\rho(\Diamond)}$. Furthermore, we say that $\mathfrak{F}'$ is a *generated subframe* of $\mathfrak{F}$ ($\mathfrak{M}$) if it is a subframe and the following closure condition is fulfilled for all $\Diamond \in O$:

$$\text{if } u \in W' \text{ and } (u, v_1, \ldots, \varphi_{\rho(\Diamond)}) \in R_\Diamond \text{ then } v_1, \ldots, \varphi_{\rho(\Diamond)} \in W'.$$

Let $W' \subseteq W$, the *subframe generated by* W' is the smallest generated subframe of $\mathfrak{F}$ whose domain contains W'. A *rooted or point-generated* frame is a frame that is generated by a singleton set, the element of which is called the *root* of the frame. All these concepts are extended to models $\mathfrak{M} = (\mathfrak{F}, V)$ and $\mathfrak{M}' = (\mathfrak{F}', V')$, if we further have that for $p \in \Pi$, $V' = V(p) \cap W'$.

Definition 2.1.11 (p-Morphism)**.** Given two frames $\mathfrak{F} = (W, R_\Diamond)_{\Diamond \in O}$ and $\mathfrak{F}' = (W', R'_\Diamond)_{\Diamond \in O}$, by a *p-morphism* (or bounded morphism) we intend a function $f : W \to W'$ satisfying the following properties:

- if $(v_0, \ldots, v_{\rho(\Diamond)}) \in R_\Diamond$ then $(f(v_0), \ldots, f(v_{\rho(\Diamond)})) \in R'_\Diamond$, for each $\Diamond \in O$ (forward or homomorphism condition),

- if $(f(v_0), v'_1, \ldots, v'_{\rho(\Diamond)}) \in R'_\Diamond$ then there exist $v_1, \ldots, v_{\rho(\Diamond)} \in W$ such that $(v_0, \ldots, v_{\rho(\Diamond)}) \in R_\Diamond$, for each $\Diamond \in O$ (backward condition).

To extend this concept to models $\mathfrak{M} = (\mathfrak{F}, V)$ and $\mathfrak{M}' = (\mathfrak{F}', V')$ we require that if $w \in V(p)$ then $f(w) \in V'(p)$.

Standard translation

Definition 2.1.12 (Standard Translation)**.** For a similarity type τ and a set of propositional letters Π, let $\mathcal{L}^1_\tau(\Pi)$ be the first-order language which has unary predicates $P_0, P_1, P_2 \ldots$ corresponding to the proposition letters $p_0, p_1, p_2, \cdots \in \Pi$, and $(n+1)$-ary symbol $R_\Diamond$ for each (n-ary) modal operator $\Diamond$ in our similarity type. Let $\mathcal{L}^2_\tau(\Pi)$ be the second-order language extending $\mathcal{L}^1_\tau(\Pi)$ where the P_i are the second-order variables. We call ST the *standard translation* from modal language to first-order logic:

$$
\begin{aligned}
ST_x(p) &= Px,\\
ST_x(\neg\varphi) &= \neg ST_x(\varphi),\\
ST_x(\varphi \wedge \varphi') &= ST_x(\varphi) \wedge ST_x(\varphi'),\\
ST_x(\Diamond(\varphi_1, \ldots, \varphi_{\rho(\Diamond)})) &= \exists y_1 \ldots y_{\rho(\Diamond)} \, ((x, y_1, \ldots, y_{\rho(\Diamond)}) \in R_\Diamond \wedge\\
&\qquad ST_{y_1}(\varphi_1) \wedge \ldots \wedge ST_{y_{\rho(\Diamond)}}(\varphi_{\rho(\Diamond)})),
\end{aligned}
$$

where $y_1 \ldots y_{\rho(\Diamond)}$ are fresh variables (that is, variables that have not been used so far in the translation).

Proposition 2.1.13 (Local and global correspondence: Propositions 2.47 and 3.12, [11])**.** *Given a model $\mathfrak{M} = (\mathfrak{F}, V)$ and $\varphi \in \mathcal{L}_\tau$, then:*

- $\mathfrak{M}, w \vDash \varphi$ *iff* $\mathfrak{M}, w \vDash ST_x(\varphi)[w]$ *(local model),*

- $\mathfrak{M} \vDash \varphi$ *iff* $\mathfrak{M} \vDash \forall x\, ST_x(\varphi)$ *(global model),*

- $\mathfrak{F}, w \vDash \varphi$ *iff* $\mathfrak{F}, w \vDash \forall P_1 \ldots P_n\, ST_x(\varphi)[w]$*(local frame),*

- $\mathfrak{F} \vDash \varphi$ *iff* $\mathfrak{F} \vDash \forall P_1 \ldots P_n \forall x\, ST_x(\varphi)$ *(global frame),* *where P_i bind the second-order variables P_i corresponding to the proposition letters p_i occurring in φ.*

Definition 2.1.14 (Modal definability)**.** We say a formula φ *defines* (or characterises) a class of frames F if for all frames $\mathfrak{F}$, $\mathfrak{F} \in \mathsf{F}$ iff $\mathfrak{F} \vDash \varphi$. Similarly if Γ is a set of formulas we say that Γ defines F if $\mathfrak{F} \in \mathsf{F}$ iff $\mathfrak{F} \vDash \Gamma$. We say a class of frames is *(modally) definable* if there is a set of modal formulas that defines it. We say a formula *defines a property* if it satisfies the class of frames that satisfies that property.

2.1.3 Some standard modal logics

Schema is a collection of formulas that share the same syntactic form, it can be given by all the uniform substitutions of a certain formula, it is useful to present specific uniform normal logics. the set of formulas is usually represented by the schemata formula that give origin to it. For example to represent the minimal (unary unimodal) normal logic we just need the schemata

$$\mathbf{K} : \Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi).$$

It as become customary to use the notation $K\Sigma_1 \ldots \Sigma_n$ to refer to the smallest normal logic containing the schemata $\Sigma_1, \ldots, \Sigma_n$.

Set theoretically this logic is defined as

$$\bigcap\{\Gamma : \Gamma \text{ is normal and } \Sigma_1 \cup \ldots \cup \Sigma_n \subseteq \Gamma\}$$

Historical names for some well-known schemata are:

$$\begin{array}{ll}
\mathbf{T}: & \Box\varphi \to \varphi \\
\mathbf{B}: & \varphi \to \Box \Diamond \varphi \\
\mathbf{4}: & \Box\varphi \to \Box \Box \varphi \\
\mathbf{5}: & \Diamond\varphi \to \Box \Diamond \varphi \\
\mathbf{D}: & \Box\varphi \to \Diamond\varphi \\
\mathbf{L}: & \Box(\varphi \wedge \Box\varphi \to \psi) \vee \Box(\psi \wedge \Box\psi \to \varphi) \\
\mathbf{W}: & \Box(\Box\varphi \to \varphi) \to \Box\varphi \\
\mathbf{Z}: & \Box(\Box\varphi \to \varphi) \to (\Diamond \Box \varphi \to \Box\varphi)
\end{array}$$

$$\mathbf{X}: \quad \Box\Box\varphi \to \Box\varphi$$

$$\mathbf{Dum}: \quad \Box(\Box(\varphi \to \Box\varphi) \to \varphi) \to (\Diamond\Box\varphi \to \varphi)$$

$$\mathbf{grz}: \quad \Box(\Box(\varphi \to \Box\varphi) \to \varphi) \to \varphi$$

Some of these schemata define (first-order) properties of binary relations:

T	Reflexive	$\forall x\ (xRx)$
B	Symmetric	$\forall x, y\ (xRy \to yRx)$
4	Transitive	$\forall x, y, z\ (xRyRz \to xRz)$
5	Euclidean	$\forall x, y, z\ (xRy \wedge xRz \to yRz)$
D	Serial	$\forall x \exists y\ (xRy)$
L	Weakly connected	$\forall x, y, z\ (xRy \wedge xRz \to (yRz \vee y = z \vee zRy))$
X	Weakly dense	$\forall x, y\ (xRy \to \exists z\ (xRzRy))$

Names of some well-known logics are

$$\begin{aligned} \mathbf{S4} &= \mathbf{KT4} \\ \mathbf{S5} &= \mathbf{KT4B} \\ \mathbf{Gl} &= \mathbf{KW} \\ \mathbf{K4.3} &= \mathbf{K4L} \\ \mathbf{S4.3} &= \mathbf{KT4L} \\ \mathbf{Grz} &= \mathbf{Kgrz} \end{aligned}$$

The logics of the linear orders (routed transitive weakly connected frames) most used to model the flow of time are the following

$$\begin{aligned} \mathbf{K4DLZ} &= \mathsf{Log}\{(\omega, <)\} \\ \mathbf{K4DLX} &= \mathsf{Log}\{(\mathbb{Q}, <)\} = \mathsf{Log}\{(\mathbb{R}, <)\} \\ \mathbf{KT4LDum} &= \mathsf{Log}\{(\omega, \leq)\} \\ \mathbf{S4.3} &= \mathsf{Log}\{(\mathbb{Q}, \leq)\} = \mathsf{Log}\{(\mathbb{R}, \leq)\} \end{aligned}$$

2.1.4 Some general results

Invariance under p-morphisms

Proposition 2.1.15 (Proposition 2.14 and Theorem 3.14, [11]). *Modal satisfaction is invariant under p-morphisms. That is, if $f : \mathfrak{M} \to \mathfrak{M}'$ is a p-morphism, then:*

$$\mathfrak{M}, w \vDash \varphi \textit{ iff } \mathfrak{M}', f(w) \vDash \varphi.$$

As a simple consequence we have that if $f : \mathfrak{F} \to \mathfrak{F}'$ is a surjective p-morphism then:

$$\mathfrak{F} \vDash \varphi \textit{ then } \mathfrak{F}' \vDash \varphi,$$

that is, $\mathsf{Log}\{\mathfrak{F}\} \subseteq \mathsf{Log}\{\mathfrak{F}'\}$.

Finite model property

Definition 2.1.16. We say that the logic of a certain class of models has the *finite model property* (f.m.p) if every formula $\varphi \in \mathcal{L}_\tau$ satisfiable in some model in that class is also satisfiable in a finite model of that class.

Finite models via filtrations

Definition 2.1.17. A set of formulas $\Gamma \subseteq \mathcal{L}_\tau$ is *closed under subformulas* if for all formulas $\varphi, \varphi', \varphi_0, \dots, \varphi_n$: $\neg\varphi \in \Gamma$ then $\varphi \in \Gamma$; $\varphi \wedge \varphi' \in \Gamma$ then $\varphi, \varphi' \in \Gamma$; $\Diamond(\varphi_0, \dots, \varphi_n) \in \Gamma$ then $\varphi_0, \dots, \varphi_n \in \Gamma$.

Given a τ-model $\mathfrak{M} = (W, R_\Diamond, V)_{\Diamond \in O}$ and a set $\Gamma \subseteq \mathcal{L}_\tau$ closed under subformulas, let $\mathfrak{M}_\Gamma = (W_\Gamma, R'_\Diamond, V')_{\Diamond \in O}$ such that:

$$W_\Gamma = \{|w| : w \in W\},$$

$$\text{If } (v_0, v_1, \dots, \varphi_{\rho(\Diamond)}) \in R_\Diamond \text{ then } (|v_0|, |v_1|, \dots, |\varphi_{\rho(\Diamond)}|) \in R'_\Diamond,$$

where

$$\Gamma_v = \{\varphi : \varphi \in \Gamma \;\&\; \mathfrak{M}, v \vDash \varphi\},$$

$$v \sim_\Gamma w \text{ iff } \Gamma_v = \Gamma_w,$$

$$|w| = \{t : w \sim_\Gamma t\},$$

$$V'(p) = \{|w| : \mathfrak{M}, w \vDash p\}.$$

We say that $\mathfrak{M}_\Gamma$ is a *filtration* of $\mathfrak{M}$ through Γ.

Proposition 2.1.18 (Proposition 2.38, [11]). *Let* $\Gamma \subseteq \mathcal{L}_\tau$ *be finite and closed under subformulas, then for all* $\mathfrak{M}$, $\mathfrak{M}_\Gamma$ *contains at most* $2^{|\Gamma|}$ *worlds.*

Theorem 2.1.19 (Filtration theorem: Theorem 2.39, [11]). *Let* $\mathfrak{M}_\Gamma$ *be a filtration of* $\mathfrak{M}$ *through* $\Gamma \subseteq \mathcal{L}_\tau$ *closed under subformulas. Then for all* $\varphi \in \Gamma$ *and* $w \in \mathfrak{M}$ *we have that*

$$\mathfrak{M}, w \vDash \varphi \textit{ iff } \mathfrak{M}_\Gamma, |w| \vDash \varphi.$$

Soundness and completeness

Definition 2.1.20. Given a class of frames (or models) F and a logic L, we say L is *sound* with respect to F if $L \subseteq L_\mathsf{F}$, and *complete* if the converse holds.

Canonical models

Definition 2.1.21 (Canonical model). Given a normal modal logic $L \subseteq \mathcal{L}_\tau$ we define the *canonical model* $\mathfrak{M}^L = (W^L, R^L_\diamond, V^L)_{\diamond \in O}$ for L where

$$\begin{aligned} W^\Gamma &= \{\Gamma : \Gamma \text{ is a } L - MCS\}, \\ (w, u_1, \ldots, u_{\rho(\diamond)})R^\Gamma_\diamond &\text{ iff for all formulas } \varphi_1 \in u_1, \ldots, \varphi_{\rho(\diamond)} \in u_{\rho(\diamond)} \diamond (\varphi_1, \ldots, \varphi_{\rho(\diamond)}) \in w, \\ V^\Gamma(p) &= \{w \in W^\Gamma : p \in w\}. \end{aligned}$$

$\mathfrak{F}^L = (W^L, R^L_\diamond)_{\diamond \in O}$ is called the *canonical frame* for L.

Lemma 2.1.22 (Truth lemma: Lemma 4.21, [11]). *For any normal modal logic* Γ *and formula* φ*,*

$$\mathfrak{M}^L, w \vDash \varphi \textit{ iff } \varphi \in w.$$

Theorem 2.1.23 (Canonical model theorem: Theorem 4.22, [11]). *Any normal logic is complete in relation to its canonical model.*

Definition 2.1.24 (Canonicity). A formula φ is *canonical* if, for any normal modal logic L, $\varphi \in L$ implies that φ is valid on the canonical frame for L. Let φ be a formula and P a property. If the canonical frame for any normal logic L containing φ has the property P, and φ is valid on any class of frames with property P, then φ is canonical for P.

Sahlqvist formulas and theorem

Definition 2.1.25 (Sahlqvist formulas)**.**

- A formula is *positive* (resp. *negative*) if every occurrence of a variable is in the scope of an even (resp. odd) number of negations.
- *Boxed atom* is a formula $\Box_1 \ldots \Box_n p$ where $\Box_1 \ldots \Box_n$ is a (possibly empty) string of unary boxes and $p \in \Pi$.
- A *Sahqvist antecedent* is a formula constructed from propositional constants, boxed atoms and negative formulas by applying $\wedge$, $\vee$, and diamonds of arbitrary arities. A *Sahlqvist implication* is a formula of the form $\varphi \to \psi$, where φ is a Sahlqvist antecedent and ψ is a positive formula.
- A *Sahlqvist formula* is a formula constructed from Sahlqvist implications by freely applying unary boxes and conjunctions, and applying polyadic boxes and disjunctions to formulas sharing no common variables. The Sahlqvist formula is monadic if no polyadic modalities occur in it.

Theorem 2.1.26 (Sahlqvist theorem: Theorem 3.54 and Theorem 4.42, [11])**.** *Every Sahlqvist formula defines a first-order formula and it is canonical in relation to the first-order formula property it defines.*

Definition 2.1.27 (Simply generalised monadic Sahlqvist formulas)**.**

- A *box formula* is a formula

$$\varphi = \Box_{1,1} \ldots \Box_{1,n_1} (\varphi_1 \to (\Box_{2,1} \ldots \Box_{2,n_2} \varphi_2 \to \ldots \Box_{k,1} \ldots \Box_{k,n_k} p) \ldots)$$

  where φ_i's are positive formulas and $p \in \Pi$ is called the *head* of φ.
- An occurrence of a variable in a box formula φ is *essential* in φ if it is a head of φ, otherwise it is *inessential* in φ. A variable in a box formula φ is *essential in* φ if it has at least one essential occurrence in it, otherwise it is *inessential* in φ.

- A set of box formulas is said *independent* if no head of a formula from the set occurs as an inessential variable in any headed box from the set.

- *Simply generalised monadic Sahlqvist formulas* are defined by replacing in the definition of classical modal Sahlqvist formulas boxed atoms by box-formulas, and further requiring that the set of all these box-formulas occurring in the construction of the formula, is independent.

Theorem 2.1.28 (Simply generalised Sahlqvist theorem, [41])**.** *Every simply generalised Sahlqvist formula defines a first-order formula and it is canonical in relation to the first-order formula property it defines.*

2.1.5 Hybrid logic

The basic hybrid logic is simply a multimodal language, whose atomic symbols are subdivided in two sorts. Beside the propositional variables, we have also the set of nominals, NOM. Furthermore for each nominal i we have the operator $@_i$. Given a similarity type τ, the basic hybrid language, $\mathcal{H}_\tau(@)$, is given by:

$$\varphi \ ::= \ i \mid p \mid \neg\varphi \mid \varphi_1 \wedge \varphi_2 \mid \Diamond(\varphi_1, \ldots, \varphi_{\rho(\Diamond)}) \,,$$

where $i \in NOM$ and everything else is as before. The novelties in the semantics are that the valuation of each nominal is a singleton, and that the operator $@_i$ changes the valuation world to the world satisfying i. Indeed $@_i\varphi$ is true in any world if ϕ is true at the world satisfying i. These logics are quite simple (the satisfability problem is not more complex than the one for modal logic) but their expressivity is impressively stronger. It can express many properties not definable in the basic modal language, e.g. irreflexivity, asymmetry, antisymmetry, intransitivity, universality, trichotomy, at most n states, etc. For more details see: [11, Section 7.3].

2.2 Notation

Our notation is mostly standard but here we introduce the most used in this thesis.

ω denotes both set of naturals and the first uncountable ordinal number.

Given a sequence over a set A, $\theta = a_1 \ldots a_n$, $|\theta| = n$ is the *length* of θ.

The empty sequence is denoted by ϵ and $|\epsilon| = 0$.

For $n < \omega$, let A^n denote the set of all sequences of length n over A, and let:

- $A^* = \bigcup_{n<\omega} A^n$ and
- $W^+ = W^* - \{\epsilon\}$.

For any $\theta, \sigma \in A^*$, we denote their *concatenation* simply by $\theta\sigma$ (with $\theta\epsilon = \epsilon\theta = \theta$).

For $n < \omega$, $\theta \in A^*$, let $\theta^n = \overbrace{\theta \dots \theta}^{n}$ (with $\theta^0 = \epsilon$).

A *prefix* of a sequence $\theta = a_1 \dots a_n$ is any sequence of the form $a_1 \dots a_k$ for $0 \leq k \leq n$.

We use consider & to be interchangeable with the word 'and'. Moreover, we consider it has priority over the word 'or' (like $\wedge$ over $\vee$ or $\times$ over $+$). So when we write A & B or C & D & E, we mean $(A$ & $B)$ or $(C$ & D & $E)$. By 'iff' we mean 'if and only if'.

Part I

‘Time’-Dependent Systems

3

Products with Linear Orders

In this chapter we tackle the problem of axiomatising product logics where one of the components is linear, standing for time. Our results cover the common choices for the models of a linear flow of time: $(\omega, <)$, $(\mathbb{Q}, <)$ and $(\mathbb{R}, <)$, as well as their reflexive counterparts $(\omega, \leq)$ $(\mathbb{Q}, \leq)$ and $(\mathbb{R}, \leq)$. In the first section we introduce the main concepts around 2-products of modal logics and list some general results in the area. In the second section we solve some open problems regarding the (non)-finite axiomatisability of product logics where one of the components is linear. In particular, we give the first examples of recursively enumerable (even decidable) two-dimensional products of finitely axiomatisable modal logics that are not finitely axiomatisable. We show that any axiomatisation of some bimodal logics that are determined by classes of product frames with a linearly ordered component must be infinite in two senses: it should contain infinitely many propositional variables, and formulas of arbitrarily large modal nesting-depth. In the last section we present some further results regarding the problem of finding an explicit axiomatisation for the considered logics.

3.1 Products of modal logics

We start by recalling the concept of product logics in its two-dimensional version (following [23, Chapter 5]).

Definition 3.1.1. Given two Kripke complete unimodal logics L_0 and L_1, the *product logic* $L_1 \times L_2$ is defined as

$$L_0 \times L_1 = \mathsf{Log}\left\{\mathfrak{F}_0 \times \mathfrak{F}_1 : \mathfrak{F}_i \text{ is a frame for } L_i, \text{ for } i < 2\right\}.$$

where the product $\mathfrak{F}_0 \times \mathfrak{F}_1$ of frames $\mathfrak{F}_0 = (W_0, R_0)$ and $\mathfrak{F}_1 = (W_1, R_1)$ is the 2-frame

$$\mathfrak{F}_0 \times \mathfrak{F}_1 = (W_0 \times W_1, \overline{R}_0, \overline{R}_1),$$

where $W_0 \times W_1$ is the Cartesian product of W_0 and W_1 and, for all $u, u' \in W_0$, $v, v' \in W_1$,

$$(u,v)\overline{R}_0(u',v') \quad \text{iff} \quad uR_0u' \text{ and } v = v',$$
$$(u,v)\overline{R}_1(u',v') \quad \text{iff} \quad vR_1v' \text{ and } u = u'.$$

2-frames of this form will be called *product frames* throughout.

Product logics are defined in a semantical way: they are logics determined by classes of product frames. Thus, a good start to understand their behaviour is to find properties that hold in every product frame. The most obvious ones are given by the three diagrams in Figure 3.1 the meaning of which can be described by the following first-order sentences:

- *left commutativity*: $\forall xyz(xR_1yR_0z \to \exists w(xR_0wR_1z))$,
- *right commutativity*: $\forall xwz(xR_0wR_1z \to \exists y(xR_1yR_0z))$,
- *Church-Rosser property*: $\forall xwy(xR_0w \wedge xR_1y \to \exists z(yR_0z \wedge wR_1z))$.

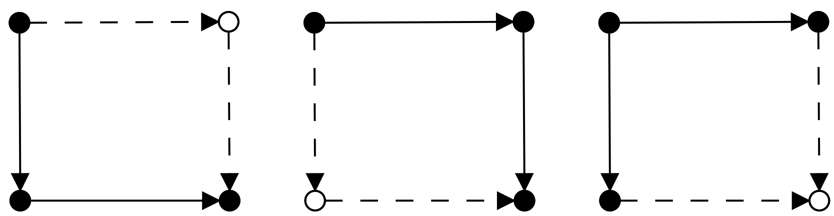

Figure 3.1: Left and right commutativity and Church-Rosser properties.

These properties can also be expressed by modal formulas. One can easily check that a 2-frame is left commutative iff it validates the formula

$$com^l = \Box_1 \Box_0 p \leftarrow \Box_0 \Box_1 p,$$

it is right commutative iff it validates

$$com^l = \Box_0 \Box_1 p \leftarrow \Box_1 \Box_0 p,$$

and it is Church-Rosser iff it validates

$$chr = \Diamond_0 \Box_1 p \to \Box_1 \Diamond_0 p.$$

The left and right commutativity axioms can be combined into a single commutativity axiom

$$com = com^l \wedge com^r.$$

But these axioms are not enough to characterise product frames. See in Figure 3.2 an example of a commutative and Church-Rosser 2-frame that is not isomorphic to the product of any frames.

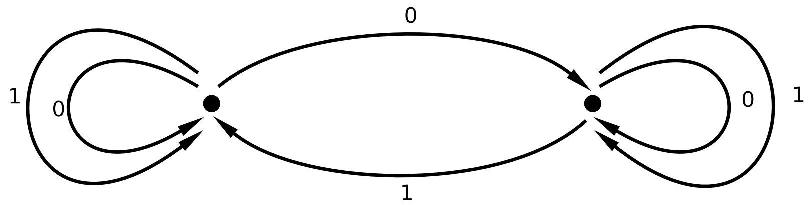

Figure 3.2: Commutative and Church-Rosser but not product frame.

The first syntactic approach to $L_0 \times L_1$ is given by:

Definition 3.1.2. Given two Kripke complete unimodal logics L_0 and L_1 the *commutator* is

$$[L_0, L_1] = (L_0, L_1) \oplus com \oplus chr.$$

It is easy to prove that we indeed have the following inclusion:

$$[L_0, L_1] \subseteq L_0 \times L_1. \tag{3.1}$$

When the converse also holds, that is, when we have:

$$L_0 \times L_1 = [L_0, L_1],$$

we say that $L_0 \times L_1$ is *product-matching*.

It turns out that many pairs of standard modal logics are indeed product-matching; however, there are many counterexamples as well. In the following sections we consider the problem of finding an axiomatisation for logics of this kind, in the particular case where L_0 is a unimodal logic for linear orders.

Let us first summarise the known general results related to this problem:

(1) If both unimodal logics L_0 and L_1 are such that their classes of Kripke frames are definable by recursive sets of first-order sentences, then their product $L_0 \times L_1$ is a recursively enumerable bimodal logic [25]. E.g. $L_0, L_1 \in \{\mathbf{K}, \mathbf{K4}, \mathbf{S4}, \mathbf{S5}, \mathbf{K4.3}, Log\{(\omega, <)\}, \mathsf{Log}\{(\mathbb{Q}, <)\} = \mathsf{Log}\{(\mathbb{R}, <)\}, \mathsf{Log}\{(\mathbb{Q}, \leq)\} = \mathsf{Log}\{(\mathbb{R}, \leq)\}\}$.

(2) If both L_0 and L_1 are finitely axiomatisable by modal formulas having universal Horn first-order correspondents, then $L_0 \times L_1$ is finitely axiomatisable [25]. In fact, these product logics are *product-matching*. For example, if each L_i is either **K** (the logic of all frames), or **K4** (the logic of all transitive frames), or **S4** (the logic of all reflexive and transitive frames), or **S5** (the logic of all equivalence frames), then $L_0 \times L_1$ is product-matching.

(3) The result in 2 cannot be generalised to products of logics axiomatised by formulas having universal (but not necessarily Horn) first-order components. Such an example is the finitely axiomatisable modal logic **K4.3**, determined by frames (W, R), where R is transitive and *weakly connected*:

$$\forall x, y, z \in W \left(xRy \wedge xRz \rightarrow (y = z \vee yRz \vee zRy) \right).$$

(Rooted transitive and weakly connected relations are *linear orders*.) As is shown in [23, Theorem .5.15], no product logic of the form $\mathbf{K4.3} \times L$ is product-matching, whenever L is any Kripke complete modal logic containing **K4** and having the two-element reflexive chain among its frames. So, say, $\mathbf{K4.3} \times \mathbf{K4}$ is an example of a recursively enumerable but not product-matching product of two finitely axiomatisable logics. However, it was left open whether any of these product logics were finitely axiomatisable.

(4) 2-product logics where at least one component logic is determined by a class of frames of finite bounded depth (e.g. **S5**), are usually decidable. Actually, product logics are often decidable when, in order to check satisfiability of a formula φ, it is enough to consider only those product frames where the depth of one of the components is bounded by some finite number which can be effectively computed from φ. This result covers multi-modal **K** and **S5** as well as products with tense extensions of multi-modal **K** or temporal logics of metric spaces [25, 35, 23, 67, 51].

(5) Products of two 'linear transitive' logics are undecidable whenever the depth of frames for both component logics cannot be bounded by any fixed $n < \omega$: examples are products of **K4.3**, **S4.3** or $\mathsf{Log}\{(\omega, <)\}$, [61, 68, 72].

(6) Products of more than two modal logics[1] are usually undecidable. In fact, no logic between $\mathbf{K} \times \mathbf{K} \times \mathbf{K}$ and $\mathbf{S5} \times \mathbf{S5} \times \mathbf{S5}$ is decidable [45].

(7) Finally, note that the product construction may result in quite complex bimodal logics. There are several examples of non-recursively enumerable, even Π^1_1-complete, products of finitely axiomatisable logics, e.g. $L_0 \times L_1$ where $L_0 \in \{\mathbf{K4}, \mathbf{K4.3}, \mathbf{S4}, \mathbf{GL}, \mathbf{Grz}, \mathsf{Log}\{(\omega, <)\}, \mathsf{Log}\{(\omega, \leq)\}\}$ and $L_0 \in \{\mathsf{Log}\{(\omega, <)\}, \mathsf{Log}\{(\omega, \leq)\}\}$, [36, 68, 72]..

Notation. We are interested in product logics with a 'linear' first component, that is, where frames for L_0 are frames for **K4.3**. To emphasise this fact, the transitive and weakly connected relations in the 2-frames we deal with are always denoted by $\leq_a$, for some subscript a. This does not necessarily mean that $\leq_a$ is reflexive. However, we use the following notation:

$$u <_a v \qquad \text{iff} \qquad u \leq_a v \text{ and } v \nleq_a u.$$

3.2 Non-finite axiomatisability

Throughout, an *irreflexive ω-fan* is a unimodal Kripke frame isomorphic to $\mathfrak{H}_\omega = (\omega + 1, R)$, where $R = \{(\omega, i) : i < \omega\}$. Similarly, a *reflexive ω-fan* is any frame isomorphic to $\mathfrak{H}^+_\omega = (\omega + 1, R^+)$, where $R^+ = R \cup \{(i, i) : i \leq \omega\}$.

Theorem 3.2.1. *Let L be any bimodal logic such that*

- *L contains* $\mathbf{K4.3} \times \mathbf{K}$, *and*
- *the product of $(\omega, \leq)$ and an (irreflexive or reflexive) ω-fan is a frame for L.*

Then L is not axiomatisable using finitely many propositional variables.

[1] Not defined but easy to figure out from the 2-product definition: n-product logics are the logics for the n-product frames where each component is a frame for the logic in the correspondent component.

Well-known examples of unimodal logics having an ω-fan among their frames are **K**, **K4**, **S4**, Gödel-Löb logic **GL** (the logic of irreflexive and transitive frames without infinite ascending chains), and Grzegorczyk logic **Grz** (the logic of reflexive and transitive frames without infinite ascending chains of distinct points). So we have the following:

Corollary 3.2.2. *Let L_0 be the logic of the most common models of the flow of time*

$$\mathsf{Log}\{(\omega,<)\},\ \mathsf{Log}\{(\mathbb{Q},<)\} = \mathsf{Log}\{(\mathbb{R},<)\},\ \mathsf{Log}\{(\omega,\le)\},\ \mathbf{S4.3} = \mathsf{Log}\{(\mathbb{Q},\le)\} = \mathsf{Log}\{(\mathbb{R},\le)\},$$

*or the logic of all linear orders, **K4.3**, and L_1 be any of the logics **K**, **K4**, **S4**, **GL**, **Grz**. Then $L_0 \times L_1$ is not axiomatisable using finitely many propositional variables.*

Note that both **K4.3** × **K** and **K4.3** × **K4** are known to be recursively enumerable [25], **K4.3** × **K** is even decidable [23, 77]. (The same holds for reflexive versions.)

Our next result shows that some of these possible axiomatisations should also have a different kind of infinity. We define the *vertical depth* $vd(\varphi)$ of a bimodal formula φ inductively by taking

$$\begin{aligned}
&vd(p) = 0,\\
&vd(\psi_1 \wedge \psi_2) = \max(vd(\psi_1), vd(\psi_2)),\\
&vd(\neg\psi) = vd(\psi),\\
&vd(\Diamond_0\psi) = vd(\psi),\\
&vd(\Diamond_1\psi) = vd(\psi) + 1.
\end{aligned}$$

Theorem 3.2.3. *Let L be a bimodal logic such that*

$$\mathbf{K4.3} \times \mathbf{K} \subseteq L \subseteq \mathsf{Log}\{(\omega,\le)\} \times \mathbf{K}.$$

Then every axiomatisation of L must contain formulas of arbitrarily large vertical depth.

3.2.1 Infinitely many propositional variables are needed

In this section we prove Theorem 3.2.1. So let L be any bimodal logic containing **K4.3** × **K** such that the product of $(\omega,\le)$ and an ω-fan is a frame for L. In order to show that L is not axiomatisable using finitely many propositional variables, we plan to proceed as follows. Given $m < \omega$, we call a Kripke model $\mathfrak{M} = (\mathfrak{F},\vartheta)$ *m-generated* if there are at most m different propositional variables p such that $\vartheta(p) \neq \emptyset$. For every $0 < k < \omega$, we will define a 2-frame $\mathfrak{F}_k$ such that:

(a) $\mathfrak{F}_k$ is not a frame for **K4.3** $\times$ **K**.

(b) If $k > 2^{4m} + 1$ then $\mathfrak{M} \models L$, for every m-generated model $\mathfrak{M}$ based on $\mathfrak{F}_k$.

This will prove Theorem 3.2.1 because of the following. Suppose that Σ axiomatises L and Σ contains m propositional variables, for some $m < \omega$. Let $k > 2^{4m} + 1$ and take a 2-frame $\mathfrak{F}_k$ satisfying **(b)**. Let $\mathfrak{M}$ be an arbitrary model based on $\mathfrak{F}_k$. Let $\mathfrak{M}_m$ be another model over $\mathfrak{F}_k$ that is the same as $\mathfrak{M}$ on propositional variables occurring in Σ, and $\varnothing$ otherwise. Then $\mathfrak{M}_m$ is clearly m-generated and $\mathfrak{M}_m \models \Sigma$ iff $\mathfrak{M} \models \Sigma$. So by **(b)**, we have $\mathfrak{M}_m \models L$. As $\Sigma \subseteq L$, we obtain $\mathfrak{M}_m \models \Sigma$, and so $\mathfrak{M} \models \Sigma$. This holds for any model $\mathfrak{M}$ over $\mathfrak{F}_k$, so $\mathfrak{F}_k$ is a frame for Σ. Therefore, $\mathsf{Log}\{\mathfrak{F}_k\}$ is a bimodal logic containing Σ, and so we have that $\mathfrak{F}_k$ is a frame for L. As **K4.3** $\times$ **K** $\subseteq L$, this implies that $\mathfrak{F}_k$ is a frame for **K4.3** $\times$ **K**, contradicting **(a)**.

We fix some $0 < k < \omega$, and begin with the definition of $\mathfrak{F}_k = (W, \leq_h, R_v)$, for the case when $(\omega, \leq) \times \mathfrak{H}_\omega$ is a frame for L:

$$W = \{y\} \cup \{x_i, u_i, v_i, w_i, z_i \;:\; i < k\},$$

$\leq_h$ is the reflexive and transitive closure of

$$\{(u_i, v_i), (v_i, w_i), (w_i, z_i) : i < k\} \cup \{(x_i, x_j), (x_i, y), (y, x_i) : i, j < k\},$$

$$R_v = \{(x_i, u_j), (x_i, z_j) : i, j < k\} \cup \{(x_i, v_j) : i, j < k, i \neq j\} \cup \{(y, u_i), (y, w_i), (y, z_i) : i < k\},$$

see Figure 3.3. Note that in Figure 3.3 (as well as in further figures) the reflexive, transitive and weakly connected $\leq_h$ is depicted by 'horizontal' arrows and its clusters by 'horizontal' ellipses, and R_v by kind of 'vertical' arrows.

If L is such that $(\omega, \leq) \times \mathfrak{H}_\omega$ is not a frame for L, but $(\omega, \leq) \times \mathfrak{H}_\omega^+$ is, then we should add the pairs $\{(w, w) : w \in W\}$ to R_v. From now on, we discuss in detail the vertically irreflexive case only. The very similar proof of the reflexive case is left to the reader.

First, we prove **(a)**. Let us begin with showing a general property of p-morphic images of weakly connected frames.

Claim 1. *Let f be a p-morphism from some weakly connected frame $\mathfrak{G}_0 = (W_0, \leq_0)$ onto a frame $\mathfrak{G}_1 = (W_1, \leq_1)$. For all $a, b \in W_0$, $x \in W_1$, if $a \leq_0 b$ and $f(a) \leq_1 x <_1 f(b)$ then there exists $c \in W_0$ such that $a \leq_0 c <_0 b$ and $f(c) = x$. Moreover, if $f(a) <_1 x <_1 f(b)$ then c can be chosen such that $a <_0 c <_0 b$.*

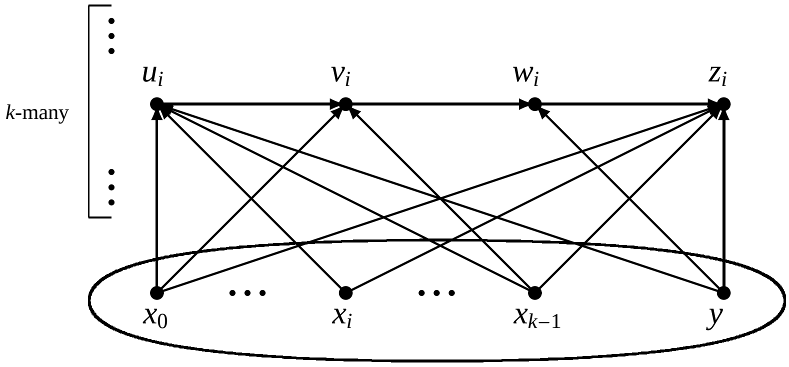

Figure 3.3: The frame $\mathfrak{F}_k$.

Proof. Take some $a, b \in W_0$, $x \in W_1$ such that $a \leq_0 b$ and $f(a) \leq_1 x <_1 f(b)$. By the backward condition on f, there exists $c \in W_0$ such that $a \leq_0 c$ and $f(c) = x$. Moreover, as f is a homomorphism, if $f(a) <_1 x$ then $a <_0 c$. As $\leq_0$ is weakly connected, we have either $c = b$, or $b \leq_0 c$, or $c \leq_0 b$. But $f(c) <_1 f(b)$, so the first two cases cannot hold. Therefore, $c <_0 b$ follows. □

As being transitive and weakly connected is first-order definable, the class of all frames for **K4.3** is closed under ultraproducts. As **K4.3** is a modal logic, its class of frames is also closed under point-generated subframes. So, by [59, Theorem .2.10], we obtain:

Claim **2.** *For every finite rooted* 2*-frame* $\mathfrak{F}$, $\mathfrak{F}$ *is a frame for* ***K4.3*** $\times$ ***K*** *iff* $\mathfrak{F}$ *is a p-morphic image of a product frame for* ***K4.3*** $\times$ ***K***.

Therefore, in order to prove **(a)** it is enough to show the following:

Lemma 3.2.4. $\mathfrak{F}_k$ *is not a p-morphic image of a product frame for* ***K4.3*** $\times$ ***K***.

Proof. Suppose that there is a p-morphism f from a product frame $\mathfrak{H} = (U, \leq_0, R_1)$ with transitive and weakly connected $\leq_0$ onto $\mathfrak{F}_k = (W, \leq_h, R_v)$. Take some $i_0 < k$. As $x_{i_0} \leq_h y R_v w_{i_0}$, there are $a_0, b, c_0 \in U$ such that $a_0 \leq_0 b R_1 c_0$, $f(a_0) = x_{i_0}$, $f(b) = y$, and $f(c_0) = w_{i_0}$. As $\mathfrak{H}$ is a product frame, there exists $d_0 \in U$ such that $a_0 R_1 d_0 \leq_0 c_0$. As f is a p-morphism, we must have that $f(d_0) = u_{i_0}$. As $u_{i_0} <_h v_{i_0} <_h w_{i_0}$, by Claim 1, there exists $e_0^1 \in U$ such that $d_0 <_0 e_0^1 <_0 c_0$ and $f(e_0^1) = v_{i_0}$. As $\mathfrak{H}$ is a product frame, there exists $a_1 \in U$ such that $a_1 R_1 e_0^1$. As f is a p-morphism, we must have that $f(a_1) = x_{i_1}$, for some $i_1 < k$, $i_1 \neq i_0$.

Next, as $yR_v w_{i_1}$, there exists $c_1 \in U$ such that bR_1c_1 and $f(c_1) = w_{i_1}$. As $\mathfrak{H}$ is a product frame, there exists $d_1 \in U$ such that $a_1R_1d_1 \leq_0 c_1$. As f is a p-morphism, we must have that $f(d_1) = u_{i_1}$. As $u_{i_1} <_h v_{i_1} <_h w_{i_1}$, by Claim 1, there exists $e_1^2 \in U$ such that $d_1 <_0 e_1^2 <_0 c_1$ and $f(e_1^2) = v_{i_1}$. As $\mathfrak{H}$ is a product frame, there exist $e_0^2, a_2 \in U$ such that $e_0^1 <_0 e_0^2 <_0 c_0$, $a_2R_1e_0^2$, and $a_2R_1e_1^2$. As f is a p-morphism, we must have that $f(e_0^2) = v_{i_0}$, and so $f(a_2) = x_{i_2}$, for some $i_2 < k$, $i_2 \notin \{i_0, i_1\}$ (see Figure 3.4).

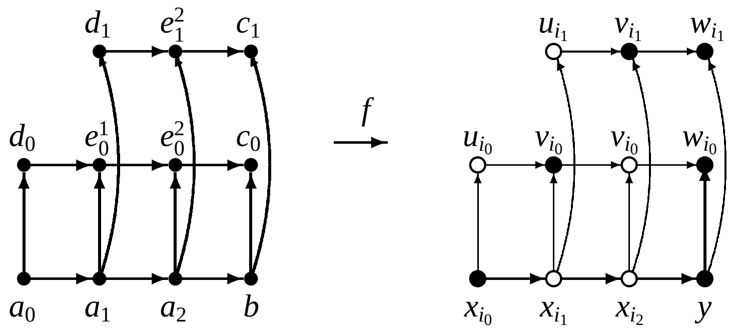

Figure 3.4: Building a p-morphism from a product frame to $\mathfrak{F}_k$: the first steps.

And so on, using that $yR_v w_{i_j}$ for all $j < k-1$, after $k-1$ steps we end up having $a_{k-1}, e_0^{k-1}, \ldots, e_{k-2}^{k-1} \in U$ and pairwise distinct $i_0, \ldots, i_{k-1} < k$ such that $f(a_{k-1}) = x_{i_{k-1}}$, $a_{k-1}R_1 e_j^{k-1}$, and $f(e_j^{k-1}) = v_{i_j}$, for all $j < k-1$. Now, as $yR_v w_{i_{k-1}}$, there exists $c_{k-1} \in U$ such that bR_1c_{k-1} and $f(c_{k-1}) = w_{i_{k-1}}$. As $\mathfrak{H}$ is a product frame, there exists $d_{k-1} \in U$ such that $a_{k-1}R_1d_{k-1} \leq_0 c_{k-1}$. As f is a p-morphism, we must have that $f(d_{k-1}) = u_{i_{k-1}}$. As $u_{i_{k-1}} <_h v_{i_{k-1}} <_h w_{i_{k-1}}$, by Claim 1, there exists $e_{k-1} \in U$ such that $d_{k-1} <_0 e_{k-1} <_0 c_{k-1}$, and $f(e_{k-1}) = v_{i_{k-1}}$. As $\mathfrak{H}$ is a product frame, there exist $e_0, \ldots, e_{k-2}, a_k \in U$ such that $e_j^{k-1} <_0 e_j <_0 c_j$ for all $j < k-1$, and $a_kR_1e_j$ for all $j < k$. As f is a p-morphism, we must have that $f(e_j) = v_{i_j}$, for all $j < k$. But there is no $x \in W_k$ such that $xR_v v_j$ for all $j < k$, so we cannot find a proper value for $f(a_k)$, a contradiction (see Figure 3.5). □

Remark 3.2.5. Instead of proving that the existence of a p-morphism from a product frame for **K4.3**×**K** onto $\mathfrak{F}_k$ leads to a contradiction, we could have proved Lemma 3.2.4 by playing a two-player 'p-morphism game.' Versions of such games are widely used in connection with axiomatisation problems in algebraic logic and many-dimensional modal logics, see e.g. [43, 57]. In this game, two players ∃ and ∀ are building step-by-step homomorphisms from larger and larger product frames for **K4.3** × **K** to a finite or countably infinite 2-frame

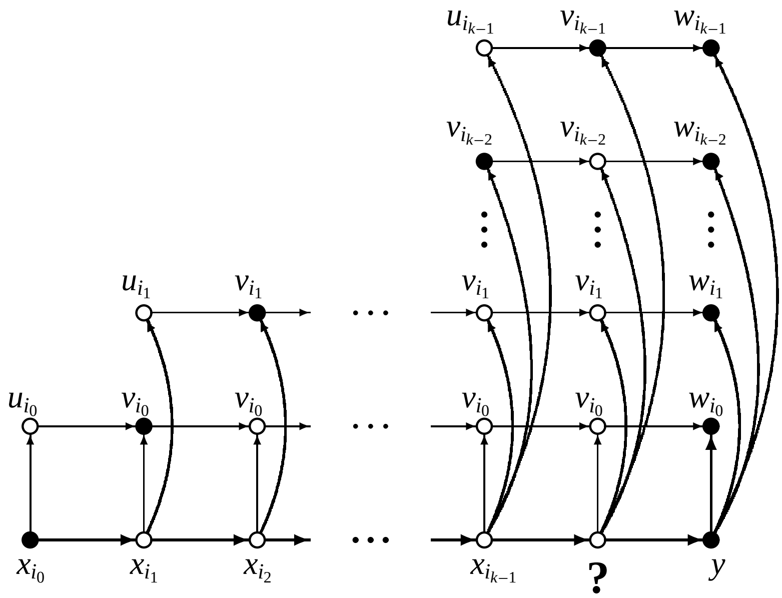

Figure 3.5: Building a p-morphism from a product frame to $\mathfrak{F}_k$: the contradiction.

$\mathfrak{F}$. At each step, $\forall$ can choose a 'defect' showing that the actual homomorphism is not a p-morphism: an instance of points failing the backward condition. If $\exists$ can reply with a larger homomorphism 'fixing' the chosen defect, then the game goes on, otherwise $\forall$ wins the game. If $\exists$ can always go on infinitely long, no matter what $\forall$'s choices are, then we say that $\exists$ *has a winning strategy in the ω-step game over* $\mathfrak{F}$. It is not hard to prove, using Claim 1, that $\exists$ has such a winning strategy iff $\mathfrak{F}$ is a p-morphic image of a product frame for **K4.3** $\times$ **K**. Figures 3.4 and 3.5 can be read as descriptions of a particular play of this game over $\mathfrak{F}_k$, won by $\forall$: the black dots show the choices of $\forall$, and the empty circles are the points in possible replies of $\exists$, until she fails to continue.

Let us now turn to the proof of **(b)**. We define a new frame $\mathfrak{G}_k = (V, \leq_h, S_v)$ by adding some points and arrows to $\mathfrak{F}_k$:

$V = W \cup \{x, u, u', v, w, z\}$,

$\leq_h$ is the reflexive and transitive closure of

$$\leq_h \cup \{(x,y),(y,x),(u,v),(u',v),(v,w),(w,z)\},$$

$$S_v = R_v \cup \{(x,u),(x,u'),(x,v),(x,z),(y,u),(y,u'),(y,w),(y,z)\} \cup \{(x,u_i),(x,v_i),(x,z_i),(x_i,u),(x_i,u'),(x_i,v),(x_i,z) : i < k\},$$

see Figure 3.6.

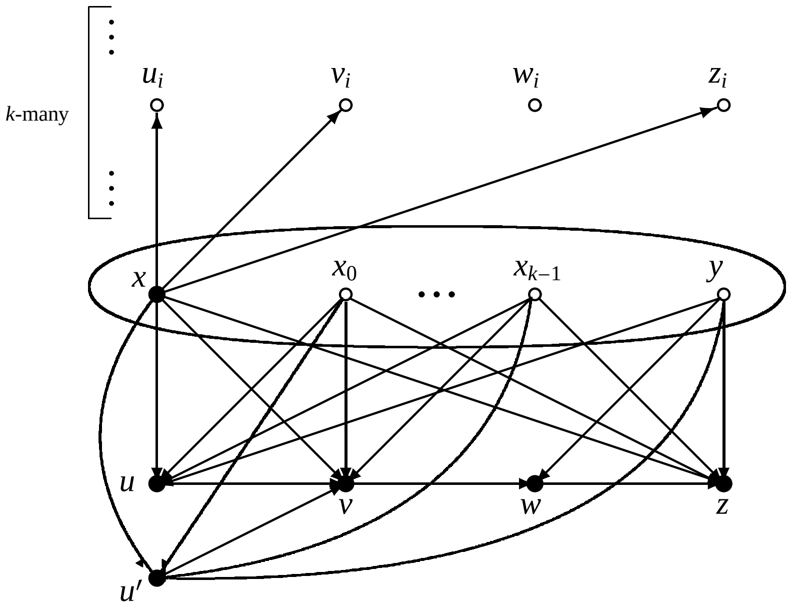

Figure 3.6: The *new* points and arrows of frame $\mathfrak{G}_k$.

Lemma 3.2.6. *$\mathfrak{G}_k$ is a p-morphic image of $(\omega, \leq) \times \mathfrak{H}_\omega$.*

Proof. Let $\mathfrak{F}_\omega$ be the irreflexive ω-fan with r as its root and $\omega \times (k+2)$ as its set of leaves. We define a function f from $(\omega, \leq) \times \mathfrak{F}_\omega$ to $\mathfrak{G}_k$. To begin with, for all $n < \omega$, we let

$$f(n, r) = \begin{cases} x_i, & \text{if } n = 3(\ell \cdot k + i),\ \ell < \omega,\ i < k, \\ x, & \text{if } n = 3\ell + 1,\ \ell < \omega, \\ y, & \text{if } n = 3\ell + 2,\ \ell < \omega. \end{cases}$$

Then, for all $n, m < \omega$, $i < k+2$, we define $f(n,(m,i))$ by taking

$$
\begin{array}{ll}
u_i, & \text{if } i < k,\ m \geq 2,\ n < 3m-3, \\
u_i, & \text{if } i < k,\ m = \ell \cdot k + i + 1 \text{ for some } \ell < \omega,\ n = 3m-3, \\
v_i, & \text{if } i < k,\ m \geq 1, m \neq \ell \cdot k + i + 1 \text{ for any } \ell < \omega,\ \ n = 3m-3, \\
v_i, & \text{if } i < k,\ m \geq 1,\ n = 3m-2, \\
w_i, & \text{if } i < k,\ m \geq 1,\ n = 3m-1, \\
z_i, & \text{if } i < k,\ n \geq 3m, \\
u, & \text{if } i = k,\ m \geq 2,\ n < 3m-3, \\
u', & \text{if } i = k+1,\ m \geq 2,\ n < 3m-3, \\
v, & \text{if } i = k \text{ or } k+1,\ m \geq 1,\ n = 3m-3 \text{ or } 3m-2, \\
w, & \text{if } i = k \text{ or } k+1,\ m \geq 1,\ n = 3m-1, \\
z, & \text{if } i = k \text{ or } k+1,\ n \geq 3m.
\end{array}
$$

It is tedious but straightforward to check that f is an onto p-morphism. Here is part of the trickiest case only. Take some $n < \omega$ such that $n = 3(\ell \cdot k + i)$ for some $\ell < \omega$, $i < k$. Then $f(n,r) = x_i$. Now take some $j < k$, $j \neq i$. We claim that

$$\{f(n,(m,j)) : m < \omega\} = \{u_j, v_j, z_j\}. \tag{3.2}$$

Indeed, on the one hand, we have $\subseteq$, as n is divisible by 3. For $\supseteq$: First, for any $m > \frac{n+3}{3}$, we have $f(n,(m,j)) = u_j$. Second, for any $m \leq \frac{n}{3}$, we have $f(n,(m,j)) = z_j$. Finally, observe that $\frac{n+3}{3} = \ell \cdot k + i + 1 \neq \ell' \cdot k + j + 1$ for any $\ell' < \omega$, as $i, j < k$ and $i \neq j$. So if $m = \frac{n+3}{3}$, then $f(n,(m,j)) = v_j$, completing the proof of (3.2).

This last observation also shows that $f(n,(m,i)) \neq v_i$ for any $m < \omega$, proving that

$$\{f(n,(m,i)) : m < \omega\} = \{u_i, z_i\},$$

as required. The remaining cases are similar, but simpler. □

Lemma 3.2.7. *Let $\mathfrak{F}$ be a 2-frame, and suppose that $f : \mathfrak{F}_k \to \mathfrak{F}$ is an onto p-morphism such that $f(x_i) = f(x_j)$, $f(v_i) = f(v_j)$, $f(w_i) = f(w_j)$, and $f(z_i) = f(z_j)$, for some $i, j < k$, $i \neq j$, $k > 2$. Then $\mathfrak{F}$ is a p-morphic image of $(\omega, \leq) \times \mathfrak{H}_\omega$.*

Proof. Let $n_0, \ldots, n_{k-3}$ be an enumeration of $\{\ell < k : \ell \neq i, j\}$. We define a function g from

$\mathfrak{G}_{k-2}$ to $\mathfrak{F}$ as follows:

$$\begin{aligned}
g(a_\ell) &= f(a_{n_\ell}), \qquad \text{for } a \in \{x,u,v,w,z\},\ \ell < k-2\\
g(y) &= f(y),\\
g(x) &= f(x_i) = f(x_j),\\
g(u) &= f(u_i),\\
g(u') &= f(u_j),\\
g(v) &= f(v_i) = f(v_j),\\
g(w) &= f(w_i) = f(w_j),\\
g(z) &= f(z_i) = f(z_j).
\end{aligned}$$

It is straightforward to check that g is an onto p-morphism, so the statement follows from Lemma 3.2.6. □

Lemma 3.2.8. *Let L be a bimodal logic such that $(\omega, \leq) \times \mathfrak{H}_\omega$ is a frame for L. If $k > 2^{4m} + 1$ then $\mathfrak{M} \models L$ for every m-generated model $\mathfrak{M}$ over $\mathfrak{F}_k$.*

Proof. Fix some k, m with $k > 2^{4m} + 1$. Let $\mathfrak{M} = (\mathfrak{F}_k, \vartheta)$ be a model such that $\vartheta(p_j) = \varnothing$ for every propositional variable p_j with $j \geq m$.

We call two points in $\mathfrak{F}_k$ $\equiv$-*equivalent* iff no bimodal formula can distinguish them in $\mathfrak{M}$, that is, for all $a, b \in W$, we let

$$a \equiv b \qquad \Longleftrightarrow \qquad \forall \text{ formula } \varphi\ (\mathfrak{M}, a \models \varphi \ \leftrightarrow\ \mathfrak{M}, b \models \varphi). \tag{3.3}$$

For every $a \in W$, let $[a]$ denote the $\equiv$-class of a, and let $A = \{[a] : a \in W\}$. We define a 2-frame $\mathfrak{A}_\mathfrak{M} = (A, S_0, S_1)$ by taking, for $a, b \in W$,

$$\begin{aligned}
[a] S_0 [b] &\qquad \Longleftrightarrow \qquad \exists a' \in [a], b' \in [b],\ a' \leq_h b',\\
[a] S_1 [b] &\qquad \Longleftrightarrow \qquad \exists a' \in [a], b' \in [b],\ a' R_v b'.
\end{aligned}$$

We claim that the function

$$f(a) = [a], \qquad a \in W$$

is a p-morphism from $\mathfrak{F}_k$ onto $\mathfrak{A}_\mathfrak{M}$. This is a straightforward consequence of duality theory and the finiteness of $\mathfrak{F}_k$, but we give a short direct proof here. First, f is a homomorphism by the definition of S_i. For the backward condition, observe that since $\mathfrak{F}_k$ is finite, there are finitely many formulas $\varphi_0, \ldots, \varphi_{n-1}$ such that

$$a \equiv b \qquad \Longleftrightarrow \qquad \forall j < n\ (\mathfrak{M}, a \models \varphi_j \ \leftrightarrow\ \mathfrak{M}, b \models \varphi_j) \tag{3.4}$$

(the sets $\{w \in W : \mathfrak{M}, w \models \varphi_j\}$ are the atoms of the algebra of $\mathfrak{M}$-definable subsets of W). Now take some $a, b \in W$ such that $[a]S_0[b]$, and let $a' \in [a]$. (The case when $[a]S_1[b]$ is similar.) Then there are $a'' \in [a]$, $b'' \in [b]$ with $a'' \leq_h b''$. Let φ be the 'atomic type' of b'', that is,

$$\varphi = \bigwedge_{j<n, \mathfrak{M}, b'' \models \varphi_j} \varphi_j \quad \wedge \bigwedge_{j<n, \mathfrak{M}, b'' \not\models \varphi_j} \neg\varphi_j.$$

Then $\mathfrak{M}, b'' \models \varphi$. Therefore $\mathfrak{M}, a'' \models \Diamond_0\varphi$, and so $\mathfrak{M}, a' \models \Diamond_0\varphi$. So there is some b' such that $a' \leq_h b'$ and $\mathfrak{M}, b' \models \varphi$. Now $b' \equiv b''$ follows by (3.4).

We define an equivalence relation $\sim_m$ on the set $\{0, 1, \dots, k-1\}$ by taking, for all $i, j < k$,

$$i \sim_m j \qquad \Longleftrightarrow \qquad \forall \ell < m \Big(\big(x_i \in \vartheta(p_\ell) \leftrightarrow x_j \in \vartheta(p_\ell)\big) \wedge \big(v_i \in \vartheta(p_\ell) \leftrightarrow v_j \in \vartheta(p_\ell)\big) \wedge \\ \big(w_i \in \vartheta(p_\ell) \leftrightarrow w_j \in \vartheta(p_\ell)\big) \wedge \big(z_i \in \vartheta(p_\ell) \leftrightarrow z_j \in \vartheta(p_\ell)\big)\Big).$$

Now recall the definition of $\equiv$ from (3.3). A straightforward induction on formulas shows that

$$\forall\, i, j < k, \quad \big(i \sim_m j \;\rightarrow\; (x_i \equiv x_j) \wedge (v_i \equiv v_j) \wedge (w_i \equiv w_j) \wedge (z_i \equiv z_j)\big). \tag{3.5}$$

As there are 2^{4m} many $\sim_m$-classes on $\{0, 1, \dots, k-1\}$ and $k > 2^{4m}+1 > 2^{4m}$, by the pigeonhole principle and (3.5), there exist $i, j < k$, $i \neq j$ such that $x_i \equiv x_j$, $v_i \equiv v_j$, $w_i \equiv w_j$, $z_i \equiv z_j$, and so $f(x_i) = f(x_j)$, $f(v_i) = f(v_j)$, $f(w_i) = f(w_j)$, $f(z_i) = f(z_j)$. Therefore, as $k > 2$, the 2-frame $\mathfrak{A}_\mathfrak{M}$ and the p-morphism f satisfy the conditions of Lemma 3.2.7, and so $\mathfrak{A}_\mathfrak{M}$ is a p-morphic image of $(\omega, \leq) \times \mathfrak{H}_\omega$. As by assumption $(\omega, \leq) \times \mathfrak{H}_\omega$ is a frame for L, we obtain that $\mathfrak{A}_\mathfrak{M}$ is a frame for L as well. In particular, $\mathfrak{M}' \models L$ for the model $\mathfrak{M}' = (\mathfrak{A}_\mathfrak{M}, \vartheta')$ defined by taking, for each propositional variable p, $\vartheta'(p) = \{f(a) : a \in \vartheta(p)\}$. As f is a p-morphism between models $\mathfrak{M}$ and $\mathfrak{M}'$, $\mathfrak{M} \models L$ follows, as required. $\square$

3.2.2 Arbitrarily large vertical depth is needed

In this section we prove Theorem 3.2.3. Let L be a bimodal logic such that

$$\mathbf{K4.3} \times \mathbf{K} \;\subseteq\; L \;\subseteq\; \mathsf{Log}\{(\omega, \leq)\} \times \mathbf{K}.$$

In order to show that every axiomatisation of L must contain formulas of arbitrarily large vertical depth, let us introduce some notions. Given a 2-frame $\mathfrak{G} = (W, R_h, R_v)$ and $x, y \in W$, a *path in* $\mathfrak{G}$ *from* x *to* y is a sequence of points $w_0, \dots, w_n$ in W such that $w_0 = x$, $w_n = y$, and for each $i < n$, either $w_i R_h w_{i+1}$ or $w_i R_v w_{i+1}$. The *vertical length* of such a path is the number of R_v-edges in it. Now, for every $x \in W$ and every $k < \omega$, we define

$$W^{x,k} = \{x\} \cup \{y \in W : \text{there is a path in } \mathfrak{G} \text{ from } x \text{ to } y \text{ of vertical length} \leq k\},$$
$$(\mathfrak{G})^{x,k} = (W^{x,k}, R'_h, R'_v),$$

where R'_h and R'_v are the respective restrictions of R_h and R_v to $W^{x,k}$ (that is, $(\mathfrak{G})^{x,k}$ is the subframe of $\mathfrak{G}$ having $W^{x,k}$ as its universe). Clearly, for every bimodal formula φ with $vd(\varphi) \leq k$,

$$\text{if } \varphi \text{ is valid in } (\mathfrak{G})^{x,k} \text{ for every } x \in W, \text{ then } \varphi \text{ is valid in } \mathfrak{G}. \tag{3.6}$$

Now we plan to proceed as follows. For every $0 < k < \omega$, we will define a 2-frame $\mathfrak{G}_k$ such that:

(a) $\mathfrak{G}_k$ is not a frame for $\mathbf{K4.3} \times \mathbf{K}$.

(b) For every point x in $\mathfrak{G}_k$, $(\mathfrak{G}_k)^{x,k}$ is a frame for $\mathsf{Log}\{(\omega, \leq)\} \times \mathbf{K}$.

This will prove Theorem 3.2.3 because of the following. Suppose that Σ axiomatises L, and there is some $k < \omega$ such that $vd(\varphi) \leq k$ for every φ in Σ. Take a 2-frame $\mathfrak{G}_k$ satisfying **(b)**. As $\Sigma \subseteq L \subseteq \mathsf{Log}\{(\omega, \leq)\} \times \mathbf{K}$, for every point x in $\mathfrak{G}_k$, $(\mathfrak{G}_k)^{x,k}$ is a frame for Σ. Thus by (3.6), $\mathfrak{G}_k$ is a frame for Σ. Therefore, $\mathsf{Log}\{\mathfrak{G}_k\}$ is a bimodal logic containing Σ, and so we have that $\mathfrak{G}_k$ is a frame for L. As $\mathbf{K4.3} \times \mathbf{K} \subseteq L$, this implies that $\mathfrak{G}_k$ is a frame for $\mathbf{K4.3} \times \mathbf{K}$, contradicting **(a)**.

Let us begin with the definition of $\mathfrak{G}_k = (W, \leq_h, R_v)$, for $0 < k < \omega$:

$W = \{u_1, u_2, u_3, v_1, v_2, v_3, v_4\} \cup \{w_i^1, w_i^2, w_i^3 : 1 \leq i \leq k\}$,

$\leq_h$ is the reflexive and transitive closure of

$\{(u_1, u_2), (u_2, u_3), (v_1, v_2), (v_2, v_3), (v_3, v_4)\} \cup \{(w_i^1, w_i^2), (w_i^2, w_i^3), (w_i^3, w_i^1) : 1 \leq i \leq k\}$,

$R_v = \{(u_1, w_1^1), (u_2, w_1^2), (u_3, w_1^1), (u_3, w_1^2), (u_3, w_1^3)\} \cup$
$\{(w_k^1, v_1), (w_k^1, v_4), (w_k^2, v_1), (w_k^2, v_3), (w_k^2, v_4), (w_k^3, v_1), (w_k^3, v_2), (w_k^3, v_4)\} \cup$
$\{(w_i^1, w_{i+1}^1), (w_i^2, w_{i+1}^2), (w_i^3, w_{i+1}^3) : 1 \leq i < k\}$,

see Figure 3.7. Note that a vertically reflexive version of $\mathfrak{G}_k$ would also do.

Next, let us prove **(a)**. Actually, we give two different proofs for $\mathfrak{G}_k$ not being a frame for $\mathbf{K4.3} \times \mathbf{K}$: one using Claim 2 above, and another, more 'direct' one. For the first: As

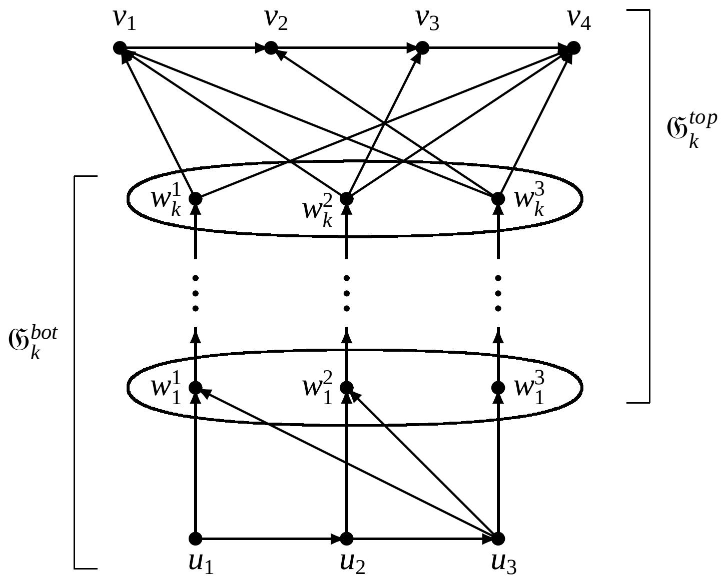

Figure 3.7: The frames $\mathfrak{G}_k$, $\mathfrak{G}_k^{bot}$, and $\mathfrak{G}_k^{top}$.

$\mathfrak{G}_k$ is a finite rooted 2-frame, by Claim 2, $\mathfrak{G}_k$ is a frame for **K4.3** × **K** iff $\mathfrak{G}_k$ is a p-morphic image of a product frame for **K4.3** × **K**. Therefore, it is enough to show the following:

Lemma 3.2.9. *$\mathfrak{G}_k$ is not a p-morphic image of a product frame for **K4.3** × **K**.*

Proof. Suppose that there is a p-morphism f from a product frame $\mathfrak{H} = (U, \leq_0, R_1)$ with transitive and weakly connected $\leq_0$ onto $\mathfrak{G}_k = (W, \leq_h, R_v)$. As $u_1 \leq_h u_2 R_v w_1^2 R_v \ldots R_v w_k^2 R_v v_3$, there exist $a, b, c_1, \ldots, c_k, c \in U$ such that $a \leq_0 b R_1 c_1 R_1 \ldots R_1 c_k R_1 c$, $f(a) = u_1$, $f(b) = u_2$, $f(c) = v_3$, and $f(c_i) = w_i^2$, for all $1 \leq i \leq k$. As $\mathfrak{H}$ is a product frame, there exist points $d_1, \ldots, d_n, d \in U$ such that $a R_1 d_1 R_1 \ldots R_1 d_k R_1 d \leq_0 c$. As f is a p-morphism, we must have that $f(d) = v_1$. As $v_1 <_h v_2 <_h v_3$, by Claim 1, there exists $e \in U$ such that $d <_0 e <_0 c$ and $f(e) = v_2$. As $\mathfrak{H}$ is a product frame, there exist $e_1, \ldots, e_k, x \in U$ such that $d_i <_0 e_i <_0 c_i$ for $1 \leq i \leq k$, $a <_0 x <_0 b$, and $x R_1 e_1 R_1 \ldots R_1 e_k R_1 e$. As f is a p-morphism, we must have that $f(e_i) = w_i^3$, for all $1 \leq i \leq k$. So we should also have that $u_1 \leq_h f(x) \leq_h u_2$ and $f(x) R_v w_1^3$. But there is no such $f(x)$, a contradiction. □

As a second proof for **(a)**, observe that in fact the proof of Lemma 3.2.9 shows that the following first-order (Π_2) sentence Φ_k fails in $\mathfrak{G}_k$:

$$\Phi_k : \quad \forall a, b, c \Big[a \leq_h b R_v^{k+1} c \to \exists d \Big(a R_v^{k+1} d \leq_h c \wedge$$

$$\forall e\big(d \leq_h e <_h c \to \exists x\,(a \leq_h x \leq_h b \wedge xR_v^{k+1}e)\big)\big)\Big]$$

(throughout, we use $uR_v^{k+1}w$ as a shorthand for $\exists w_1 \dots w_k\ (uR_v w_1 R_v \dots R_v w_k R_v w)$), see Figure 3.8. (Note that Figure 3.8 also shows the steps in a particular play of the p-morphism game over $\mathfrak{G}_k$ that player $\forall$ wins, cf. Remark 3.2.5.)

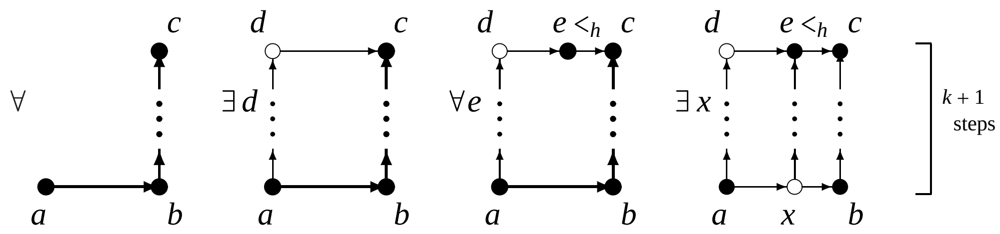

Figure 3.8: The property Φ_k.

It is straightforward to see that Φ_k holds in every product frame for **K4.3** × **K**. Now we will show that Φ_k is definable in the bimodal language, giving us an explicit bimodal formula showing that $\mathfrak{G}_k$ is not a frame for **K4.3** × **K**. To this end, for every $n < \omega$ and bimodal formula ψ, we define $\Diamond_1^n\psi$ by taking

$$\Diamond_1^0\psi = \psi \quad \text{and} \quad \Diamond_1^{n+1}\psi = \Diamond_1 \Diamond_1^n \psi.$$

Claim 3. *There is a bimodal formula*

$$\varphi_k: \quad \Diamond_0\big(p \wedge \Diamond_1^{k+1}(r \wedge q \wedge \Diamond_0 q)\big) \wedge \Box_0(\Diamond_0 p \to \Box_1^{k+1}q) \ \to\ \Diamond_1^{k+1}(\Diamond_0 r \wedge \Box_0 q)$$

such that, for all 2-frames $\mathfrak{F} = (W, \leq_0, R_1)$ with weakly connected $\leq_0$, φ_k is valid in $\mathfrak{F}$ iff Φ_k holds in $\mathfrak{F}$.

Remark 3.2.10. Notice that even if this formula is not Sahlqvist it is a simply generalised monadic Sahlqvist formula, see Definition 2.1.27.

Proof. Let $\mathfrak{F} = (W, \leq_0, R_1)$ be a 2-frame such that $\leq_0$ is weakly connected.

$\Leftarrow$: Let $\mathfrak{M}$ be a model based on $\mathfrak{F}$ and suppose that, for some $a \in W$,

$$\mathfrak{M}, a \models \Diamond_0\big(p \wedge \Diamond_1^{k+1}(r \wedge q \wedge \Box_0 q)\big) \quad \text{and}$$
$$\mathfrak{M}, a \models \Box_0(\Diamond_0 p \to \Box_1^{k+1} q). \tag{3.7}$$

So, there exist $b, c \in W$ such that $a \leq_0 bR_1^{k+1}c$ and

$$\mathfrak{M}, b \vDash p, \tag{3.8}$$

$$\mathfrak{M}, c \vDash r, \tag{3.9}$$

$$\mathfrak{M}, c \vDash q \wedge \Box_0 q. \tag{3.10}$$

Therefore, by Φ_k, there exists $d \in W$ such that $aR_1^{k+1}d \leq_0 c$ and for all $e \in W$, if $d \leq_0 e <_0 c$ then there exists $x \in W$ with $a \leq_0 x \leq_0 b$ and $xR_v^{k+1}e$. Then, by (3.9), we have $\mathfrak{M}, d \vDash \Diamond_0 r$. We claim that $\mathfrak{M}, d \vDash \Box_0 q$ also holds. Indeed, take some $e \in W$ with $d \leq_0 e$. As $\leq_0$ is weakly connected, either $e = c$, or $c \leq_0 e$, or $e \leq_0 c$. In the first two cases, $\mathfrak{M}, e \vDash q$ holds by (3.10). If $e <_0 c$ then there is some $x \in W$ with $a \leq_0 x \leq_0 b$ and $xR_v^{k+1}e$. So $\mathfrak{M}, e \vDash q$ holds by (3.7) and (3.8).

$\Rightarrow$: Suppose that Φ_k fails in $\mathfrak{F}$, that is, there exists $a, b, c \in W$ such that $a \leq_0 bR_1^{k+1}c$ and all $d \in W$ are 'bad.' We define a model $\mathfrak{M} = (\mathfrak{F}, \vartheta)$ by taking

$$\vartheta(p) = \{b\},$$
$$\vartheta(q) = \{c\} \cup \{e : c \leq_0 e\} \cup \{e : \exists x \,(a \leq_0 x \leq_0 b \wedge xR_1^{k+1}e)\},$$
$$\vartheta(r) = \{c\}.$$

Then clearly $\mathfrak{M}, a \vDash \Diamond_0\big(p \wedge \Diamond_1^{k+1}(r \wedge q \wedge \Box_0 q)\big)$. We claim that $\mathfrak{M}, a \vDash \Box_0(\Diamond_0 p \to \Box_1^{k+1} q)$. Indeed, if $a \leq_0 x$ and $\mathfrak{M}, x \vDash \Diamond_0 p$, then $a \leq_0 x \leq_0 b$ should hold, and so $\mathfrak{M}, x \vDash \Box_1^{k+1} q$ follows. Next, we claim that $\mathfrak{M}, a \vDash \Box_1^{k+1}(\Diamond_0 r \to \neg\, \Box_0\, q)$. Indeed, take some d such that $aR_1^{k+1}d$ and $\mathfrak{M}, d \vDash \Diamond_0 r$. Then $d \leq_0 c$ and, by assumption, there is some e such that $d \leq_0 e <_0 c$ and $xR_1^{k+1}e$ holds for no x with $a \leq_0 x \leq_0 b$. Therefore, $\mathfrak{M}, e \not\vDash q$ and $\mathfrak{M}, d \not\vDash \Box_0 q$ follows, as required. ◻

Let us now turn to the proof of **(b)**. We define two special subframes of $\mathfrak{G}_k$:

- Let $\mathfrak{G}_k^{top}$ be the subframe of $\mathfrak{G}_k$ with universe $W^{top} = W - \{u_1, u_2, u_3\}$, and
- let $\mathfrak{G}_k^{bot}$ be the subframe of $\mathfrak{G}_k$ with universe $W^{bot} = W - \{v_1, v_2, v_3, v_4\}$,

see Figure 3.7. It is straightforward to see the following:

Claim **4.** *For every point x in $\mathfrak{G}_k$, either $(\mathfrak{G}_k)^{x,k}$ is a generated subframe of $\mathfrak{G}_k^{top}$, or it is a generated subframe of $\mathfrak{G}_k^{bot}$.*

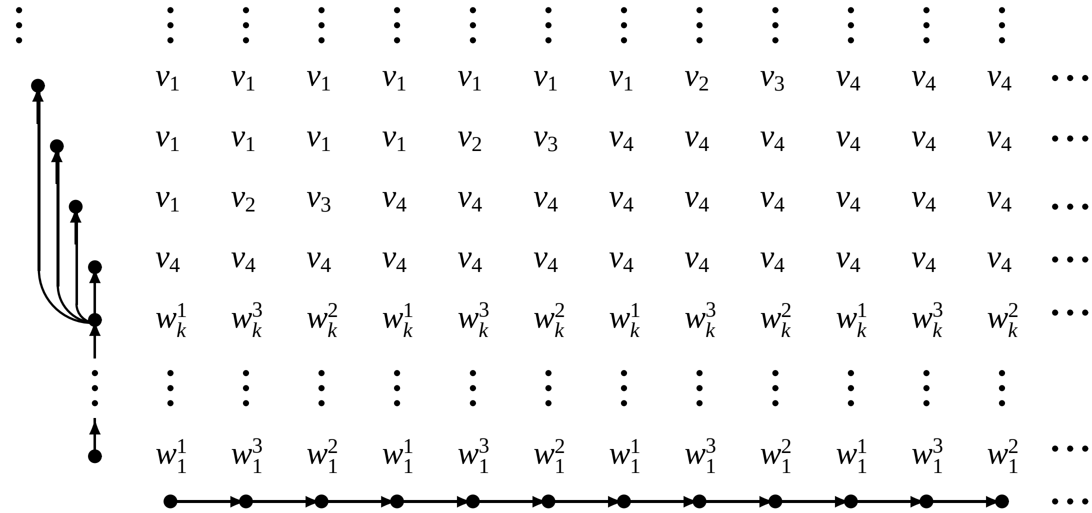

Figure 3.9: The p-morphism f from $(\omega, \leq) \times \mathfrak{F}$ onto $\mathfrak{G}_k^{top}$.

We complete the proof by showing that both $\mathfrak{G}_k^{top}$ and $\mathfrak{G}_k^{bot}$ are frames for L.

Lemma 3.2.11. *$\mathfrak{G}_k^{top}$ is a p-morphic image of $(\omega, \leq) \times \mathfrak{F}$, for some frame $\mathfrak{F}$.*

Proof. See Figure 3.9 for a function f from $(\omega, \leq) \times \mathfrak{F}$, where $\mathfrak{F}$ is a kind of 'rake:' an irreflexive and intransitive $k-1$-long path, followed by an irreflexive ω-fan. (In Figure 3.9 each point of $(\omega, \leq) \times \mathfrak{F}$ is labelled by its f-image.) It is not hard to check that f is a p-morphism from $(\omega, \leq) \times \mathfrak{F}$ onto $\mathfrak{G}_k^{top}$. □

Lemma 3.2.12. *$\mathfrak{G}_k^{bot}$ is a p-morphic image of $(\omega, \leq) \times \mathfrak{F}$, for some frame $\mathfrak{F}$.*

Proof. See Figure 3.10 for a function g from $\mathfrak{F}_0 \times \mathfrak{F}_1$, where $\mathfrak{F}_0$ is a 'balloon' (a two-point reflexive linear order, followed by a 3-point cluster), and $\mathfrak{F}_1$ is a kind of 'comb:' a root seeing three irreflexive and intransitive k-long branches. (In Figure 3.10 each point of $\mathfrak{F}_0 \times \mathfrak{F}_1$ is labelled by its g-image.) It is not hard to check that g is a p-morphism from $\mathfrak{F}_0 \times \mathfrak{F}_1$ onto $\mathfrak{G}_k^{bot}$. As $\mathfrak{F}_0$ is a p-morphic image of $(\omega, \leq)$, the lemma follows. □

3.3 Towards an explicit axiomatisation

In the previous section we presented cases of bimodal logics with linear components that cannot be finitely axiomatisable. Nonetheless, most of these logics are known to be recursively enumerable (e.g. if both logics are such that their classes of Kripke frames are

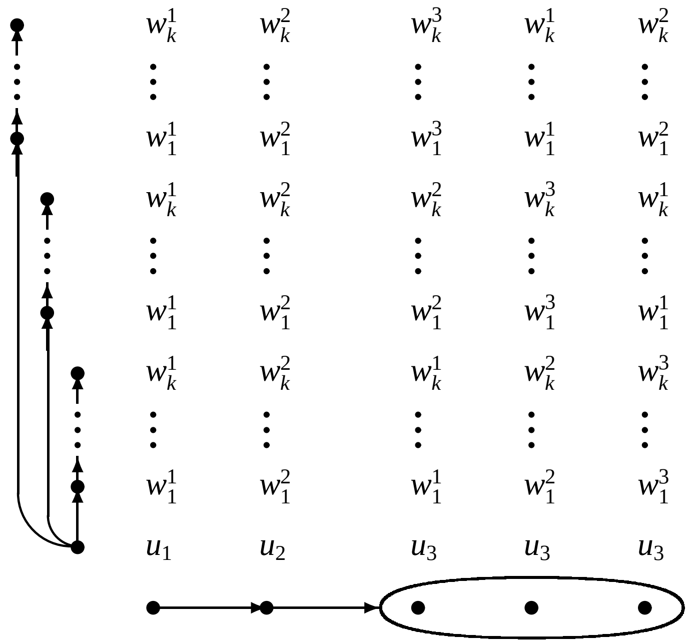

Figure 3.10: The p-morphism g from $\mathfrak{F}_0 \times \mathfrak{F}_1$ onto $\mathfrak{G}_k^{bot}$.

definable by recursive sets of first-order formulas, Section 3.1(1)) and so it makes sense to look for an explicit axiomatisation. Moreover, Theorems 3.2.1 and 3.2.3 above do not apply to product logics **K4.3**$\times L$, where L has no ω-fan among its frames. Important 'standard' logics of this kind are **K4.3** and **S5**. So the questions whether the recursively enumerable logics **K4.3** $\times$ **K4.3** and **K4.3** $\times$ **S5** are finitely axiomatisable remain open. (The same applies to products with **S4.3**.) Furthermore, so far no finitely axiomatisable but not product-matching product logic has been found. Clearly these two tasks are connected and while hunting for axioms for **K4.3**$\times$**K** we may discover that finite of these are enough to axiomatise **K4.3**$\times$**S5** or **K4.3** $\times$ **K4.3**.

Given a particular set of axioms, new formulas can be directly obtained by finding modal correspondents of first-order formulas that are valid in product frames but not in all 2-frames for the axioms in question. In Claim 3 we described an infinite set of those. The first-order formula corresponds to the guaranty that $\exists$ can successfully answer to a particular sequence of $\forall$'s plays in the p-morphism game, see Remark 3.2.5 and Figure 3.8. This is the very beginning of this research but the hope is to find a set of canonical axioms that guarantees that $\exists$ can win the ω-round global (not only in a square) game over finite frames. That is, that the logic is Kripke complete and the finite frames for the logic are bounded images of product frames. Moreover, if the considered logic has the finite model property (not known), this would guarantee that the resulting logic has the finite model property, and thus completeness is a direct consequence. Though, even if the obtained logic is not Kripke complete, the obtained information may be useful to work at the model level. Therefore finding independent properties of this kind is the natural first step for any attempt to obtain an explicit axiomatisation.

It is interesting to notice that for both considered sequences of bimodal frames that are not bounded images of product frames, in order to detect the 'defects', ∀ needs to consider more and more linear branches (through vertical branching). In this section we concentrate on what can 'go wrong' when the game is limited to a square. This problem may be subdivided into the two basic cases:

- ∀ plays the vertical direction based on the minimal point (regarding the linear order) played till then, the left squares, see Figure 3.11.

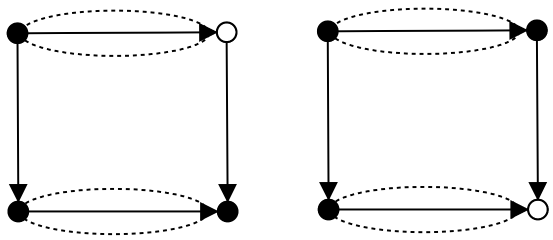

Figure 3.11: Left square game.

- ∀ plays the vertical direction based on the maximal point (regarding the linear order) played till then, the right squares, see Figure 3.12.

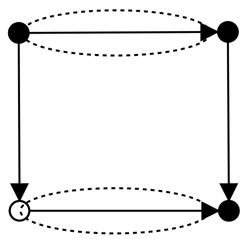

Figure 3.12: Right square game.

In both cases ∀ and ∃ are then limited to play inside the square, that is, they can only pick elements within the area limited by dashed lines.

In sub-Section 3.3.1 we prove it is enough to add two more modal formulas to guarantee that ∃ wins all ω-round plays on the left square games over finite frames. In sub-Section 3.3.2 we prove that despite the fact the similarities between the conditions imposed by φ_1 (see Claim 3) and the ones imposed by formulas used for the left case (they both guarantee that ∃ can answer to the following attack from ∀), it is not enough to guarantee a winning strategy for ∃ in the correspondent right square ω-game.

Interestingly the considered formulas are all simply generalised monadic Sahlqvist formulas in the sense of [41], thus, even if not Sahlqvist, they are canonical.

We start by introducing some more notation.

Definition 3.3.1. Let $\mathfrak{F} = (W, \leq_0, R_1)$ be a finite frame for $(\mathbf{K4.3}, \mathbf{K})$.

The *cluster* of x is

$$C_x = \{x' \,:\, x' = x \text{ or } (x \leq_0 x' \text{ and } x' \leq_0 x)\}.$$

Also we use the following:

$$\begin{aligned} x < y \quad &\text{iff} \quad x \leq_0 y \nleq_0 x, \\ x << y \quad &\text{iff} \quad x < y \;\&\; \forall x' \,(x \leq_0 x' < y \;\rightarrow\; x' \in C_x)). \end{aligned}$$

If $\mathfrak{F}$ is finite, for all $x, y \in W$ with $x \leq_0 y$, the *horizontal distance* $hd(x,y)$ *of* x *and* y is defined by taking

$$\begin{aligned} hd(x,y) = 0 \quad &\text{iff} \quad x \in C_y, \\ hd(x,y) = n \quad &\text{iff} \quad \text{there exist } z_0, \ldots, z_n \text{ such that } x = z_0 << z_1 << \cdots << z_n = y. \end{aligned}$$

Let A be a finite nonempty set of points linearly ordered by $\leq_0$. We say that $a^* \in A$ is *minimal in* A, if for all $a \in A$, either $a = a^*$ or $a^* \leq_0 a$. We say that $a^* \in A$ is *maximal in* A, if for all $a \in A$, either $a = a^*$ or $a \leq_0 a^*$.

We also introduce the following first-order formulas:

$$\begin{aligned} sq(x,y,z,w) :&\quad x \leq_0 wR_1z \;\&\; xR_1y \leq_0 z, \\ \psi_u(x,y,z,w) :&\quad sq(x,y,z,w) \;\&\; \forall b \,\big(y \leq_0 b < z \rightarrow \exists a \,(x \leq_0 a \leq_0 w \;\&\; aR_1b)\big), \end{aligned}$$

$$\psi_d(x,y,z,w): \quad sq(x,y,z,w) \;\&\; \forall a\,\big(x \leq_0 a < w \to \exists b\,(y \leq_0 b \leq_0 z \;\&\; aR_1 b)\big),$$
$$\psi_b(x,y,z,w): \quad \psi_u(x,y,z,w) \;\&\; \psi_d(x,y,z,w),$$
$$\Phi_{l,i}: \quad \forall x,y,z\,\big(xR_1 y \leq_0 z \to \exists w\,\psi_i(x,y,z,w)\big),$$
$$\Phi_{c,i}: \quad \forall x,y,w\,\big(x \leq_0 w \;\&\; xR_1 y \to \exists z\,\psi_i(x,y,z,w)\big),$$
$$\Phi_{r,i}: \quad \forall x,y,w\,\big(x \leq_0 wR_v z \to \exists y\,\psi_i(x,y,z,w)\big).$$

Let us also formalise the restricted version of the **K4.3**×**K**-p-morphism game to a square.

Definition 3.3.2 (Playing in a square)**.** Given a rooted frame $\mathfrak{F} = (W, \leq_0, R_1)$ for (**K4.3**, **K**), and $x,y,z,w \in W$ with $sq(x,y,z,w)$, let $\mathcal{G}_\omega(\mathfrak{F},x,y,z,w)$ denote the following version of the **K4.3** × **K**-p-morphism game: The two players $\forall$ and $\exists$ build a countable sequence of $(\mathfrak{F},x,y,z,w)$-*networks*

$$N_0 \subseteq N_1 \subseteq \cdots \subseteq N_k \subseteq$$

where, for each $k < \omega$, $N_k = (U_k, <_k, f_k)$ is such that

- $U_k = \{u_0, u_1, \ldots, u_{i_k}, u_\omega\}$ for some $i_k \leq k$, and $<_k$ is an irreflexive linear order on U_k with smallest element u_0 and largest element u_ω,
- f_k is a homomorphism from $(U_k, <_k) \times \big(\{0,1\}, \{(0,1)\}\big)$ to $\mathfrak{F}$ such that $f_k(u_0,0) = x$, $f_k(u_0,1) = y$, $f_k(u_\omega,1) = z$, $f_k(u_\omega,0) = w$.

(Here $N_{k-1} \subseteq N_k$ means that $U_{k-1} \subseteq U_k$, $<_{k-1} \subseteq <_k$, and $f_{k-1} \subseteq f_k$.)

In round 0, they begin with N_0 such that $U_0 = \{u_0, u_\omega\}$. In round $i+1$, for $i < \omega$, some sequence $N_0 \subseteq \cdots \subseteq N_i$ of $(\mathfrak{F},x,y,z,w)$-networks has already been built. $\forall$ picks some $s \in W$ such that

(a) either $x \leq_0 s < w$,

(b) or $y \leq_0 s < z$.

In case (a), if there is some $u \in U_i$ with $f_i(u,0) = s$, then $\exists$ responds with $N_{i+1} = N_i$. Otherwise, she takes the $<_i$-largest element u of U_i such that $f_i(u,0) \leq_0 s$, adds a new point u^* to U_i as the immediate $<_{i+1}$-successor of u, and defines $f_{i+1}(u^*,0)$ as s. Then $\exists$ also has to define (if she can) $f_{i+1}(u^*,1)$ such that, for all $u \in U_i$,

- if $u <_{i+1} u^*$ then $f_{i+1}(u,1) \leq_0 f_{i+1}(u^*,1)$, and
- if $u^* <_{i+1} u$ then $f_{i+1}(u^*,1) \leq_0 f_{i+1}(u,1)$.

In case (b), if there is some $u \in U_i$ with $f_i(u,1) = s$, then $\exists$ responds with $N_{i+1} = N_i$. Otherwise, she takes the $<_i$-largest element u of U_i such that $f_i(u,1) \leq_0 s$, adds a new point u^* to U_i as the immediate $<_{i+1}$-successor of u, and defines $f_{i+1}(u^*,1)$ as s. Then $\exists$ also has to define (if she can) $f_{i+1}(u^*,0)$ such that, for all $u \in U_i$,

- if $u <_{i+1} u^*$ then $f_{i+1}(u,0) \leq_0 f_{i+1}(u^*,0)$, and
- if $u^* <_{i+1} u$ then $f_{i+1}(u^*,0) \leq_0 f_{i+1}(u,0)$.

If $\exists$ can respond in each round i for $i < \omega$, then *she wins the play*. We say that $\exists$ *has a winning strategy in* $\mathcal{G}_\omega(\mathfrak{F}, x, y, z, w)$ if she can win all the plays, whatever moves $\forall$ takes in the rounds.

We will also consider the following version $\mathcal{G}'_\omega(\mathfrak{F}, x, y, z, w)$ of our game: now in each round i, $\forall$ is also allowed just to pass (that is, do nothing). In this case, $\exists$ should respond with $N_i = N_{i-1}$. Clearly, if $\exists$ has a winning strategy in $\mathcal{G}_\omega(\mathfrak{F}, x, y, z, w)$, then she has one in $\mathcal{G}'_\omega(\mathfrak{F}, x, y, z, w)$ as well.

3.3.1 Left Squares

Lemma 3.3.3. *There is a modal formula*

$$\varphi_{l,d}: \quad \Diamond_1\big(\Diamond_0 p \wedge \Box_0((p \vee \Diamond_0 p) \to q)\big) \wedge \Box_0(\Diamond_1 p \to \Box_0 r) \ \to\ \Box_0(\Diamond_1 q \vee r)$$

such that, for all finite frames $\mathfrak{F}$ for $(\mathbf{K4.3}, \mathbf{K}) \oplus com^l$,

$$\mathfrak{F} \models \Phi_{l,d} \qquad \Longleftrightarrow \qquad \mathfrak{F} \models \varphi_{l,d}.$$

Proof. $\Rightarrow$: Suppose that $\mathfrak{F} = (W, \leq_0, R_1)$ is such that $\mathfrak{F} \models \varphi_{l,d}$, and take some model $\mathfrak{M}$ based on $\mathfrak{F}$. Suppose that $x \in W$ is such that

$$\mathfrak{M}, x \models \Diamond_1\big(\Diamond_0 p \wedge \Box_0((p \vee \Diamond_0 p)p \to q)\big) \wedge \Box_0(\Diamond_1 p \to \Box_0 r).$$

Then there are $y, z \in W$ such that $xR_1 y \leq_0 z$, $\mathfrak{M}, y \models \Box_0((p \vee \Diamond_0 p) \to q)$ and $\mathfrak{M}, z \models p$. We claim that $\mathfrak{M}, x \models \Box_0(\Diamond_1 q \vee r)$ follows. Indeed, let a be such that $x \leq_0 a$. By $\varphi_{l,d}$, there exists w with $\psi_d(x, y, z, w)$. There are three cases:

1. $x \leq_0 a < w$. Then aR_1b for some b with $y \leq_0 b \leq_0 z$, and so $\mathfrak{M}, a \vDash \Diamond_1 q$.

2. $a = w$. Then aR_1z, so $\mathfrak{M}, a \vDash \Diamond_1 q$.

3. $w \leq_0 a$. As $\mathfrak{M}, z \vDash p$ and wR_1z, we have $\mathfrak{M}, w \vDash \Diamond_1 p$. As $x \leq_0 w$ and $\mathfrak{M}, x \vDash \Box_0(\Diamond_1 p \to \Box_0 r)$, we have $\mathfrak{M}, w \vDash \Box_0 r$, and so $\mathfrak{M}, a \vDash r$, as required. (Note that in this direction we did not use that $\mathfrak{F}$ is finite.)

$\Leftarrow$: Suppose that $\mathfrak{F} = (W, \leq_0, R_1)$ is such that $\mathfrak{F} \nvDash \Phi_{l,d}$, that is, there exist x, y, z such that $xR_1y \leq_0 z$, but for all $w \in W$, $\neg\psi_d(x,y,z,w)$. Let w be minimal with $sq(x,y,z,w)$ (here we use that $\mathfrak{F}$ is finite and validates com^l), and let a^* be such that $x \leq_0 a^* < w$ and there is no b with aR_1b and $y \leq_0 b \leq_0 z$. We define a model $\mathfrak{M}$ by taking

$$
\begin{aligned}
\mathfrak{M}, u \vDash p &\quad \text{iff} \quad u = z, \\
\mathfrak{M}, u \vDash r &\quad \text{iff} \quad w \leq_0 u, \\
\mathfrak{M}, u \vDash q &\quad \text{iff} \quad u = z \text{ or } y \leq_0 u \leq_0 z.
\end{aligned}
$$

Then $\mathfrak{M}, y \vDash \Diamond_0 p \wedge \Box_0((p \vee \Diamond_0 p) \to q)$, and so $\mathfrak{M}, x \vDash \Diamond_1\big(\Diamond_0 p \wedge \Box_0((p \vee \Diamond_0 p) \to q)\big)$. First, we claim that $\mathfrak{M}, x \vDash \Box_0(\Diamond_1 p \to \Box_0 r)$. Indeed, let a, a' be such that $x \leq_0 a \leq_0 a'$ and $\mathfrak{M}, a \vDash \Diamond_1 p$. Then $a \leq_0 z$ and so $sq(x,y,z,a)$. By the minimality of w, either $w = a$ or $w \leq_0 a$ follows. Thus $w \leq_0 a'$, and we obtain $\mathfrak{M}, a' \vDash r$, as required.

Next, we claim that $\mathfrak{M}, a^* \nvDash \Diamond_1 q \vee r$. Indeed, as $a^* < w$, we have $w \not\leq_0 a^*$, and so $\mathfrak{M}, a^* \nvDash r$. On the other hand, a^* by definition is such that there is no b with $a^* R_1 b$ and $y \leq_0 b \leq_0 z$, and by the minimality of w, $a^* \not R_1 z$. Therefore, $\mathfrak{M}, a^* \nvDash \Diamond_1 q$ follows.

So we obtained $\mathfrak{M}, x \nvDash \Box_0(\Diamond_1 q \vee r)$, and so $\mathfrak{M}, x \nvDash \varphi_{l,d}$ as required.

Remark 3.3.4. It is easy to see that the real first-order correspondent is:

$$\forall x, y, z\,\big(xR_1y \leq_0 z \to (\exists w\, \psi_d(x,y,z,w) \vee (\forall x'\, x \leq_0 x' \to (\exists w'\, x \leq_0 w' \leq_0 x' \wedge w'R_1z)))\big).$$

The only difference in the proof would be the right to left direction, where we have that $\mathfrak{M}, x \vDash \Box_0 r$ in the case that x is an accumulation point of points R_1-seeing z, in such a way that there is no minimal element completing the square.

This formula is canonical since it is simply generalised monadic Salhqvist formula, see Definition 2.1.27.

$\square$

Lemma 3.3.5. *For every finite frame $\mathfrak{F}$ for $(\textbf{K4.3}, \textbf{K})$, if $\mathfrak{F}$ validates $\Phi_{l,d}$, then $\mathfrak{F}$ validates $\Phi_{l,b}$ as well.*

Proof. Suppose that $\mathfrak{F} = (W, \leq_0, R_1)$ validates $\varphi_{l,d}$, and let $x, y, z \in W$ be such that $xR_1y \leq_0 z$. We claim that there exists w such that $\psi_b(x, y, z, w)$ holds. We prove the statement by induction on $hd(y, z)$. If $hd(y, z) = 0$ then the statement is obvious. Now suppose that $hd(y, z) = n$ for some $n > 0$ and the statement holds for every z' with $y \leq_0 z'$ and $hd(y, z') < n$. If $n = 1$ and $y \not\leq_0 y$, then the statement is obvious. Otherwise, let z' be such that $y \leq_0 z' << z$. Then $hd(y, z') < n$, so by the IH, there exists w' such that $\psi_b(x, y, z', w')$. Now if $z' \not\leq_0 z'$, then the statement obviously follows.

So we may assume that $z' \leq_0 z'$, that is, $|C_{z'}| = m > 0$. We will define sequences $w_0 < \cdots < w_N$ and $C^0 \supset \cdots \supset C^N$ for some $N \leq m$ such that, for every i:

(1) $\psi_d(w', z', z, w_i)$

(2) $C^i = \{c \in C_{z'} \;:\; \text{there is no } a \text{ with } w' \leq_0 a \leq_0 w_i \;\&\; aR_1c\}$.

As the C^is are decreasing, it follows that $C^N = \varnothing$ for some $N \leq m$, and then $\psi_b(w', z', z, w_N)$ holds. It follows that $\psi_b(x, y, z, w_N)$ as required.

To begin with, let w_0 be any w such that $\psi_d(w', z', z, w)$ (by $\varphi_{l,d}$ there is such). Now suppose that we have defined w_i and C^i such that (1)-(2) hold, and $C^i \neq \varnothing$. Take some $c \in C^i$. Then there exists v such that $\psi_d(w', z', c, v)$ and either $v = w_i$ or $w_i < v$. By $\varphi_{l,d}$, there exists w_{i+1} such that $\psi_d(v, c, z, w_{i+1})$. Then it is not hard to check that $\psi_d(w', z', z, w_{i+1})$ holds. Also, $C^{i+1} \subseteq C^i$ and $c \notin C^{i+1}$. ◻

Lemma 3.3.6. *There is a modal formula*

$$\varphi_{c,u}: \quad \Diamond_0(p \wedge \Box_1(q' \vee \Box_0 p')) \wedge \Box_0((p \vee \Diamond_0 p) \to \Box_1 q) \to \Box_1(\Diamond_0 q' \vee \Box_0(q \vee p'))$$

such that, for all finite frames $\mathfrak{F}$ for $(\textbf{K4.3}, \textbf{K}) \oplus crh$,

$$\mathfrak{F} \models \Phi_{c,u} \qquad \Longleftrightarrow \qquad \mathfrak{F} \models \varphi_{c,u}.$$

Proof. $\Rightarrow$: Suppose that $\mathfrak{F} = (W, \leq_0, R_1)$ is such that $\mathfrak{F} \models \varphi_{c,u}$, and take some model $\mathfrak{M}$ based on $\mathfrak{F}$. Suppose that $x \in W$ is such that

$$\mathfrak{M}, x \models \Diamond_0(p \wedge \Box_1(q' \vee \Box_0 p')) \wedge \Box_0((p \vee \Diamond_0 p) \to \Box_1 q).$$

Then there is w such that $x \leq_0 w$ and

$$
\begin{aligned}
&w \in V(p),\\
&\{b : \exists a\ (x \leq_0 a \leq_0 w \text{ or } a = w)\ \&\ aR_1 b\} \subseteq V(q),\\
&\{b : \exists z'\ wR_1 z' \leq_0 b\ \&\ z' \notin V(q')\} \subseteq V(p').
\end{aligned}
$$

Since for all y such that xR_1y there is z such that $\psi_u(x,y,z,w)$, so $z \in V(q)$, plus for all b, $y \leq_0 b < z$ implies $\mathfrak{F}, V, b \vDash q$. Thus, if $\mathfrak{M}, y \vDash \neg \Diamond_0 q'$ then we get that $\mathfrak{M}, z \vDash \neg q'$ and $\mathfrak{M}, z \vDash \Box_0 p'$, so for all c such that $z \leq_0 c$ $\mathfrak{M}, c \vDash p'$. Therefore $\mathfrak{M}, y \vDash \Box_0(q \vee p')$ and hence $\mathfrak{M}, x \vDash \Box_1(\Diamond_0 q' \vee \Box_0(q \vee p'))$. (Again we do not use finiteness in this direction.)

$\Leftarrow$: Suppose that $\mathfrak{F} = (W, \leq_0, R_1)$ is such that $\mathfrak{F} \nvDash \Phi_{c,u}$, that is, there exist x, y, w such that $x \leq_0 w$ and xR_1y and

$$\forall z\ sq(x,y,z,w) \rightarrow \exists b\ (y \leq_0 b < z\ \&\ \forall a\ x \leq_0 a \leq_0 w \rightarrow a\not{R}_1 b).$$

We pick the minimal z such that $sq(x,y,z,w)$ (there is at least one by CR and from finiteness and boundness, a minimal, obviously $y \neq z$) and $\mathfrak{M}$ over $\mathfrak{F}$ with V such that

$$
\begin{aligned}
V(p) &= \{w\},\\
V(p') &= \{b : \exists z'\ wR_1 z' \leq_0 b\ \&\ z' \notin V(q')\}\\
V(q) &= \{b : \exists a\ (x \leq_0 a \leq_0 w \text{ or } a = w)\ \&\ aR_1 b\},\\
V(q') &= \{c : y \nleq_0 c\}.
\end{aligned}
$$

So,

$$\mathfrak{M}, x \vDash \Diamond_0(p \wedge \Box_1(q' \vee \Box_0 p')) \wedge \Box_0((p \vee \Diamond_0 p) \rightarrow \Box_1 q).$$

We know that there exists b such that $y \leq_0 b < z$ and $\forall a\ x \leq_0 a \leq_0 w \rightarrow a\not{R}_1 b$ thus, since z is minimal,

$$\mathfrak{M}, y \vDash \neg \Diamond_0 q' \text{ and } \mathfrak{M}, b \vDash \neg q \wedge \neg p'.$$

Hence $\mathfrak{M}, y \vDash \neg \Diamond_0 q' \wedge \Diamond_0(\neg q \wedge \neg p')$ and therefore $\mathfrak{M}, x \vDash \neg\varphi_{c,u}$.

Remark 3.3.7. It is easy to see that the real first-order correspondent is:

$$\Phi'_{c,u} : \forall x, y, w \big(x \leq_0 w\ \& xR_1 y \rightarrow (\exists z\, \psi_d(x,y,z,w) \vee (\forall y'\ y \leq_0 y' \rightarrow (\exists z'\ y \leq_0 z' \leq_0 y' \wedge wR_1 z')))\big).$$

Since $\Phi_{c,u} \to \Phi'_{c,u}$, the only thing missing in the proof is in the left to right direction, where we conclude that $\mathfrak{M}, x \vDash \Diamond_0 q' \vee \Box_0 p'$ in the case that y is an accumulation point of points R_1-seen by w (so there is no minimal element completing the square). This formula is canonical since it is simply generalised monadic Salhqvist formula, see Definition 2.1.27.

□

Lemma 3.3.8. *Let $\mathfrak{F} = (W, \leq_0, R_1)$ be a finite frame for $(\textbf{K4.3}, \textbf{K})$ that validates $\Phi_{l,b}$ and $\Phi_{c,u}$. For all $x, y, z \in W$, if $y \leq_0 y$ and $xR_1y << z$, then there exist w^* and a^* such that $\psi_b(x,y,z,w^*)$, $x \leq_0 a^* \leq_0 w^*$ and a^*R_1b for some $b \in C_y$, and the following hold:*

(1) $a^* \leq_0 a^*$ & $\forall b \in C_y\ \exists a \in C_{a^*}\ aR_1b$,

(2) *either* **(2a)***:* $\forall a\ (x \leq_0 a < w^* \to \exists b \in C_y\ aR_1b\ \&\ a \leq_0 a^*)$,

or **(2b)***:* $\exists r \big(x \leq_0 r < w^*\ \&\ (\neg\exists b \in C_y\ rR_1b)\ \&\ (\exists b \in C_z\ rR_1b)\ \&\ (r \in C_{a^*} \vee a^* << r)\ \&.$

$\&\ \forall a\,(x \leq_0 a < r \to \exists b \in C_y\ aR_1b)\ \&\ \forall a\,(r \leq_0 a < w^* \to \exists b \in C_z\ aR_1b)\big)$.

Proof. By $\Phi_{l,b}$, there exists w with $\psi_b(x,y,z,w)$. If $w \in C_x$ then $w^* = a^* = w$ clearly have **(1)** and **(2a)**. So suppose there is no such w, and let w be maximal such that $\psi_b(x,y,z,w)$ and $x < w$. We consider two cases:

Case 1. $\forall a\ (x \leq_0 a < w \to \exists b \in C_y\ aR_1b)$.
As $\psi_b(x,y,z,w)$ and $y \leq_0 y < z$, there is a with $x \leq_0 a \leq_0 w$ and aR_1b for some $b \in C_y$. Let a^* be a maximal such a. We show that $w^* = w$ and a^* satisfy **(1)** and **(2a)**. For **(1)**: Let $b^* \in C_y$ be such that $a^*R_1b^*$. Then by $\Phi_{l,b}$, there exists w' with $\psi_b(a^*, b^*, z, w')$.

Claim 5. $\psi_b(x,y,z,w')$.

Proof. First, if $y \leq_0 b < z$ then $b^* \leq_0 b < z$, so by $\psi_b(a^*, b^*, z, w')$ there exists a such that $x \leq_0 a^* \leq_0 a \leq_0 w'$ and aR_1b. Second, if $x \leq_0 a < w'$ then either (a) $a^* \leq_0 a < w'$ when, by $\psi_b(a^*, b^*, z, w')$, aR_1b for some b with $b^* \leq_0 b \leq_0 z$; or (b) $a = a^*$ when aR_1b^* and $y \leq_0 b^* \leq_0 z$; or (c) $x \leq_0 a < a^* \leq_0 w^*$ when, by $\psi_b(x,y,z,w^*)$, there is some b with aR_1b and $y \leq_0 b \leq_0 z$. □

So, by the maximality of w^*, we have either $w' = w^*$ or $w' \leq_0 w^*$. Now take some $b \in C_y$. Then $b^* \leq_0 b < z$, so by $\psi_b(a^*, b^*, z, w')$ there is some a with aR_1b and $a^* \leq_0 a \leq_0 w^*$. By the maximality of a^*, $a^* \leq_0 a^*$ and $a \in C_{a^*}$ follows, as required in **(1)**. For **(2a)**: Let a be such that $x \leq_0 a < w^* = w$. By assumption, there is some $b \in C_y$ with aR_1b. By the maximality of a^*, either $a = a^*$ or $a \leq_0 a^*$. As $a^* \leq_0 a^*$, we have $a \leq_0 a^*$, as required.

Case 2. $\exists a\ (x \leq_0 a < w\ \&\ \neg\exists b \in C_y\ aR_1b)$.
Let r be a minimal such a. By $y \leq_0 y$ and $\Phi_{l,b}$, there exists a such that $\psi_b(x, y, y, a)$. By the definition of r, $a \leq_0 r$ must hold. Now let a^* be a maximal a such that $x \leq_0 a \leq_0 r$ and aR_1b for some $b \in C_y$.

Claim **6.** *If $a^* \leq_0 a < r$ then $a \in C_{a^*}$ and aR_1b for some $b \in C_y$.*

Proof. Let a be such that $a^* \leq_0 a < r$. By the definition of r and $\psi_b(x, y, z, w)$, it follows that there is a $b \in C_y$ with aR_1b. So, by the maximality of a^*, either $a = a^*$ or $a \leq_0 a^*$, and so $a \in C_{a^*}$. □

Now let $b^* \in C_y$ be such that $a^*R_1b^*$. By $\Phi_{c,u}$, there exists r' with $\psi_u(a^*, b^*, r', r)$. By the definition of r, we have $b^* < r'$. By $\psi_b(x, y, z, w)$ and $x \leq_0 r < w$, we can choose some $r' \in C_z$. Now, by Claim 6, we actually have $\psi_b(a^*, b^*, r', r)$. Next, by $\Phi_{l,b}$, there exists w' with $\psi_b(r, r', z, w')$.

Claim **7.** $\psi_b(x, y, z, w')$.

Proof. First, if $y \leq_0 b < z$ then $b^* \leq_0 b < r'$, so by $\psi_b(a^*, b^*, r', r)$ there exists a such that $x \leq_0 a^* \leq_0 a \leq_0 r \leq_0 w'$ and aR_1b. Second, if $x \leq_0 a < w'$ then either (a) $r \leq_0 a < w'$ when aR_1b for some b with $y \leq_0 r' \leq_0 b \leq_0 z$ by $\Psi_b(r, r', z, w')$; or (b) $a = r$ when aR_1r'; or (c) $x \leq_0 a < r$. In case (c), by $\psi_b(x, y, z, w)$, we have some b such that aR_1b and $y \leq_0 b \leq_0 z$. □

So, by the maximality of w, we have either $w' = w$ or $w' \leq_0 w$. We will show that $w^* = w'$ and a^* will do. First, let us show that they satisfy **(1)**. Let $b \in C_y$. Then $b^* \leq_0 b < r'$, so by $\psi_b(a^*, b^*, r', r)$, there is some a such that $x \leq_0 a^* \leq_0 a \leq_0 r$ and aR_1b. By the maximality of a^*, we have either $a = a^*$ or $a \leq_0 a^*$, and so $a^* \leq_0 a^*$ and $a \in C_{a^*}$, as required. As concerns **(2)**, there are two cases, depending on the location of r vs. w^*:

Case 2.1. $r \in C_{w^*}$. We show that in this case **(2a)** holds. Indeed, let a be such that $x \leq_0 a < w^*$. By Claim 6, there is no a such that $a^* < a < r$, so $a \leq_0 a^*$ follows from $a^* \leq_0 a^*$. Now suppose that there is no $b \in C_y$ with aR_1b. As $x \leq_0 a < w$, by the minimality of r, either $a = r$ or $r \leq_0 a$. So $w^* \leq_0 a$, contradicting $a < w^*$.

Case 2.2. $r < w^*$. We show that in this case **(2b)** holds for our r. Indeed, now $x \leq_0 r < w^*$, and $\neg\exists b \in C_y\ rR_1b$ holds by the definition of r. By Claim 6, we have either $r \in C_{a^*}$ or $a^* << r$. Next, let a be such that $x \leq_0 a < r < w$. Then, by the minimality of r and $\psi_b(x, y, z, w)$, there is some $b \in C_y$ with aR_1b. Finally, let a be such that $r \leq_0 a < w^*$. Then, by $\psi_b(r, r', z, w^*)$, there is $b \in C_z$ with aR_1b. □

Lemma 3.3.9. *Let $\mathfrak{F} = (W, \leq_0, R_1)$ be a finite frame for $(\mathbf{K4.3}, \mathbf{K})$ that validates $\Phi_{l,b}$ and $\Phi_{c,u}$. For all $x, y, z \in W$, if $xR_1y << z$, then there exists w^* such that $sq(x, y, z, w^*)$ and $\exists$ has a winning strategy in $\mathcal{G}_\omega(\mathfrak{F}, x, y, z, w^*)$.*

Proof. If $y \not\leq_0 y$, then by $\Phi_{l,b}$, there exists w^* such that $\psi_b(x, y, z, w^*)$. We claim that this w^* will do. First, there is no b such that $y \leq_0 b < z$. Second, if a is such that $x \leq_0 a < w^*$, then there is some b with $y \leq_0 b \leq_0 z$, and so $b \in C_z$. Thus the statement follows.

Now suppose that $y \leq_0 y$, and let w^* and a^* be like in Lemma 3.3.8. Let us consider first the case when they satisfy **(1)** and **(2a)**.

Claim **8.** *$\exists$ has a winning strategy in $\mathcal{G}_\omega(\mathfrak{F}, x, y, z, w^*)$ such that, for all $n < \omega$,*

- *$f_n(u, 0) \leq_0 a^*$ and*
- *$f_n(u, 1) \in C_y$,*

for all $u \in U_n - \{u_\omega\}$.

Proof. It is by induction on n. For $n = 0$ it is obvious. Assume that we have the claim for n. Suppose first that $\forall$ picks an s such that $x \leq_0 s < w^*$. Suppose also that there is no $u \in U_n$ with $f_n(u, 0) = s$, so $\exists$ has to respond with a new point u^*, and set $f_{n+1}(u^*, 0) = s$. Then, by **(2a)**, $f_{n+1}(u^*, 0) \leq_0 a^*$, and $\exists$ can choose some $f_{n+1}(u^*, 1) \in C_y$ such that $sR_1 f_{n+1}(u^*, 1)$. By the IH, $f_{n+1}(u, 1) \in C_y$ for all $u \in U_n - \{u_\omega\}$. So whenever $u, u' \in U_{n+1}$ and $u <_{n+1} u'$, then $f_{n+1}(u, 1) \leq_0 f_{n+1}(u', 1)$, as required.

Next, suppose that $\forall$ picks an s' such that $y \leq_0 s' < z$, that is, $s' \in C_y$. Suppose also that there is no $u \in U_n$ with $f_n(u, 1) = s'$, so $\exists$ has to respond with a new point u^*, and set $f_{n+1}(u^*, 1) = s'$. Note that, by the IH, $f_n(u, 1) \in C_y$ for all $u \in U_n - \{u_\omega\}$, so u^* is the $<_{n+1}$-largest non-u_ω element of U_{n+1}. As $s' \in C_y$, by **(1)**, $a^* \leq_0 a^*$ and $\exists$ can choose some

$f_{n+1}(u^*,0) \in C_{a^*}$ such that $f_{n+1}(u^*,0) R_1 s'$. By the IH, for all $u \in U_n - \{u_\omega\}$, $f_{n+1}(u,0) \leq_0 a^*$. Therefore, $f_{n+1}(u,0) \leq_0 f_{n+1}(u^*,0)$, for all $u \in U_n - \{u_\omega\}$, as required. □

Now consider the case when w^* and a^* satisfy **(1)** and **(2b)**, and let r be like in **(2b)**.

Claim 9. *$\exists$ has a winning strategy in $\mathcal{G}_\omega(\mathfrak{F}, x, y, z, w^*)$ such that, for all $n < \omega$,*

- *if $f_n(u,0) < r$ then $f_n(u,1) \in C_y$, and*
- *if either $f_n(u,0) = r$ or $r \leq_0 f_n(u,0)$ then $f_n(u,1) \in C_z$,*

for all $u \in U_n$.

Proof. It is by induction on n. For $n = 0$ it is obvious. Assume that we have the claim for n. Suppose first that $\forall$ picks s such that $x \leq_0 s < w^*$. Suppose also that there is no $u \in U_n$ with $f_n(u,0) = s$, so $\exists$ has to respond with a new point u^*, and set $f_{n+1}(u^*,0) = s$. As concerns how $\exists$ should choose $f_{n+1}(u^*,1)$, there are two cases, depending on the location of s vs. r:

Case 1. $x \leq_0 s < r$. Then by **(2b)**, $\exists$ can choose some $f_{n+1}(u^*,1) \in C_y$ such that $s R_1 f_{n+1}(u^*,1)$ holds. Then $f_{n+1}(u^*,1) \leq_0 f_{n+1}(u,1)$ always holds, for any $u \in U_n$. If $u <_{n+1} u^*$ for some $u \in U_n$, then $f_{n+1}(u,0) \leq_0 s < r$. So, by the IH, $f_{n+1}(u,1) \in C_y$, and so $f_{n+1}(u,1) \leq_0 f_{n+1}(u^*,1)$.

Case 2. $s = r$ or $r \leq_0 s < w^*$. Then, by **(2b)**, $\exists$ can choose some $f_{n+1}(u^*,1) \in C_z$ such that $s R_1 f_{n+1}(u^*,1)$ holds. Then $f_{n+1}(u,1) \leq_0 f_{n+1}(u^*,1)$ always holds, for any $u \in U_n$. Now let $u^* <_{n+1} u$, for some $u \in U_n$. Then $r \leq_0 s \leq_0 f_{n+1}(u,0) \leq_0 w^*$. So, by the IH, $f_{n+1}(u,1) \in C_z$, and so $f_{n+1}(u^*,1) \leq_0 f_{n+1}(u,1)$.

Next, suppose that $\forall$ picks s' such that $y \leq_0 s' < z$, that is, $s' \in C_y$. Suppose also that there is no $u \in U_n$ with $f_n(u,1) = s'$, so $\exists$ has to respond with a new point u^*, and set $f_{n+1}(u^*,1) = s'$. By **(1)**, $\exists$ can choose some $f_{n+1}(u^*,0) \in C_{a^*}$ such that $f_{n+1}(u^*,0) R_1 s'$. Now let $u <_{n+1} u^*$ for some $u \in U_n$. Then $f_{n+1}(u,1) \leq_0 s'$, so $f_{n+1}(u,1) \in C_y$. By the IH, we have $f_{n+1}(u,0) < r$, so by **(2b)**, $f_{n+1}(u,0) \leq_0 a^* \leq_0 f_{n+1}(u^*,0)$ follows, as required. Next, let $u^* <_{n+1} u$ for some $u \in U_n$. Then $s' < f_{n+1}(u^*,1)$, and so $f_{n+1}(u^*,1) \in C_z$. So, by the IH, either $f_{n+1}(u,0) = r$ or $r \leq_0 f_{n+1}(u,0)$. Therefore, by **(2b)**, $f_{n+1}(u^*,0) \leq_0 a^* \leq_0 f_{n+1}(u,0)$, as required. □

Now Lemma 3.3.9 clearly follows. □

Lemma 3.3.10. *Let $\mathfrak{F} = (W, \leq_0, R_1)$ be a finite frame for $(\textbf{K4.3}, \textbf{K})$ that validates $\Phi_{l,b}$ and $\Phi_{c,u}$. For all $x, y, z \in W$, if $xR_1 y \leq_0 z$, then there exists w^* such that $sq(x, y, z, w^*)$ and $\exists$ has a winning strategy in $\mathcal{G}_\omega(\mathfrak{F}, x, y, z, w^*)$.*

Proof. It is by induction on $hd(y, z)$. If $hd(y, z) = 0$ then, by $\Phi_{l,b}$, there exists w such that $\psi_b(x, y, z, w)$. Such a w will clearly do as w^*. Now suppose that the lemma holds for all z' with $hd(y, z') \leq m$, and let $x, y, z', z \in W$ be such that $xR_1 y \leq_0 z' << z$ and $hd(y, z') = m$. By the IH, there is some w' such that $sq(x, y, z', w')$ and $\exists$ has a winning strategy in $\mathcal{G}_\omega(\mathfrak{F}, x, y, z', w')$. Also, by Lemma 3.3.9, there is some w^* such that $sq(w', z', z, w^*)$ and $\exists$ has a winning strategy in $\mathcal{G}_\omega(\mathfrak{F}, w', z', z, w^*)$.

In order to formulate our claim, we need the following notion. Suppose that we are given an $(\mathfrak{F}, x, y, z', w')$-network $N^1 = (U^1, <^1, f^1)$ and an $(\mathfrak{F}, w', z', z, w^*)$-network $N^2 = (U^2, <^2, f^2)$, where $U^1 = \{u_0^1, \ldots, u_\omega^1\}$ and $U^2 = \{u_0^2, \ldots, u_\omega^2\}$ are disjoint sets. Then

$$N^1 \sqcup N^2 = (U, <, f)$$

is defined by taking $U = U^1 \cup U^2 - \{u_\omega^1, u_0^2\}$, linearly ordered by $<$ such that first come the points from U^1 as ordered by $<^1$, followed by the points from U^2 as ordered by $<^2$, and f is the restriction of $f^1 \cup f^2$ on $U \times \big(\{0, 1\}, \{(0, 1)\}\big)$. Further, take some fresh point u^+, not in U. Then let

$$N^1 \sqcup^+ N^2 = (U^+, <^+, f^+),$$

where $(U^+, <^+)$ is defined by adding u^+ to $(U, <)$ 'in the middle', that is, such that it is $<^+$-larger than all points coming from U^1, and $<^+$-smaller than all points coming from U^2, and f^+ is the extension of f such that $f^+(u^+, 0) = w'$ and $f^+(u^+, 1) = z'$. It is easy to check that both $N^1 \sqcup N^2$ and $N^1 \sqcup^+ N^2$ are $(\mathfrak{F}, x, y, z, w^*)$-networks.

Claim 10. *$\exists$ has a winning strategy in $\mathcal{G}_\omega(\mathfrak{F}, x, y, z, w^*)$ such that, for all $n < \omega$, her response $N_n = (U_n, <_n, f_n)$ has the following properties: there exist N_n^1 and N_n^2 such that*

- *either $N_n = N_n^1 \sqcup N_n^2$ or $N_n = N_n^1 \sqcup^+ N_n^2$,*
- *$N_0^1 \subseteq \cdots \subseteq N_n^1$ is an n round play in $\mathcal{G}'_\omega(\mathfrak{F}, x, y, z', w')$,*
- *$N_0^2 \subseteq \cdots \subseteq N_n^2$ is an n round play in $\mathcal{G}'_\omega(\mathfrak{F}, w', z', z, w^*)$.*

Proof. It is by induction on n. For $n = 0$ it is obvious. Assume that we have the claim for n, and now we are in round $n+1$ of a play of $\mathcal{G}_\omega(\mathfrak{F}, x, y, z, w^*)$. Suppose that $\forall$ picks either an s with $x \leq_0 s < w^*$, or an s' with $y \leq_0 s' < z$. There are three cases:

Case 1. Either $x \leq_0 s < w'$ or $y \leq_0 s' < z'$.
$\exists$ uses her winning strategy in $\mathcal{G}'_\omega(\mathfrak{F}, x, y, z', w')$ to obtain $N^1_{n+1} \supseteq N^1_n$, sets $N^2_{n+1} = N^2_n$ and

$$N_{n+1} = \begin{cases} N^1_{n+1} \sqcup^+ N^2_{n+1}, & \text{if } u^+ \in U_n, \\ N^1_{n+1} \sqcup N^2_{n+1}, & \text{otherwise.} \end{cases}$$

Case 2. Either $s = w'$ and $w' \not\leq_0 w'$, or $s' = z'$ and $z' \not\leq_0 z'$.
$\exists$ sets $N^1_{n+1} = N^1_n$, $N^2_{n+1} = N^2_n$ and $N_{n+1} = N^1_{n+1} \sqcup^+ N^2_{n+1}$.

Case 3. Either $w' \leq_0 s < w^*$ or $z' \leq_0 s' < z$.
$\exists$ sets $N^1_{n+1} = N^1_n$, uses her winning strategy in $\mathcal{G}'_\omega(\mathfrak{F}, w', z', z, w^*)$ to obtain $N^2_{n+1} \supseteq N^2_n$, and sets

$$N_{n+1} = \begin{cases} N^1_{n+1} \sqcup^+ N^2_{n+1}, & \text{if } u^+ \in U_n, \\ N^1_{n+1} \sqcup N^2_{n+1}, & \text{otherwise.} \end{cases}$$

It is straightforward to check that N^1_{n+1}, N^2_{n+1}, and N_{n+1} are as required. □

Now Lemma 3.3.10 clearly follows. □

Corollary 3.3.11. *Let $\mathfrak{F} = (W, \leq_0, \equiv_1)$ be a finite frame for $(\textbf{K4.3}, \textbf{S5})$ that validates $\Phi_{l,b}$. For all $x, y, z \in W$, if $x \equiv_1 y \leq_0 z$, then there exists w^* such that $sq(x, y, z, w^*)$ and $\exists$ has a winning strategy in $\mathcal{G}_\omega(\mathfrak{F}, x, y, z, w^*)$.*

In the vertical transitive cases we do not have to define a less restricted version of the **K4.3** $\times$ **K** to consider a wider range of defects than the one that may be detected by playing over a square, indeed we can ask the question:

Question 3.3.12. *Let $\mathfrak{F} = (W, \leq_0, R_1)$ be a finite frame for $(\textbf{K4.3}, \textbf{K4.3})$ (or $(\textbf{K4.3}, \textbf{S5})$) that validates $\Phi_{l,b}$ and $\Phi_{c,u}$. Let $x, x', y, z, w \in W$ be such that $x' R_1 x R_1 y$ and $\exists$ has a winning strategy in $\mathcal{G}_\omega(\mathfrak{F}, x, y, z, w)$. Does it follow that there exists w' such that $\exists$ has a winning strategy both in $\mathcal{G}_\omega(\mathfrak{F}, x', y, z, w')$ and $\mathcal{G}_\omega(\mathfrak{F}, x', x, w, w')$?*

3.3.2 Right Squares

In Claim 3 we have seen that there is a modal formula

$$\varphi_1 = \varphi_{r,u} : \quad \Diamond_0(p \wedge \Diamond_1(p' \wedge q \wedge \Box_0 q)) \wedge \Box_0(\Diamond_0 p \to \Box_1 q) \to \Diamond_1(\Diamond_0 p' \wedge \Box_0 q)$$

such that, for all frames $\mathfrak{F}$ for $(\mathbf{K4.3}, \mathbf{K})$,

$$\mathfrak{F} \models \Phi_{r,u} \qquad \Longleftrightarrow \qquad \mathfrak{F} \models \varphi_{r,u}.$$

And even if a similar lemma to Lemma 3.3.5 can be proved:

Claim **11.** *For every finite frame $\mathfrak{F}$ for $(\boldsymbol{K4.3}, \boldsymbol{K})$, if $\mathfrak{F}$ validates $\Phi_{r,u}$, then $\mathfrak{F}$ validates $\Phi_{r,b}$ as well.*

Proof. Suppose that $\mathfrak{F} = (W, \leq_0, R_1)$ validates $\Phi_{r,d}$, and let $x, z, w \in W$ be such that $x \leq_0 zR_1 w$. We claim that there exists y such that $\psi_b(x, y, z, w)$ holds. We prove the statement by induction on $hd(x, w)$. If $hd(x, w) = 0$ then the statement is obvious. Now suppose that $hd(x, w) = n$ for some $n > 0$ and the statement holds for every x' with $x' \leq_0 w$ and $hd(x', w) < n$. If $n = 1$ and $x \not\leq_0 x$, then the statement is obvious. Otherwise, let x' be such that $x \leq_0 x' << w$. Then $hd(x', w) < n$, so by the IH, there exists y' such that $\psi_b(x', y', z, w)$. Now if $x \not\leq_0 x$, then the statement obviously follows.

So we may assume that $x \leq_0 x$, that is, $|C_x| = m > 0$. We will define sequences $y_0, \dots, y_N$ and $C^0 \supset \cdots \supset C^N$ for some $N \leq |C_x|$ such that, for every i:

(1) $\psi_u(x, y_i, z, w)$

(2) $C^i = \{a \in C_x \ : \text{ there is no } b \text{ with } y_i \leq_0 b \leq_0 z \ \& \ aR_1 b\}$.

As the C^is are decreasing, it follows that $C^N = \varnothing$ for some $N \leq |C_x|$, and then $\psi_b(x, y_N, z, w)$ holds, as required.

To begin with, let y_0 be any y such that $\psi_u(x, y, z, w)$ (by $\Phi_{r,u}$ there is such). Now suppose that we have defined y_i and C^i such that (1)-(2) hold, and $C^i \neq \varnothing$. Take some $a^* \in C^i$. By $\Phi_{r,u}$, there exists b^* such that $\psi_u(a^*, b^*, y_i, x)$. By $\Phi_{r,u}$ again, there exists y_{i+1} such that $\psi_u(x, y_{i+1}, b^*, a^*)$.

Claim **12.** *$\psi_u(x, y_{i+1}, z, w)$, that is, for every b with $y_{i+1} \leq_0 b < z$, there exists a with $x \leq_0 a \leq_0 w$ and $aR_1 b$,* *Proof.* As $y_{i+1} \leq_0 b^* \leq_0 y_i \leq_0 z$, there are five cases:

Case 1. $y_{i+1} \leq_0 b < b^*$. Then the claim holds by $\psi_u(x, y_{i+1}, b^*, a^*)$.

Case 2. $b = b^*$. Then the claim holds by $a^* R_1 b^*$.

Case 3. $b^* \leq_0 b < y_i$. Then the claim holds by $\psi_u(a^*, b^*, y_i, x)$.

Case 4. $b = y_i$. Then the claim holds by $x R_1 y_i$ and $x \leq_0 x$.

Case 5. $y_i \leq_0 b < z$. Then the claim holds by $\psi_u(x, y_i, z, w)$. □

As $y_{i+1} \leq_0 v \leq_0 y_i$, we have that $C^{i+1} \subseteq C^i$ and $a \notin C^{i+1}$, so the lemma follows. □

Therefore, imposing $\varphi_{r,u}$ assures us that $\exists$ can answer successfully to $\forall$'s first attack after the formation of the square. In the case of left squares this was enough to guarantee that $\exists$ had a winning strategy for all further attacks. In Figure 3.13 is depicted a frame satisfying $\Phi_{l,b}$ and $\Phi_{r,b}$ but $\exists$ cannot answer to the next round. The horizontal arrows and ellipses represent the reflexive, transitive and weakly connected $\leq_0$, and the boxes, triangles and circles the $\equiv_1$-equivalence classes. It is not hard to check that $\leq_0$ and $\equiv_1$ commute. (In case of a symmetric second relation, the Church-Rosser property follows from commutativity.)

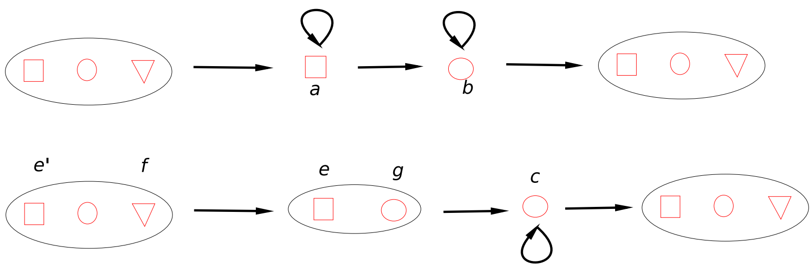

Figure 3.13: $\Phi_{l,b}$, $\Phi_{c,u}$ and $\Phi_{r,b}$ but $\exists$ loses in the second round after the formation of the (right) square.

Consider the following attack from $\forall$: he opens with the sequence a, b, c, forcing a right square. If $\exists$ answers with e' then she loses immediately if $\forall$ plays f. If $\exists$ plays e then $\forall$ plays g to which $\exists$ must respond with b and then $\exists$ loses if $\forall$ plays e. Figure 3.14 depicts these two cases.

Theorem 3.3.13. ***K4.3*** $\times$ ***S5*** *is not product matching.*

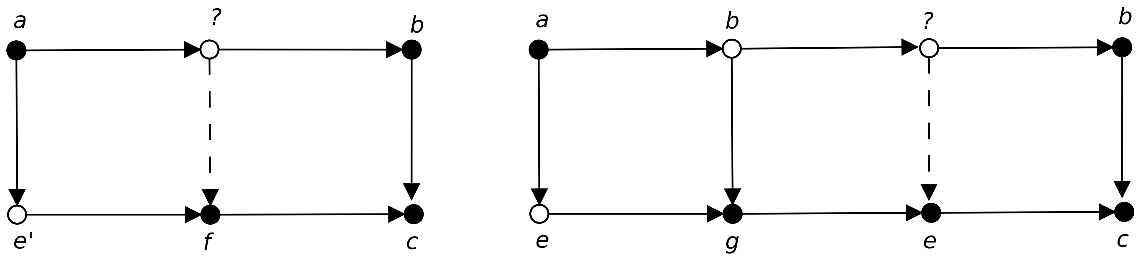

Figure 3.14: ∃ loses.

Proof. It is straightforward to see that the (finite) frame represented in Figure 3.13 is a frame for $[\mathbf{K4.3} \times \mathbf{S5}]$ but it is not the image of any product frame. ◻

Remark 3.3.14. The frame represented in Figure 3.15 (using the same notation as in the previous figure) is also a frame for the commutator and it is straightforward to see that property $\Phi_{r,u}$ $(= \Phi_0)$ (see Figure 3.8) fails in $\mathfrak{F}$. This is an alternative proof of the previous theorem. Moreover, it shows that the two properties (corresponding to the two different ∀'s attacks) are independent also at the $[\mathbf{K4.3} \times \mathbf{S5}]$ level.

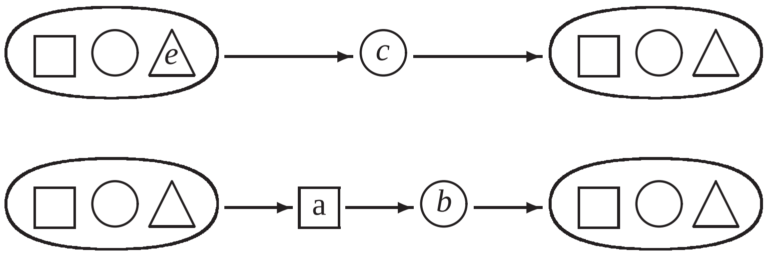

Figure 3.15: A frame $\mathfrak{F}$ showing that $\mathbf{S4.3} \times \mathbf{S5}$ is not product-matching.

We now give an example of a frame where ∃ can resist two rounds after the formation of the square, but loses in the third, see Figure 3.16. The circle around the points a and b mean they are a R_0 cluster. ∃ loses if ∀ plays a, c, e, d, b, a. In particular this shows that even if we add a modal formula corresponding to the attack in Figure 3.13, it would still not be enough to axiomatise $\mathbf{K4.3} \times \mathbf{K}$.

Since this last frame is not a frame for $[\mathbf{K4.3}, \mathbf{K4.3}]$ nor $[\mathbf{K4.3}, \mathbf{S5}]$ we may ask:

Question 3.3.15. *Would a formula φ corresponding to the answer to the attack in Figure*

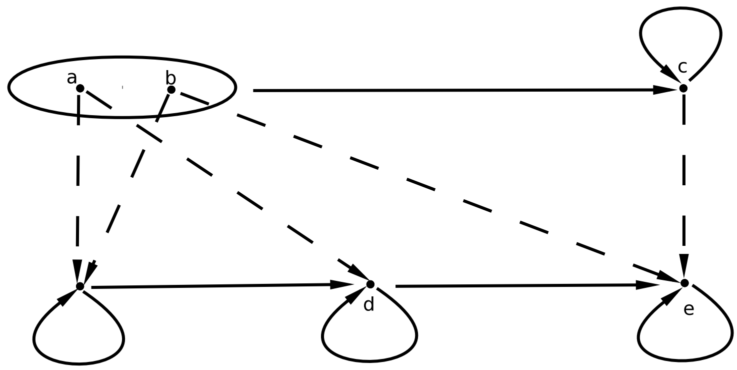

Figure 3.16: ∃ loses in the third.

3.14 be enough to guarantee that ∃ wins the ω-round game in squares over frames for $[\textbf{\textit{K4.3}} \times \textbf{\textit{K4.3}}] \oplus \varphi_{l,d} \oplus \varphi_{r,u} \oplus \varphi$*? And over frames for* $[\textbf{\textit{K4.3}} \times \textbf{\textit{S5}}] \oplus \varphi_{l,d} \oplus \varphi_{r,u} \oplus \varphi$*?*

3.4 Discussion

We conclude the chapter with a few remarks on further research about axiomatising two-dimensional product logics with a 'linear' component.

Can it work? The first question we may pose is if the approach in Section 3.3 can be successful. As is shown in [59, Theorem .2.10], if each L_i, $i < 2$, is a logic such that its class of frames is closed under ultraproducts, then $L_0 \times L_1$ is a canonical bimodal logic. So, say, the r.e. logics, **K4.3** × **K**, **K4.3** × **K4**, **K4.3** × **K4.3**, and **K4.3** × **S5** are such. However, the following questions are open:

Question 3.4.1. *Does any of these product logics have a canonical axiomatisation?*

Question 3.4.2. *Is the class of* all *frames for any of these product logics closed under ultraproducts?*

It is quite difficult to think about modally expressible properties that do not have first-order correspondents. So answers to the above would directly be relevant to finding explicit, possibly infinite, axiomatisations for the logics in question. In case of higher dimensional product logics (and of algebras of relations) similar questions have negative answers [44, 48, 59]. It is not known, however, whether the techniques used to achieve these results are applicable to two-dimensional cases. In the case that any of the previous questions has a negative answer, the approach in Section 3.3 is unlikely to yield an explicit axiomatisation without the use of some substantially new ideas.

The cases of K4.3 $\times$ K4.3 and K4.3 $\times$ S5 As we said above, Theorems 3.2.1 and 3.2.3 do not apply to product logics **K4.3** $\times L$, where L has no ω-fan among its frames, e.g. **K4.3** and **S5**.

K4.3$\times$**K4.3** is known to be not product-matching (see (3) in Section 3.1) and in Theorem 3.3.13 we prove that neither is **K4.3** $\times$ **S5**, so the question remains:

Question 3.4.3. *Are the recursively enumerable logics **K4.3** $\times$ **K4.3** and **K4.3** $\times$ **S5** finitely axiomatisable?*

The Log $\{(\omega,<)\} \times$ S5 case It is known that the logic

$$\mathsf{Log}\,\{(\omega,<)\} \times \mathbf{S5}$$

is decidable [23, Theorem .6.50], being open if it is finitely axiomatisable. However, there is a related positive axiomatisation result. As is shown in [78] (see also [23, Theorem .11.78]), if we have in the language of the first component logic not only $\Box_0$ and $\Diamond_0$ but also a next time operator $\bigcirc_0$, then the resulting logic $\mathsf{Log}_{\Box,\circ}\,\{(\omega,<)\} \times \mathbf{S5}$ is kind of product-matching: one needs to take the formulas axiomatising the components, plus the one describing that $\bigcirc_0$ and the **S5**-box $\Box_1$ commute

$$\bigcirc_0 \Box_1 p \leftrightarrow \Box_1 \bigcirc_0 p.$$

Part II

History-Dependent Systems

4

Reactive frames: the local view

Reactivity. Given a system with states and the possibility of transitions between states, we can imagine a path beginning at an initial state and moving along the path following allowed transitions. If our starting point is s_0, and the path is $s_0 \ldots s_n$, then the system is an ordinary non-reactive system if the options available at s_n (i.e. which states t we can go to from s_n) do not depend on the path $s_0 \ldots s_n$ (i.e. do not depend on how we got to s_n). Otherwise if there is such dependence then the system is *reactive*. A simple example would be to consider as worlds the configurations on a chess board and the allowed moves as transitions. It is clear that this system is reactive in the above sense. To be able to castle one must not have moved either the king or the rook, it is not enough to check their current positions. Moving the king or the rook corresponds to a higher order state transition, changing its nature.

One can take a reactive system and turn it into an ordinary system by taking the new states as the paths. This is true but from the point of view of applications there is serious loss of information, as the applicability of the reactive system may come from the manner in which the change occurs along the path. In any specific application, the states have meaning, the transitions have meaning and the paths have meaning. Therefore the changes in the system as we go along a path can have very important meaning in the context, which enhances the usability of the model.

Figure 4.1 presents a simple transition system, allowing the following transitions:

$$s \to t_1, s \to t_2, t_1 \to s', t_2 \to s', s' \to w.$$

A system becomes reactive when the transition table changes as we move along the graph. We can make the above system reactive by, for example, saying that if we start in s,

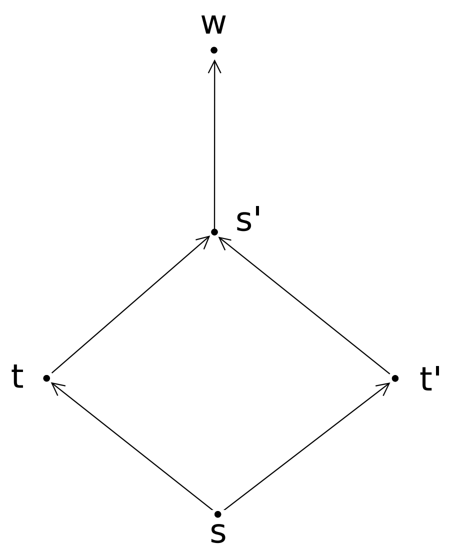

Figure 4.1: A transition system.

when we reach s', it sees w if and only if we pass through t. A way of depicting this situation can be found in Figure 4.2.

The double arrow from the edge $s \to t'$ to the edge $s' \to w$ indicates that if we go from s to t' then the connection $s' \to w$ is no longer available. The double arrow expresses the dependence on the path by codifying the transitions effects on the system configuration. We can make double arrows to act over other double arrows, getting higher and higher levels of dependence.

As observed above, a reactive system can be seen in a static point of view by considering their paths as the actual states. Figure 4.3 shows the conversion of our example. The points in Figure 4.3 are the paths. So from the point sts', there is a transition to $sts'w$ and from the point $st's'$ there is no transition to $st's'w$. That the accessible points vary when one moves around the graph is already a property of a static graph. What reactivity adds is the possibility of the accessible points from a point to be different depending on how we got there. In this sense the Figure 4.3 is not a proper representation of Figure 4.2. The two paths sts' and $st's'$ share an endpoint (i.e. same state s') and this can be important. This information can be recovered by an equivalence relation identifying the paths with the same endpoint. In Figure 4.4 is represented the unfolded version of Figure 4.2.

Next we introduce an interpretation of modal logic over reactive graphs (frames), in a

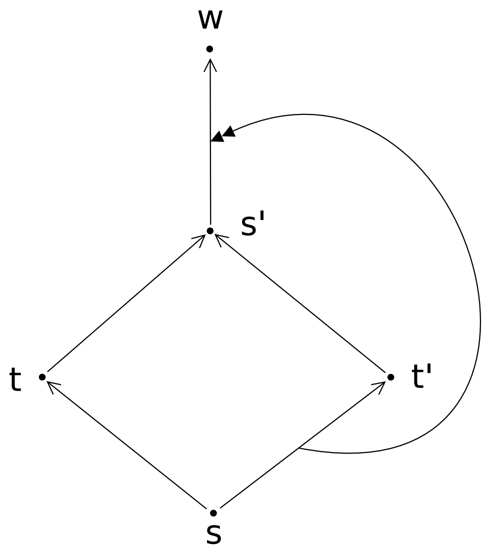

Figure 4.2: A higher order arrow is added yielding it reactive.

way that generalises the usual Kripke semantics taking in consideration these last remarks.

Reactive modal logic. We were naturally led to the choice of a bimodal language, where $\Diamond_R$ deals, as usual, with the dynamics and $\Diamond_P$ with the relation that identifies all the paths with the same endpoint. As we stressed before, reactivity is about the change in the set of accessible points of a given point. $\Diamond_P$ ranges through the various relational states of each point. As we shall see, many reactive properties can be axiomatised by the interaction between $\Diamond_R$ and $\Diamond_P$.

Like the accessibility relation, we let the value of (part of) the variables evolve when we move along the frame's points. The subset X of variables which we consider fixed, possibly all or none of them, will be a parameter of our logics. So the value of the variables in X depends only on the position in the graph, i.e. the path endpoint. The inclusion of this features in the models gives the results more generality, more possibilities of application and allows us to better understand the influence of each component of the generalisation in the results.

We can now ask the obvious questions:

- which relation between $\Box_P, \Box_R$ is imposed by this semantics?

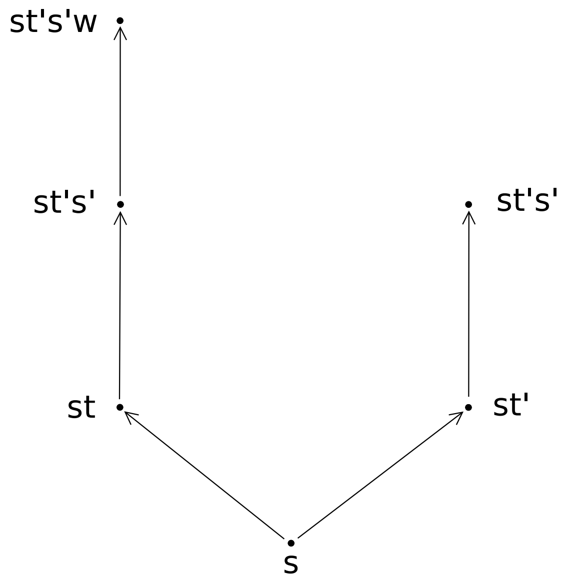

Figure 4.3: The unfolded version.

- if we add the reflexivity or transitivity or other usual axioms on $\Box_R$, what are the corresponding conditions on the models?
- which other properties can we express with this language?

As an answer we present a procedure to prove that, given a logic and some reactive properties, the first axiomatises the second. While soundness is proved directly, to prove completeness we use the static view on reactivity - the unfolded models. This allows us to use the canonical model theorem and other classic techniques. When we unfold the notion of a reactive model we obtain a classic bimodal Kripke model, (W, R, P, V) (let R be the dynamics and P the equivalence relation relating the paths with the same end point), satisfying three additional properties:

- there is a family $I \subseteq W$ picking one element on each P-equivalence class that R-generates all graph;
- xRy, xRz and yPz imply that $y \neq z$;
- the worlds related by P satisfy the same variables in X.

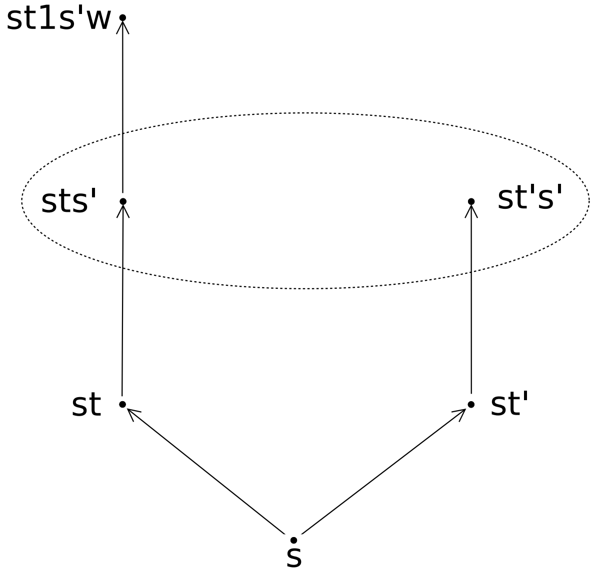

Figure 4.4: Identifying paths with the same endpoint.

A *shattered* frame is a frame satisfying the first property, and if it satisfies also the second we say it is a *coherently shattered (cs)* frame. The completeness proofs are done in two steps:

- we use the canonical model theorem to obtain completeness to a certain subclass of shattered frames corresponding to the reactive one (usually the first-order correspondent of the added axiom);
- we find a truth preserving model transformation that given shattered frame gives a *cs*-frame with that property.

The procedure is successful in many cases (see Table 4.1 at page 126) but we present also its limitations in the form of an open problem. We hope the technique proves itself useful to characterise logics, possibly in extended languages, expressing reactivity properties coming from research areas where reactivity is being applied to, or properties suggested by the switch graph research. A systematically investigation of the decision problem is missing but, for some cases, a rather straightforward application of the filtration method allowed us to establish the finite model property.

4.1 Reactive models

Definition 4.1.1. We consider the *reactive similarity type* $r = (\{\Diamond_R, \Diamond_P\}, \rho = 1)$. Thus the modal language $\mathcal{L}_r$ is defined by

$$\varphi \ ::= \ p \mid \neg\varphi \mid \varphi_1 \wedge \varphi_2 \mid \Diamond_R\varphi \mid \Diamond_P\varphi\,,$$

where $p \in \Pi$. The other connectives: $\top$, $\bot$, $\vee$, $\rightarrow$, $\leftrightarrow$, $\Box_R$ and $\Box_P$ are introduced by the usual abbreviations.

Definition 4.1.2. A *path* over a set W is any finite sequence of points from W. A *prefix* of a path $w_0w_1\dots w_n$ is any path of the form $w_0w_1\dots w_k$ for $k \le n$. Given a path $\lambda = w_0w_1\dots w_n$, we let $t(\lambda) = w_n$ and n its length.

A *reactive frame* is a pair (W, Δ), where W is a non-empty set and Δ is a set of paths over W that

- contains all one-element paths, i.e., W,
- is closed under taking prefixes.

Given $X \subseteq \Pi$, a *X-reactive model* is a triple $\mathfrak{M} = (W, \Delta, \nu)$, where (W, Δ) is a reactive frame and ν is a function $\nu : \Pi \to 2^{\Delta}$ such that for $p \in X$ and $\lambda w, \lambda' w \in \Delta$ we have $\lambda w \in \nu(p)$ iff $\lambda' w \in \nu(p)$. X corresponds to the subset of variables that are fixed while we move, i.e., which value is determined by the current world.

Given a X-reactive model $\mathfrak{M}$, for every $\lambda \in \Delta$ and every $\mathcal{L}_r$-formula φ, we define the notion 'φ *is true at* λ *in* $\mathfrak{M}$ ($\mathfrak{M}, \lambda \models_X \varphi$)' inductively as follows:

- $\mathfrak{M}, \lambda \models_X p$ iff $\lambda \in \nu(p)$ for variables p,
- $\mathfrak{M}, \lambda \models_X \neg\varphi$ iff $\mathfrak{M}, \lambda \not\models_X \varphi$,
- $\mathfrak{M}, \lambda \models_X \varphi_1 \wedge \varphi_2$ iff $\mathfrak{M}, \lambda \models_X \varphi_1$ and $\mathfrak{M}, \lambda \models_X \varphi_2$,
- $\mathfrak{M}, \lambda \models_X \Diamond_R\varphi$ iff there is $w \in W$ such that $\lambda w \in \Delta$ and $\mathfrak{M}, \lambda w \models_X \varphi$,
- $\mathfrak{M}, \lambda \models_X \Diamond_P\varphi$ iff there is $\gamma \in \Delta$ such that $t(\gamma) = t(\lambda)$ and $\mathfrak{M}, \gamma \models_X \varphi$.

Notice that if $X = \Pi$, then ν can be seen as a function from the Π to 2^W and the first line of the definition of $\vDash_X$ can be equivalently replaced by $\mathfrak{M}, \lambda \vDash_\Pi p$ iff $t(\lambda) \in \nu(p)$.

We say that φ *is true in* $\mathfrak{M}$ iff $\mathfrak{M}, \lambda \vDash_X \varphi$ for every $\lambda \in \Delta$. We say that φ *is X-valid in* a reactive frame if it is true in every X-reactive model over it.

When $X = \Pi$ we may omit X.

Definition 4.1.3. Given a reactive frame $\mathfrak{F} = (W, \Delta)$ we define $\sim_F \subseteq \Delta^2$ as:

$$\lambda \sim_F \gamma \text{ iff } t(\lambda) = t(\gamma) \,\&\, \forall \beta \in W^* \lambda\beta \in \Delta \leftrightarrow \gamma\beta \in \Delta.$$

We may omit $\mathfrak{F}$ when it is clear from the context. We write $[\lambda]$ to refer to the equivalence class of $\lambda \in \Delta$.

Remark 4.1.4. We are only interested in studying properties of X-reactive frames, that is, logics valid in the whole class of X-reactive models over reactive frames. Otherwise the notion of $\sim$ could be adapted to the context of X-reactive models and many of the following results would be valid in its model version.

Proposition 4.1.5. *Let* (W, Δ) *be a reactive frame, then:*

1. *If* $\lambda \sim \gamma$ *and* $(W, \Delta, \nu), \lambda \vDash_\Pi \varphi$ *then* $(W, \Delta, \nu), \gamma \vDash_\Pi \varphi$ *and*
2. *If* $\lambda, \gamma \in \Delta$ *and* $\lambda \not\sim \gamma$ *then there exists* φ *and* ν *such that* $(W, \Delta, \nu), \lambda \vDash_\Pi \varphi$ *and* $(W, \Delta, \nu), \gamma \vDash_\Pi \neg\varphi$.

Proof. 1. Let us prove it by induction on the structure of φ:

- if φ is a variable this is trivial since $t(\lambda) = t(\gamma)$.
- if $\varphi = \neg\psi$ then $\mathfrak{M}, \lambda \nvDash_\Pi \psi$ and so by IH and symmetry of $\sim$, $\mathfrak{M}, \gamma \nvDash_\Pi \psi$ thus $\mathfrak{M}, \gamma \vDash_\Pi \neg\psi = \varphi$. The $\varphi = \varphi_1 \wedge \varphi_2$ case is trivial.
- if $\varphi = \Diamond_R \psi$ then there exists $w \in W$ such that $\lambda w \in \Delta$ and $\mathfrak{M}, \lambda w \vDash_\Pi \psi$. It is clear that we also have $\lambda w \sim \gamma w \in \Delta$ and so by I.H. $\mathfrak{M}, \gamma w \vDash_\Pi \psi$ hence $\mathfrak{M}, \lambda \vDash_\Pi \Diamond_R \psi = \varphi$.
- the case $\varphi = \Diamond_P \psi$ also comes from $t(\lambda) = t(\gamma)$.

2. If $t(\lambda) = t(\gamma)$ and there is $\beta = w_1 \ldots w_n$ such that we do not have $\lambda\beta \in \Delta$ iff $\gamma\beta \in \Delta$. Without loss of generality let us assume $\lambda\beta \in \Delta$ and $\gamma\beta \notin \Delta$ and pick a valuation ν that

distinguishes all w_i, i.e., let $p_i \in \Pi$ with $1 \leq i \leq n$ and $\alpha \in \nu(p_i)$ iff $t(\alpha) = w_i$. Let $\varphi = \Diamond_R \varphi_\beta$ where φ_β is defined recursively by:

$$\begin{aligned} \varphi_{w_n} &= p_n \\ \varphi_{w_i \lambda} &= p_i \wedge \Diamond_R \varphi_\lambda. \end{aligned}$$

It is clear that $(W, \Delta, \nu), \lambda \vDash_\Pi \varphi$ but $(W, \Delta, \nu), \gamma \nvDash_\Pi \varphi$.
If $t(\lambda) \neq t(\gamma)$ we pick a valuation that distinguishes them.

□

Definition 4.1.6. $L_{r,X}$ is the logic of all reactive frames:

$$L_{r,X} = \{\varphi : \varphi \text{ is X-valid in every reactive frame}\}.$$

4.1.1 Reactivity unfolded

Definition 4.1.7. 1. A *shattered frame* is a bimodal frame $\mathfrak{F} = (W, R, P)$ such that P is an equivalence relation over W. Given a shattered frame, we say $I \subseteq W$ is an *initial family* if it picks one element from each P-class that R-generates the whole frame.

2. A *cs-frame* (coherently shattered) is a shattered frame that admits an initial family and that is *coherent*, i.e., such that for all $w, w', w'' \in W$, if wRw', wRw'' and $w'Pw''$ then $w' = w''$.

3. A *X-shattered model* is a Kripke model $\mathfrak{M} = (\mathfrak{F}, V)$ over a shattered frame $\mathfrak{F} = (W, R, P)$ and V is *X-admissible*, i.e., for all $w, w' \in W$, if wPw' then

$$w \in V(p) \text{ iff } w' \in V(p) \text{ for all } p \in X.$$

We say φ is *X-valid* in a shattered frame if it is true in every X-shattered model over it.

4. An *X-cs-model* is a X-shattered model over a cs-frame.

Notice that in order for the restrictions on the valuations in a Π-shattered model (W, R, P, V) to correspond to the restrictions on a general frame (W, R, P, A), where A is the boolean algebra generated by the equivalence classes of P, A has to be closed under m_R (and m_P). As we shall see in Proposition 4.2.12 the *cs*-frames that satisfy this requirement are the ones coming from unfolding static reactive frames. But in this case there is no restriction to deal with!

Definition 4.1.8. $L_{cs,X}$ is the logic of cs-frames:

$$L_{cs,X} = \{\varphi : \varphi \text{ is } X\text{-valid in every cs-frame}\}.$$

Remark 4.1.9. It is straightforward to see that every reactive frame $\mathfrak{F} = (W, \Delta)$ can be regarded as cs-frame $\mathfrak{F}_{cs} = (\Delta, R^\Delta, P^\Delta)$ where W is an initial family and

- $\lambda R^\Delta \gamma$ iff there is some $w \in W$ such that $\gamma = \lambda w$,
- $\lambda P^\Delta \gamma$ iff $t(\lambda) = t(\gamma)$.

We call it the *unfolding* of $\mathfrak{F}$. Furthermore it is easy to see that there is a bijective correspondence between X-reactive models $\mathfrak{M} = (W, \Delta, v)$ and X-cs-models $\mathfrak{M}_{cs} = (\Delta, R^\Delta, P^\Delta, v)$ that preserves truthness, i.e., for every $\lambda \in \Delta$ and every $\mathcal{L}_r$-formula φ,

$$\mathfrak{M}, \lambda \vDash_X \varphi \quad \text{iff} \quad \mathfrak{M}_{cs}, \lambda \vDash_X \varphi,$$

and so we have $L_{cs,X} \subseteq L_{r,X}$.

The converse is also true.

Theorem 4.1.10. *Let (W, R, P) be a cs-frame with I as initial family. There is a reactive frame such that there is bijective correspondence between X-cs-model over the first and X-reactive models over the second that preserves truthness.*

Proof. We call an R-path $\lambda = w_0 \ldots w_n$ an *I- initial path* if $w_0 \in I$. For every such I-initial path λ, let

$$l_I(\lambda) = i_0 \ldots i_n,$$

for the unique sequence of $i_j \in I$ with $w_j P i_j$ for $1 \leq j \leq n$ (see Definition 4.1.7).

Moreover l_I is an injective function from the set of all $\mathfrak{M}$-initial paths into paths over I, that is, if $l_I(\lambda) = l_I(\gamma)$ then $\lambda = \gamma$. Indeed, as l clearly preserves the length of a path, we can do induction on the length n of $\lambda = w_0 \ldots w_n$, $\gamma = v_0 \ldots v_n$. If $n = 0$ then $w_0 = l_I(w_0) = l_I(v_0) = v_0$ follows. Now suppose that $l_I(w_0 \ldots w_n w_{n+1}) = l_I(v_0 \ldots v_n v_{n+1}) = i_1 \ldots i_n i_{n+1}$ for some $i_j \in I$, $1 \leq j \leq n+1$. Then $l_I(w_0 \ldots w_n) = l_I(v_0 \ldots v_n) = i_1 \ldots i_n$ and $w_{n+1} P i_{n+1} P v_{n+1}$. By the IH, we have $w_j = v_j$ for $1 \leq j \leq n$. Therefore not only $w_n R w_{n+1}$, but also $w_n R v_{n+1}$. So $w_{n+1} = v_{n+1}$ follows by coherence.

Let $\Delta^I = \{l_I(\lambda) : \lambda$ is an I-initial path$\}$, (I, Δ^I) is clearly a reactive frame, we call it a *folding* of (W, R, P).

Given a X-cs-model $\mathfrak{M} = (W, R, P, V)$ (with I as an initial family) let $\mathfrak{M}' = (I, \Delta^I, \nu)$ with

$$\nu(p) = \{l_I(\lambda) \in \Delta^I : t(\lambda) \in V(p)\}, \text{ for each variable } p.$$

It is straightforward to see that $\mathfrak{M}'$ is a X-reactive model. We will show that $\mathfrak{M} \vDash_X \varphi$ iff $\mathcal{M}' \vDash_X \varphi$.

According to Remark 4.1.9 we can regard $\mathfrak{M}'$ as a X-cs-model $\mathfrak{M}'_{cs}$ preserving truthness. Hence, to conclude, it is enough to prove that $\mathfrak{M}$ is a bounded morphic image of $\mathfrak{M}'_{cs}$:

We define a function $f : \Delta^I \to W$ by taking

$$f(l_I(\lambda)) = t(\lambda).$$

which is well defined since l_I is injective. We claim that f is a surjective bounded morphism from $\mathfrak{M}'_{cs}$ onto $\mathfrak{M}$:

- f is surjective since by Definition 4.1.7(1) we have that I R-generates all the frame;

- p-morphism in R. First, if $l_I(\lambda), l_I(\lambda)i \in \Delta^I$ then there is some $w \in W$ such that $t(\lambda)Rw$ and $l_I(\lambda)i = l_I(\lambda w)$. So $t(\lambda)Rw = f(l_I(\lambda)i)$. Second, if $f(l_I(\lambda)) = t(\lambda)Rw$ then $l_I(\lambda w) \in \Delta^I$ and $f(l_I(\lambda w)) = w$.

- p-morphism in P. First, if $t(l_I(\lambda)) = t(l_I(\gamma))$ then $t(\lambda)Pt(\gamma)$. Second, if $f(l_I(\lambda)) = t(\lambda)Pw$ then, by Definition 4.1.7(1), there is some $\mathfrak{M}$-initial path γ such that $t(\gamma) = w$. Then $l_I(\gamma) \in \Delta^I$ and $t(l_I(\lambda)) = t(l_I(\gamma))$ follows.

- p-morphism in V: $l_I(\lambda) \in \nu$ iff $t(\lambda) = f(l_I(\lambda)) \in V(p)$.

□

Corollary 4.1.11. $L_{cs,X} = L_{r,X}$.

Proof. From Remark 4.1.9 we get one direction. For the other, suppose that $\varphi \notin L_{cs,X}$, that is, there is some X-cs-model $\mathfrak{M} = (W, R, P, V)$ having I as initial family and such that $\mathfrak{M} \nvDash_X \varphi$. $\mathfrak{M}'$ is a X-reactive model and $\mathfrak{M}' \nvDash \varphi$ follows from the previous proposition. □

Given a reactive frame we can obtain a cs-frame by unfolding (and vice-versa by folding) where each X-model over the first corresponds to a X-model over the second and modal satisfaction is preserved.

With shattered frames we are back in the classic setting even if, in the case of $X \neq \emptyset$, we are not at the frame level. There are restrictions on the models over the correspondent Kripke frames, the situation is closer to the case of general frames.

Folding and unfolding will be the bridge from reactivity to this classical setting and will allow us to use some known techniques in the study of the axiomatisation of logics over reactive frames. The most immediate consequence of this connection is that the logics formed by formulas valid over reactive frames and cs-frames coincide.

4.2 Axiomatising

Now, using the strong relation between reactive frames and cs-frames we will present the axiomatisation of the logics obtained associated with some classes of reactive frames.

Definition 4.2.1. Let

$$L_X = (\mathbf{K}_R, \mathbf{S5}_P) +_X p \leftrightarrow \Box_P p,$$

meaning that L_X is the closure by the rules of modus ponens and necessity (for $\Box_R$ and $\Box_P$) of the set containing

$$p \leftrightarrow \Box_P p$$

for every propositional variable $p \in X$, the substitution instances of all propositional tautologies and of the following axioms:

1. $\Box_R(p \to q) \to (\Box_R p \to \Box_R q)$,
2. $\Box_P(p \to q) \to (\Box_P p \to \Box_P q)$,
3. $\Box_P p \to p$,
4. $\Box_P p \to \Box_P \Box_P p$,
5. $p \to \Box_P \Diamond_P p$.

See Definition 2.1.5.

It is clear that $L_X \subseteq L_{cs,X} = L_{r,X}$ since all the L_X axioms are sound with respect to shattered frames. Notice that the $\Diamond_R$ fragment of L_X is just K_R, i.e. K. It is obvious that $K \subseteq L_X$ and from the fact that every Kripke model is a trivial X-reactive model we get the other inclusion.

We want to show that L_X is complete with respect to all reactive frames. For that we will prove that L_X's canonical model is a X-shattered model and then prove that any such model is the bounded image of a generated subframe of a X-cs-frame. Hence concluding that $L_X = L_{cs,X} = L_{r,X}$.

If $X \neq \varnothing$ then L_X is not closed under structural substitution. One can define the canonical model of a normal logic L the usual way, $\mathfrak{M}^L = (W^L, R^L, P^L, V^L)$ where:

$$
\begin{aligned}
&W^L = \{s : s \text{ is } L - MCS\} \\
&sR^Lt \text{ iff } \{\varphi : \Box_R\varphi \in s\} \subseteq t \text{ iff } \{\Diamond_R\varphi : \varphi \in r\} \subseteq s \\
&sP^Lt \text{ iff } \{\varphi : \Box_P\varphi \in s\} \subseteq t \text{ iff } \{\Diamond_P\varphi : \varphi \in r\} \subseteq s \\
&V^L(p) = \{s \in W^{L_X} : p \in s\}.
\end{aligned}
$$

and prove the well-known Truth Lemma about it:

$$\{\varphi : \mathfrak{M}^L \vDash \varphi\} = L$$

even if the logic in question is not closed under the rule of substitution, see e.g. [38].

Proposition 4.2.2. *If a logic L contains $p \leftrightarrow \Box_P p$ and for $p \in X \subseteq \Pi$ then given vP^Lw we have that:*

$$v \in V^L(p) \;\Rightarrow\; w \in V^L(p) \textit{ for all } p \in X.$$

Proof. If sP^Lt and $p \in s$ then $\Box_P p \in s$ and so $p \in t$. □

Clearly, if P^L is an equivalence class then all the worlds related by P^L satisfy the same variables in X.

Corollary 4.2.3. *$\mathfrak{M}^{L_X}$ is a X-shattered model and so*

$$\{\,\varphi : \varphi \textit{ is } X\textit{-valid in every shattered frame}\,\} = L_X.$$

In other words, we have that L_X is sound and complete with respect to the class of X-models over shattered frames. Next, by showing that it is also complete with respect to its subclass of cs-frames we conclude the axioms generating L_X axiomatise the minimal logic over reactive frames.

Theorem 4.2.4. $L_X = L_{r,X}$.

Proof. Since $L_{r,X} = L_{cs,X}$, it is equivalent to prove that $L_X = L_{cs,X}$.

Every cs-frame is also a shattered frame, hence we have that $L_{cs,X} \subseteq L_X$.

Let $\varphi \notin L_X$ then there is a X-shattered model $\mathfrak{M} = (W, R, P, V)$ that does not satisfy φ.

We define $\mathcal{B}(\mathfrak{M}) = (W', R', P', V')$, where:

$$W' = \{(z, y) : yPz\} \cup W \times \{*\},$$
$$(x, i)R'(y, j) \text{ iff } i = * \,\&\, x = y \text{ or } i \neq * \,\&\, xRy \,\&\, y = j,$$
$$(x, i)P'(y, j) \text{ iff } i = j \,\&\, (i = * \to x = y),$$
$$(x, i) \in V'(p) \text{ iff } x \in V(p).$$

$\mathcal{B}(\mathfrak{M})$ is a X-cs-model and $\mathfrak{M}$ is a bounded morphic image of $\mathcal{B}(\mathfrak{M})$'s submodel generated by $W'' = W' - W \times \{*\}$ (and therefore $\varphi \notin L_{cs,X}$):

- let I be a any family of elements of W' picking one element from each P-class. It is clear that $W \times \{*\} \subseteq I$ and so I generates the $\mathfrak{N}$.

  V' is an X-admissible valuation: $(x, i)P'(y, j)$ implies $i = j$, hence xPy and therefore if $p \in X$,

  $$(x, i) \in V'(p) \text{ iff } z \in V(p) \text{ iff } y \in V(p) \text{ iff } (y, j) \in V'(p).$$

  Let us check that it is coherent. Given $(x, i), (y, j), (z, k) \in W'$, if $(y, j)P'(z, k)$ then $j = k$. Let us assume also $(x, i)R'(y, j)$ and $(x, i)R'(z, k)$. If $i = *$ then $x = y = z$, and if $i \neq *$ then $y = j = k = z$. So $(y, j) = (z, k)$.

- let $f : W'' \to W$ be defined by: $f((x, i)) = x$.

  It is straightforward to see that f is surjective, let us see that it is actually a p-morphism:

  – f is a p-morphism in R

    * $(x, i)R'(y, j)$ then $f((x, i)) = xRy = f((y, j))$.
    * $f((x, i))Ry$ then $(x, i)R'(y, y)$.

– f is a p-morphism in P

  * $(x,i)P'(y,j)$ then $i = j$ and so, by construction, we have $f((x,i)) = xPy = f((y,j))$.
  * $f((x,i))Py$ then $(x,i)P(y,i)$.

– f is a p-morphism in V
  $(x,y) \in V'(p)$ iff $f((x,i)) = x \in V'(p)$.

To this way of generating a X-cs-model from a given X-shattered model we call the *blow up trick* and it is inspired by the standard extensions used in [40]. A similar technique was also used in [56]. □

If we consider the basic properties of frames like reflexivity, symmetry or transitivity, we see that there are many ways of generalising it to the reactive frame level. These properties refer to the accessibility of points without referring to the changes in their relational state since it is always the same. In the reactive case this is not true and they may mean different things. Reflexivity can mean for example that we can always access loop without any change in the set of accessible points or we may require anything else. All the mentioned properties have similar variants, they are the generalised notions of the originals.

Just as the (static) graphs properties are studied in classic modal logic, we dedicate the rest of this chapter to the study of the logics of the subclasses of reactive frames satisfying some of these properties.

Our strategy is the same as we used for the general case. Given a class of reactive frames and set of candidate axioms, we first check if they correspond to the reactive frame property in question (in some cases we only have soundness). Then we prove completeness in steps:

- we establish that the logic they originate is complete with respect to class of shattered frames with a certain property by analysing its canonical model;

- we check that its subclass of cs-frames validates exactly the same formulas by presenting a transformation from shattered models to cs-models (blow up trick) preserving the shattered properties we are considering plus that each shattered model is a bounded morphic image of a generated submodel of its transformation;

- we show that folding these cs-frames gives origin to reactive frames with the required property, thus obtaining the result by applying Proposition 4.1.10 like in Corollary 4.1.11.

Reflexivity and transitivity

Now we study some subclasses of reactive frames obtained by imposing properties that generalise the notions of reflexivity and transitivity in the static case and axiomatise them.

Let us introduce some variations of the blow up trick that work for all the next cases.

Proposition 4.2.5. *Given a X-shattered model* $\mathfrak{M} = (W, R, P, V)$ *we define*

$$\mathcal{B}_i(\mathfrak{M}) = (W_i, R_i, P_i, V_i)$$

for $i = 1, 2, 3, 4$*:*

1.
   - $W_1 = \{(x,(y,0)),(x,(y,1)) : xPy\} \cup W \times \{*\}$,
   - $(x,i)R_1(y,j)$ *iff* $i = * \;\&\; x = y$ *or*
     $\big(i = (x', a) \;\&\; j = (y', b) \;\&$
     $(b = |a-1| \;\&\; y = t'(x, y')$ *or* $i = j \;\&\; y = t'(x,x))\big)$,
   - $(x,i)P_1(y,j)$ *iff* $i = j \;\&\; (i = 1 \to x = y)$,
   - $(z,i) \in V_1(p)$ *iff* $z \in V(p)$.
2.
   - $W_2 = \{(t^n(c), n, c, d, i) : cPd; i = 0, 1, *; n < \omega\}$,
   - $(a,b,c,d,i)R_2(a',b',c',d',i')$ *iff* $(c,d) = (c',d') \;\&$
     $\big(i = * \;\&\; (a,b) = (a',b')$ *or*
     $i \neq * \;\&\; aRa' \;\&$
     $(a = t(a') \;\&\; b = b' + 1 \;\&\; i = i'$ *or*
     $a' = c' = d' \;\&\; b' = 0 \;\&\; i = |i' - 1|)\big)$,
   - $w = (a,b,c,d,i)P_2(a',b',c',d',i') = w'$ *iff* $(d,i) = (d',i') \;\&$
     $(i = * \to w = w')$
   - $(a,b,c,d,i) \in V_2(p)$ *iff* $a \in V(p)$.
3.
   - $W_3 = \{((x,y),n),(x,n) : xPy, n < \omega\}$,

- $(x,i)R_3(y,j)$ *iff* $y = (y_1, y_2)$ &
  $\big(x \in W$ & $(x = y_1$ & $i = j$ *or* $j > i$ & $y_1 = y_2)$ *or*
  $x = (x_1, x_2)$ & $j > i$ & $x_1 R y_1 = y_2\big)$,
- $(x,i)P_3(y,j)$ *iff* $i = j$ & $(x \in W$ & $x = y$ *or* $x = (x_1, w)$ & $y = (y_1, w))$,
- $(x,i) \in V_3(p)$ *iff* $x \in W$ & $x \in V(P)$ *or* $x = (x_1, x_2)$ & $x_1 \in V(p)$.

4.
- $W_4 = W_3, P_4 = P_3$ & $V_4 = V_3$,
- $(x,i)R_4(y,j)$ *iff* $y = (y_1, y_2)$ &
  $\big(x \in W$ & $(x = y_1$ & $i = j$ *or* $j > i$ & $y_1 = y_2)$ *or*
  $x = (x_1, x_2)$ & $j > i$ & $t'(x_1, y_2) = y_1\big)$.

where t, t' are defined by

- *$t(w) = v$ be such that vRwPv if there is such v, otherwise it is undefined,*
- *$t'(w,v) = v'$, such that $wRv'Pv$ and $wRv \to v' = v$ if there is such a v', and is undefined otherwise.*

As usual $t^0(w) = w$ and $t^{n+1}(w) = t(t^n(w))$.

We claim that $\mathcal{B}_i(\mathfrak{M})$ is a X-cs-model and $\mathfrak{M}$ is a bounded morphic image of a generated submodel of $\mathcal{B}_i(\mathfrak{M})$ for $i = 1, 2, 3, 4$.

Remark 4.2.6. Any of these constructions could have been used in Theorem 4.2.4 instead of $\mathcal{B}$.

Proof. 1. $\mathcal{B}_1(\mathfrak{M})$ is a X-cs-model and $\mathfrak{M}$ is a bounded morphic image of the submodel of $\mathcal{B}_1(\mathfrak{M})$ generated by $W_1' = \{(x,(y,0)),(x,(y,1)) : xPy\}$:

- the existence of an initial family and that V_1 is an X-admissible valuation are dealt just like in prop. 4.2.4.
  Let us check that it is coherent. Given $(x,i),(y,j),(z,k) \in W_1$. If $(y,j)P_1(z,k)$ then $j = k$. Let us assume that we also have $(x,i)R_1(y,j)$ and $(x,i)R_1(z,k)$. If $i = *$ we get $x = y = z$. If $i = (z,a)$ then $j = (w,b) = k$. If $a = b$ then $y = t(x,x) = z$ and if $a \neq b$ then $y = t(x,w) = z$. In any case $(y,j) = (z,k)$.
- let $f : W_1' \to W$ be defined by: $f((x,i)) = x$. It is straightforward to see that it is a surjective function, let us see that it is actually a p-morphism:

- f is a p-morphism in R
  - $(x,(x',i))R_1(y,(y',j))$ then or $y = t'(x,y')$ or $y = t'(x,x)$. In both cases $f((x,i)) = xRy = f((y,j))$,
  - $f((x,(t,a)))Ry$ then we have that $t'(x,y) = y$, hence $(x,(t,a))R_1(y,(y,|a-1|))$;
- f is a p-morphism in P
  - $(x,i)P_1(y,j)$ then $i = j$ and so $f((x,i)) = xPy = f((y,j))$,
  - $f((x,i))Py$ then $(x,i)P_1(y,i)$;
- f is a p-morphism in V, is trivial since by definition $(x,i) \in V_1(p)$ iff $f((x,i)) = x \in V$.

2. $\mathcal{B}_2(\mathfrak{M})$ is a X-cs-model and $\mathfrak{M}$ is a bounded morphic image of the submodel of $\mathcal{B}_2(\mathfrak{M})$ generated by $W_2' = \{(t^n(c), n, c, d, i) : cPd; i = 0,1; n < \omega\}$:

   - it is easy to verify that V_2 is admissible and that any choice of I, picking one element from each P-class and containing $\{(t^n(c), n, c, d, *) : cPd; n < \omega\}$, works as initial family.

     Let us check that it is coherent. Given $w = (a,b,c,d,i)$, $w' = (a',b',c',d',i')$ and $w'' = (a'',b'',c'',d'',i'')$ in W_2. If $w'P'w''$ then $(d',i') = (d'',i'')$. Let us assume that we also have wR_2w' and wR_2w'' so $(c',d') = (c,d) = (c'',d'')$. If $i = *$ we get $(a,b) = (a',b') = (a'',b'')$. Otherwise, if either $i = i'$ or $i = i''$ then $i = i' = i''$, so $b' = b-1 = b''$ and $a = t^{b'}(c) = t^{b''}(c) = a''$. If $i \neq i'$ then $b = b' = 0$, $a' = c' = d' = d'' = c'' = a''$. Hence, in any case, $w' = w''$.

   - let $f : W_2' \to W$ be defined by $f((a,b,c,d,i)) = a$.
     It is straightforward to see that it is a surjective function, let us see that it is actually a p-morphism:
     The condition in the valuation is trivial as before;
     - f is a p-morphism in R
       - $w = (a,b,c,d,i)R_2(a',b',c',d',i') = w'$ then $f((x,i)) = aRb = f((y,j))$,
       - $f((a,b,c,d,i))Ry$ then or $(a,b,c,d,i)R_2(y,0,y,y,|i-1|)$;
     - f is a p-morphism in P

- - $* \ w = (a,b,c,d,i)P_2(a',b',c',d',i') = w'$ then $aPcPd = d'Pc'Pa'$. Thus, by transitivity of P, we obtain $f(w) = aPa' = f(w')$.
    - $* \ f((a,b,c,d,i))Py$ then $(a,b,c,d,i)P_2(y,0,y,d,i)$.

3. $\mathcal{B}_3(\mathfrak{M})$ is a X-cs-model and $\mathfrak{M}$ is a bounded morphic image of the submodel of $\mathcal{B}_3(\mathfrak{M})$ generated by $W_3' = \{((x,y),n) : xPy, n < \omega\}$:

   - the existence of an initial family and the admissibility of V_3 are dealt as before (in this case the generator is $\{(x,n) : n < \omega\}$).
     Let us check that it satisfies coherence. Given $(x,i),(y,j),(z,k) \in W_3$. Let us assume that we have $(x,i)R_3(y,j)$, $(x,i)R_3(z,k)$ and $(y,j)P_3(z,k)$ then $y = (y_1,w)$, $z = (z_1,w)$ and $j = k$. So, either $i = j = k$, in which case, $x \in W$ and $y_1 = x = z_1$; or $i < j = k$ and $y_1 = w = z_1$.
   - let $f : W_3' \to W$ be defined by: $f(((x,y),i)) = x$.
     It is straightforward to see that it is a function. Let us see that it is also a p-morphism (the condition in V is dealt as before):
     - f is a p-morphism in R
       - $* \ ((x_1,x_2),i)R_3((y_1,y_2),j)$ then
         $$f(((x_1,x_2),i)) = x_1Ry_1 = f(((y_1,y_2),j)),$$
       - $* \ f(((x_1,x_2),i))Ry$ then $((x_1,x_2),i)R_3((y,y),j)$ for any $j > i$;
     - f is a p-morphism in P
       - $* \ ((x_1,x_2),i)P_3((y_1,y_2),j)$ then
         $$f(((x_1,x_2),i)) = x_1Px_2 = y_2Py_1 = f(((y_1,y_2),j)),$$
       - $* \ f((x_1,x_2),i)Py$ then $((x_1,x_2),i)P_3((y,x_2),i)$.

4. Let us consider the morphism as in 3., we just need to check the conditions involving R_4:

   - initiality is dealt as in 3.
     Coherence: let us assume that we have $(x,i)R_4(y,j)$, $(x,i)R_4(z,k)$ and $(y,j)P_4$ (z,k). Then $y = (y_1,w)$, $z = (z_1,w)$ and $j = k$. If $x \in W$ then, if $i = j = k$ then $x = y_1 = z_1$ and if $i < j = k$ then $y_1 = w = z_1$. Otherwise $x = (x_1,x_2)$ and $y_1 = t(x_1,w) = z_1$.

- $((x_1,x_2),i)R_4((y_1,y_2),j)$ then $y_1 = t'(x_1,y_2)$ hence

$$f(((x_1,x_2),i)) = x_1 R y_1 = f(((y_1,y_2),j)),$$

- $f(((x_1,x_2),i))Ry$ then $t'(x_1,y) = y$ thus $((x_1,x_2),i)R_4((y,y),j)$ for any $j > i$.

□

We start by analysing the subclasses of reactive frames (or models) that correspond to the usual axioms for reflexivity and transitivity (with one operator).

Let us prove a lemma that will be useful throughout the chapter:

Lemma 4.2.7. *Given a cs-frame (W,R,P) admitting as initial family I. Let γs and $\gamma' s$ be two I-initial paths,*

$$l_I(\gamma s) \sim_{(I,\Delta_I)} l_I(\gamma' s).$$

Proof. $l_I(\gamma s)\alpha \in \Delta_I$ iff there is a β such that $\gamma s\beta$ is an I-initial path (iff $\gamma' s\beta$ is an I-initial path) and $l_I(\beta) = \alpha$ iff $l_I(\gamma' s)\alpha \in \Delta_I$. Since clearly $t(l_I(\gamma s)) = l_I(s) = t(l_I(\gamma' s))$ we get $l_I(\gamma s) \sim l_I(\gamma' s)$. □

Proposition 4.2.8. 1. *A reactive frame (W,Δ) (Π)-validates $p \to \Diamond_R p$ iff it is* strongly reflexive, *i.e. satisfies:*

$$\lambda w \in \Delta \to \lambda w \sim \lambda ww \in \Delta.$$

Let $L_X^T = L_X \oplus p \to \Diamond_R p$.

2. *L_Π^T is sound and complete with respect to the class of (all Π-reactive models over) strongly reflexive frames.*

3. *If $X \neq \Pi$, L_X^T is not sound and complete with respect to (all X-models over) any class of reactive frames.*

Proof. 1.

- Given a strongly reflexive Π-reactive model $\mathfrak{M} = (W,\Delta,v)$ and $\lambda w \in \Delta$ such that $\mathfrak{M},\lambda w \vDash_\Pi \varphi$. Thus $\lambda w \sim \lambda ww$ and, by Proposition 4.1.5, $\mathfrak{M},\lambda ww \vDash_\Pi \varphi$ and so $\mathfrak{M},\lambda w \vDash_\Pi \Diamond_R \varphi$.

- Given a non strongly reactive frame (W, Δ). There exists some $\lambda w \in \Delta$ such that $\lambda w \not\to \lambda ww$, so we have two cases.
  Either $\lambda ww \notin \Delta$ so if we take $p \in \Pi$ and pick ν such that $\lambda w \in \nu(p)$ and $\lambda ww' \notin \nu(p)$ for all $\lambda ww' \in \Delta$ in which case

$$(W, \Delta, \nu), \lambda w \vDash p \wedge \neg \Diamond_R p.$$

  Or, there is $\beta = w_1 \dots w_n$ such that we do not have $\lambda w\beta \in \Delta \leftrightarrow \lambda ww\beta \in \Delta$. We consider $w_0 = t(\lambda) = t(\gamma)$, $p_0, \dots, p_n$ and pick ν such that $\alpha \in \nu(p_i)$ iff $t(\alpha) = w_i$. Let $\varphi = \Diamond_R \varphi_\beta$ be as defined in proposition (4.1.5.2). If $\lambda w\beta \in \Delta$ but $\lambda ww\beta \notin \Delta$ then λw satisfies $\psi = (p_0 \wedge \varphi) \vee \neg p_0$ and λww does not. If $\lambda w\beta \notin \Delta$ but $\lambda ww\beta \in \Delta$ then λw satisfies $\neg\psi = \neg(p_0 \wedge \varphi) \wedge p_0$ and λww does not. Since no $\lambda ww' \in \Delta$ with $w' \neq w$ satisfies p_0 and both ψ and $\neg\psi$ imply p_0 (which is only satisfied at paths with w as end point), we know that either $(W, \Delta, \nu), \lambda w \nvDash \psi \to \Diamond_R \psi$ or $(W, \Delta, \nu), \lambda w \nvDash \neg\psi \to \Diamond_R \neg\psi$

Imposing the usual axiom for reflexivity forces a very strong notion of reflexivity in reactive frames. Strong reflexivity imposes that, no matter which path we have covered, we can always loop without any change to the accessible worlds.

2. Soundness of L_Π^T comes easily from the soundness of the new axiom which has just been established.

Let us prove that $L_\Pi^T = \{\varphi : \varphi$ is valid in every strongly reflexive reactive frame$\}$:

- using lemma (4.2.2) it is easy to check that the canonical model for L_Π^T is a Π-shattered model (W, R, P, V) where R is reflexive. Since every cs-frame is also a shattered frame we have that:

$$\begin{aligned} L_\Pi^T &= \{\varphi : \varphi \text{ is true in every } \Pi\text{-shattered model } (W, R, P, \nu) \text{ where } R \text{ is reflexive}\} \\ &\subseteq \{\varphi : \varphi \text{ is true in every } \Pi\text{-cs model } (W, R, P, \nu) \text{ where } R \text{ is reflexive}\}. \end{aligned}$$

- $\mathcal{B}_1$ preserves R-reflexivity:
  Given (x, i) the case $i = *$ is trivial and if $i = (x', a)$ we have

$$(x, (x', a)) R_2 (t'(x, x), (x', a)) = (x, (x', a))$$

  since $t'(x, x) = x$. Hence

$$L_\Pi^T = \{\varphi : \varphi \text{ is true in every } \Pi\text{-cs model } (W, R, P, \nu) \text{ where } R \text{ is reflexive}\}.$$

- given a R-reflexive cs-frame $\mathfrak{F} = (W, R, P)$ with initial family I, (I, Δ^I) is a strongly reflexive reactive frame:
  Let $\lambda w \in \Delta^I$, so there is $\gamma \in W^*$ and $s \in W$ such that γs is an I-initial path and $l_I(\gamma s) = \lambda w$. Since $\mathfrak{F}$ is R-reflexive we have that γss is also an I-initial path and clearly $l_I(\gamma ss) = \lambda ww \in \Delta$. Applying Lemma 4.2.7 we conclude that $\lambda w \sim \lambda ww$.

3. Let $X \subsetneq \Pi$ and a class of reactive frames F.

If there is a reactive frame $\mathfrak{F} = (W, \Delta) \in \mathsf{F}$ then let $\lambda \in \Delta$ and $p \in \Pi - X$. We pick ν such that $\lambda \in \nu(p)$ and $\lambda w \notin \nu(p)$ for all $\lambda w \in \Delta$. So

$$(W, \Delta, \nu), \lambda w \models p \wedge \neg \Diamond_R p.$$

If F is the empty class it validates $\perp$. Since there are Π-reactive models over strongly reflexive frames and $L_X^T \subseteq L_\Pi^T$ we conclude that

$$L_X^T \subsetneq \{\varphi : \varphi \text{ is valid in all } \mathfrak{F} \in \mathsf{F}\} \ni \perp.$$

□

Notice that to have reactive frame completeness we have to impose that $X = \Pi$. This will also happen in the next case.

Proposition 4.2.9. *1. A reactive frame (W, Δ) (Π-)validates $\Diamond_R \Diamond_R p \to \Diamond_R p$ iff it is* strongly transitive*, i.e. satisfies:*

$$\lambda ww'w'' \in \Delta \to \lambda ww'w'' \sim \lambda ww'' \in \Delta.$$

Let $L_X^4 = L_X \oplus \Diamond_R \Diamond_R p \to \Diamond_R p$.

2. *L_Π^4 is sound and complete with respect to the class of Π-reactive models over strongly transitive frames.*

3. *If $X \neq \Pi$, L_X^4 is not sound and complete with respect to (all X-models over) any class of reactive frames.*

Proof. 1.

- Given a strongly transitive reactive model $\mathfrak{M} = (W, \Delta, \nu)$ and $\lambda w \in \Delta$ such that $\mathfrak{M}, \lambda w \models \Diamond_R \Diamond_R \varphi$ then there are $w', w'' \in W$ such that $\lambda ww'w'' \in \Delta$ and $\mathfrak{M}, \lambda ww'w'' \models \varphi$. Thus $\lambda ww'w'' \sim \lambda ww'' \in \Delta$ and, by Proposition 4.1.5, $\mathfrak{M}, \lambda ww'' \models \varphi$ hence $\mathfrak{M}, \lambda w \models \Diamond_R \varphi$.

- Given a non strongly transitive reactive frame (W,Δ). There is $\lambda ww'w'' \in \Delta$ such that $\lambda ww'w'' \nsim \lambda ww''$ then we have two cases:
  Either $\lambda ww'' \notin \Delta$ which implies $w'' \neq w'$. Hence, if we take $p \in \Pi$ and pick ν such that $\lambda ww'w'' \in \nu(p)$ and $\lambda wv \notin \nu(p)$ for all $\lambda wv \in \Delta$, in which case

$$\lambda w \vDash \Diamond_R \Diamond_R p \wedge \neg \Diamond_R p.$$

  Or there is β such that we do not have $\lambda ww'w''\beta \in \Delta \leftrightarrow \lambda ww''\beta \in \Delta$. In which case we pick $w_0 = w''$ and define ψ as in proposition (4.2.8.1). As before we conclude that either $(W,\Delta,\nu),\lambda w \nvDash \Diamond_R \Diamond_R \psi \to \Diamond_R\psi$ or $(W,\Delta,\nu),\lambda w \nvDash \Diamond_R \Diamond_R \neg\psi \to \Diamond_R\neg\psi$.

Similarly to the case of strong reflexivity, strong transitivity imposes that, regardless of the path we have covered, every world accessible in two steps is accessible in one and that the set of accessible worlds is the same in both cases.

2. Soundness of L^4_Π comes easily from the soundness of the new axiom which has just been established.

Let us prove that $L^4_\Pi = \{\varphi : \varphi \text{ is valid in every strongly reflexive reactive frame}\}$:

- using lemma (4.2.2) it is easy to check that the canonical model for L^4_X is a shattered model (W,R,P,V) where R transitive.

- $\mathcal{B}_3$ preserves R-transitivity:
  If $(x_1,i_1)R_3(x_2,i_2)R_3(x_3,i_3)$ then $x_k = (x_{k,1},x_{k,2})$ for $k = 2,3$, $i_1 \le i_2 < i_3$ and $x_{3,1} = x_{3,2}$. If $x_1 \in W$ then immediately we conclude $(x_1,i_1)R_3(x_3,i_3)$. If $x_1 = (x_{1,1},x_{1,2})$ then $x_{1,1}Rx_{2,1}Rx_{3,1}$. Hence $x_{1,1}Rx_{3,1} = x_{3,2}$ and $(x_1,i_1)R_3(x_3,i_3)$.

- given a R-transitive cs-frame with initial family I, (I,Δ^I) is a strongly transitive reactive frame:
  Let $\lambda ww'w'' \in \Delta^I$, so there is $\gamma \in W^*$ and $s,s',s'' \in W$ such that $\gamma ss's''$ is an I-initial path and $l_I(\gamma ss's'') = \lambda ww'w''$. Since $\mathfrak{F}$ is R-transitive we have that $\gamma ss''$ is also an I-initial path and clearly $l_I(\gamma ss'') = \lambda ww'' \in \Delta$. Applying Lemma 4.2.7 we conclude that $\lambda ww'w'' \sim \lambda ww''$.

3. Let $X \subsetneq \Pi$ and a class of reactive frames F.

If there is a reactive frame $\mathfrak{F} = (W, \Delta) \in \mathsf{F}$ with a path of length three, $w_0 w_1 w_2$, let ν be such that $w_0 w_1 w_2 \in \nu(p)$ and $w_0 v \notin \nu(p)$ for all $w_0 v \in \Delta$. Then $(W, \Delta, \nu), w_0 \models \Diamond_R \Diamond_R p \wedge \neg \Diamond_R p$.

Given a reactive frame (W, Δ) such that there are no $w_0, w_1, w_2 \in W$ such that $w_0 w_1 w_2 \in \Delta$ then it validates $\Diamond_R \Box_R \bot$ but $\Diamond_R \Box_R \bot \notin L_X^T$ since

$$L_X^4 \subseteq L_\Pi^4 \subseteq \{\varphi : (\{a\}, \{a\}^*, \nu) \models \varphi \;\&\; \nu(p) = \{a\}^* \text{ for all } p \in \Pi\} \not\ni \Diamond_R \Box_R \bot.$$

So if there is no frame with a path of length three in F (in particular if F is empty) then

$$L_X^4 \subsetneq \{\varphi : \varphi \text{ is valid in all } F \in \mathsf{F}\}.$$

□

Let us consider some variants of these axioms and see that they axiomatise other generalised notions of reflexivity and transitivity.

Proposition 4.2.10. 1. *A reactive frame (W, Δ) X-validates $p \to \Diamond_R \Diamond_P p$ iff it is* outwardly reflexive, *i.e. satisfies:*

$$\lambda w \in \Delta \to \lambda w w \in \Delta.$$

$L_X^{T_o} = L_X \oplus p \to \Diamond_R \Diamond_P p$ is sound and complete with respect to the class of X-reactive models over outwardly reflexive frames.

2. *Let $L_X^{T_i} = L_X \oplus p \to \Diamond_P \Diamond_R p$. If $X = \Pi$, $L_\Pi^{T_i}$ is sound and complete with respect to the class of Π-reactive models over* inwardly reflexive *frames, i.e. satisfying:*

$$\lambda w \in \Delta \to \exists \lambda' \; \lambda w \sim \lambda' w w \in \Delta.$$

If $X \subsetneq \Pi$, $L_X^{T_i}$ is not sound with respect to (all X-models over) any class of reactive frames.

3. *A reactive frame (W, Δ) X-validates $\Box_P p \to \Diamond_P \Diamond_R p$ iff is* lightly reflexive, *i.e. satisfies:*

$$w \in W \to \exists \lambda \; \lambda w w.$$

$L_X^{T_l} = L_X \oplus \Box_P p \to \Diamond_P \Diamond_R p$ is sound and complete with respect to the class of X-reactive models over inwardly reflexive frames.

It is clear that strong reflexivity implies inward and outward reflexivity and both imply light reflexivity.

Proof.

1. Given a X-reactive model $\mathfrak{M} = (W, \Delta, \nu)$ over a outwardly reflexive reactive frame and $\lambda w \in \Delta$ such that $\mathfrak{M}, \lambda w \vDash \varphi$, then since $\lambda ww \in \Delta$ and $t(\lambda w) = t(\lambda ww)$ we have $\mathfrak{M}, \lambda ww \vDash \Diamond_P \varphi$ and so $\mathfrak{M}, \lambda w \vDash \Diamond_R \Diamond_P \varphi$.

If a reactive frame (W, Δ) is not outwardly reflexive then there exists some $\lambda w \in \Delta$ such that $\lambda ww \notin \Delta$. Picking a $p \in \Pi$ and choosing a ν such that $\lambda' \in \nu(p)$ iff $t(\lambda') = w$. Now we have that $(W, \Delta, \nu), \lambda w \vDash p \wedge \neg \Diamond_R \Diamond_P p$.

Soundness of $L_X^{T_o}$ comes easily from the soundness of the new axiom which has just been established.

Using the same strategy as before we will obtain the equality:

$$L_X^{T_o} = \{\varphi : \varphi \text{ is } X\text{-valid in every outwardly reflexive reactive frame}\}.$$

- $q \to \Diamond_R \Diamond_P q$ is a Sahlqvist formula and so the canonical frame (W, R, P) for $L_X^{T_o}$ satisfies:

$$\forall w \exists w' wRw'Pw \tag{a}$$

  since it is the first-order correspondent to the new axiom.

- $\mathcal{B}_1$ preserves property (a):
  Given (x, i), if $i = *$ then $(x, i)R_1(x, i)P_1(x, i)$ and if $i = (x', a)$ then by property (a) there exists y such that $xRyPx$ thus $t'(x, x)$ is defined and $(x, i)R_1(t'(x, x), i)P_1(x, i)$.

- let (W, R, P) be a cs-frame satisfying property (a) and I an initial family. (I, Δ^I) is a outward reflexive reactive frame:
  Let $\lambda w \in \Delta^I$ then there are $\gamma \in W^*$ and $s \in W$ such that γs is an I-initial path and $l_I(\gamma s) = \lambda w$. By property (a) there exists $t \in W$ such that $sRtPs$ (which implies tPw) thus γst is also an I-initial path and $l_I(\gamma st) = \lambda ww \in \Delta^I$.

2. To establish soundness we have just to check if the added axiom is sound: given a Π-reactive model $\mathfrak{M} = (W, \Delta, \nu)$ over a inwardly reflexive reactive frame and $\lambda w \in \Delta$ such

that $\mathfrak{M}, \lambda w \models \varphi$ then since exists λ' such that $\lambda' ww \in \Delta$ and $\lambda w \sim \lambda' ww$. Thus by proposition (4.1.5) we have $\mathfrak{M}, \lambda' ww \models \varphi$ and since $t(\lambda w) = t(\lambda' w)$ we have that $\mathfrak{M}, \lambda w \models \Diamond_P \Diamond_R \varphi$.

For completeness we proceed as before:

- $p \to \Diamond_P \Diamond_R q$ is a Sahlqvist formula and so the canonical frame $\mathfrak{F} = (W, R, P)$ for $L_\Pi^{T_i}$ satisfies:

$$\forall w \exists w' w P w' R w \tag{b}$$

  since it is its first-order correspondent.

- $\mathcal{B}_2$ preserves property (b): given $w = (a, b, c, d, i) \in W_2$, if $i = *$ then $wP_2 wR_2 w$ and if $i \neq *$ then $(a, b, c, d, i) P_2 (t(a), b + 1, c, d, i) R_2 (a, b, c, d, i)$.

- let (W, R, P) be a cs-frame satisfying property (b) and I an initial family. (I, Δ^I) is a inwardly reflexive reactive frame:
  Let $\lambda w \in \Delta^I$, so there is $\gamma \in W^*$ and $s \in W$ such that γs is an I-initial path and $l_I(\gamma s) = \lambda w$. By property (b) there exists $t \in W$ such that $tRsPt$. From the initiality of I we know that there is some γ' such that $\gamma' t$ is an I-initial path, thus $\gamma' ts$ is also an I-initial and $l_I(\gamma st) = l_I(\gamma) ww = \lambda' ww \in \Delta^I$. Applying Lemma 4.2.7 we conclude that $\lambda w \sim \lambda' ww$.

So

$$L_\Pi^{T_i} = \{\varphi : \varphi \text{ is valid in every inwardly reflexive reactive frame}\}.$$

Given $X \subsetneq \Pi$ and a reactive frame (W, Δ) there is a X-reactive model $\mathfrak{M} = (W, \Delta, \nu)$, $\lambda \in \Delta$ and φ such that $\mathfrak{M}, \lambda \not\models \varphi \to \Diamond_R \varphi$: given $\lambda w \in \Delta$ and some $p \in \Pi - X$, let ν such that $\lambda w \in \nu(p)$ and $\lambda' ww \notin \nu(p)$ for all $\lambda' ww \in \Delta$. So

$$(W, \Delta, \nu), \lambda w \models p \wedge \neg \Diamond_R \Diamond p.$$

3. Let $\mathfrak{M} = (W, \Delta, \nu)$ be a X-reactive model over a lightly reflexive reactive frame and $\lambda w \in \Delta$ such that $\mathfrak{M}, \lambda w \models \Box_P \varphi$. Since there exists λ' such that $\lambda' ww \in \Delta$ and $t(\lambda w) = t(\lambda' ww)$ we have $\mathfrak{M}, \lambda' ww \models \varphi$ and so $\mathfrak{M}, \lambda' w \models \Diamond_R \varphi$ hence $\mathfrak{M}, \lambda w \models \Diamond_P \Diamond_R \varphi$.

If a reactive frame (W, Δ) is not lightly reflexive then there exists some $w \in W$ for which there is no $\lambda \in \Delta$ such that $\lambda ww \in \Delta$. So we pick a $p \in \Pi$ and choose a ν such that $\lambda' \in \nu(p)$ iff $t(\lambda') = w$. Now we have that $(W, \Delta, \nu), w \models \Box_P p \wedge \neg \Diamond_P \Diamond_R p$.

Soundness of $L_X^{T_l}$ comes easily from the soundness of the new axiom which has just been established.

For completeness, as before:

- $\Box p \to \Diamond_P \Diamond_R q$ is a Sahlqvist formula and (just as before) the canonical frame (W, R, P) for $L_X^{T_o}$ satisfies:
$$\forall w \exists w' w'' \; wPw'Rw''Pw, \tag{c}$$
since it is its first-order correspondent.

- $\mathcal{B}_1$ preserves property (c): given $(x, i) \in W_1$, if $i = 1$ then
$$(x, i)P_1(x, i)R_1(x, i)P_1(x, i)$$
and if $i = (x', a)$ then by property (c) there exist y, z such that $xPyRzPx$ thus $t'(y, y)$ is defined and
$$(x, i)P_1(y, i)R_1(t(y, y), i)P_1(x, i).$$

- let (W, R, P) be a cs-frame satisfying property (c) and I an initial family. (I, Δ^I) is a lightly reflexive reactive frame:
Let $\lambda w \in \Delta^I$ then there are s, t such that $l_I(s) = w = l_I(t)$ and sRt. By initiality of I there is an I-initial path ending in s, γs, making γst also an I-initial path, hence $l_I(\gamma)ww = \lambda' ww \in \Delta$.

So
$$L_X^{T_l} = \{\varphi : \varphi \text{ is } X\text{-valid in every lightly reflexive reactive frame}\}.$$

□

Proposition 4.2.11. *1. A reactive frame (W, Δ) X-validates $\Diamond_R \Diamond_R p \to \Diamond_R \Diamond_P p$ iff it is* left transitive*, i.e. satisfies:*
$$\lambda ww'w'' \in \Delta \to \lambda ww'' \in \Delta.$$
$L_X^{4_l} = L_X \oplus \Diamond_R \Diamond_R p \to \Diamond_R \Diamond_P p$ is sound and complete with respect to the class of X-reactive models over left transitive reactive frames.

2. *A reactive frame* (W,Δ) *X-validates* $\Diamond_R \Diamond_R p \to \Diamond_P \Diamond_R \Diamond_P p$ *iff it is* middle transitive, *i.e. satisfies:*

$$\lambda ww'w'' \in \Delta \to \exists \lambda' \ \lambda' ww'' \in \Delta.$$

$L_X^{4m} = L_X \oplus \Diamond_R \Diamond_R p \to \Diamond_P \Diamond_R \Diamond_P p$ *is sound and complete with respect to the class of X-reactive models over middle transitive reactive frames.*

3. *Let* $L_X^{4r} = L_X \oplus \Diamond_R \Diamond_R p \to \Diamond_P \Diamond_R p$. L_Π^{4r} *is sound and complete with respect to the class of* Π*-reactive models over* right transitive *reactive frames, i.e. satisfying:*

$$\lambda ww'w'' \in \Delta \to \exists \lambda' \ \lambda ww'w'' \sim \lambda' ww'' \in \Delta.$$

If $X \subsetneq \Pi$, L_X^{4r} *is not sound and complete with respect to (all X-models over) any class of reactive frames.*

4. *A reactive frame* (W,Δ) *X-validates* $\Diamond_R \Diamond_P \Diamond_R p \to \Diamond_R \Diamond_P p$ *iff it is* globally left transitive, *i.e. satisfies:*

$$\lambda ww', \lambda' w'w'' \in \Delta \to \lambda ww'' \in \Delta.$$

$L_X^{4gl} = L_X \oplus \Diamond_R \Diamond_P \Diamond_R p \to \Diamond_R \Diamond_P p$ *is sound and complete with respect to the class of X-reactive models over globally left transitive reactive frames.*

5. *A reactive frame* (W,Δ) *X-validates* $\Diamond_R \Diamond_P \Diamond_R p \to \Diamond_P \Diamond_R \Diamond_P p$ *iff it is* globally middle transitive, *i.e. satisfies:*

$$\lambda ww', \lambda' w'w'' \in \Delta \to \exists \lambda'' \ \lambda'' ww'' \in \Delta.$$

$L_X^{4gm} = L_X \oplus \Diamond_R \Diamond_P \Diamond_R p \to \Diamond_P \Diamond_R \Diamond_P p$ *is sound and complete with respect to the class of X-reactive models over globally transitive reactive frames.*

6. *Let* $L_X^{4gr} = L_X \oplus \Diamond_R \Diamond_P \Diamond_R p \to \Diamond_P \Diamond_R p$. L_Π^{4gr} *is sound and complete with respect to the class of* Π*-reactive models over* globally right transitive *reactive frames, i.e. satisfies:*

$$\lambda ww', \lambda' w'w'' \in \Delta \to \exists \lambda'' \ \lambda' w'w'' \sim \lambda'' ww'' \in \Delta.$$

If $X \subsetneq \Pi$, L_X^{4gr} *is not sound and complete with respect to (all X-models over) any class of reactive frames.*

It is clear that strong transitivity implies left, middle and right transitivity. Both left and right transitivity imply middle transitivity and all of them are implied by its global version.

Proof. 1. Given a X-reactive model $\mathfrak{M} = (W, \Delta, \nu)$ over a left transitive reactive frame and $\lambda w \in \Delta$ such that

$$\mathfrak{M}, \lambda w \vDash \Diamond_R \Diamond_R \varphi,$$

i.e. there are w', w'' such that $\lambda w w' w'' \in \Delta$ and $\mathfrak{M}, \lambda w w' w'' \vDash \varphi$. By left transitivity we get that $\lambda w w'' \in \Delta$ and since $t(\lambda w w' w'') = t(\lambda w w'')$ we have

$$\mathfrak{M}, \lambda w \vDash \Diamond_R \Diamond_P \varphi.$$

If a reactive frame (W, Δ) is not left transitive then there exists $\lambda w w' w'' \in \Delta$ such that $\lambda w w'' \notin \Delta$. So we pick a $p \in \Pi$ and choose a ν such that $\gamma \in \nu(p)$ iff $t(\gamma) = w''$. Now we have that

$$(W, \Delta, \nu), \lambda w \vDash \Diamond_R \Diamond_R p \wedge \neg \Diamond_R \Diamond_P p.$$

Soundness of $L_X^{4_l}$ follows from the soundness of the new axiom which has just been established.

Let us prove that $L_X^{4_l} = \{\varphi : \varphi \text{ is } X\text{-valid in every left transitive reactive frame}\}$:

- $\Diamond_R \Diamond_R p \to \Diamond_R \Diamond_P p$ is a Sahlqvist formula and so the canonical model (W, R, P) for $L_X^{4_l}$ satisfies:

$$\forall t v w \exists w' \; tRvRw \to tRw'Pw, \tag{LT}$$

  since it is its first-order correspondent.

- $\mathcal{B}_4$ preserves property (LT):
  Given $(x_k, i_k) \in W_4$ for $k = 1, 2, 3$ such that $(x_1, i_1)R_4(x_2, i_2)R_4(x_3, i_3)$. So $i_1 \leq i_2 < i_3$, $x_k = (x_{k,1}, x_{k,2})$ for $k = 2, 3$. If $x_1 \in W$ we have that

$$(x_1, i_1)R_4((x_{3,2}, x_{3,2}), i_3)P_4(x_3, i_3).$$

  If $x_1 = (x_{1,1}, x_{1,2})$ then $x_{1,1}Rx_{2,1}Rx_{3,1}$, hence, by property (LT), there is x such that $x_{1,1}RxPx_{3,1}(Px_{3,2})$. Thus $t'(x_{1,1}, x_{3,2})$ is defined and

$$(x_1, i_1)R_4((t'(x_{1,1}, x_{3,2})), i_3)P_4(x_3, i_3).$$

- let (W, R, P) be a cs-frame satisfying property (LT) and I an initial family. (I, Δ^I) is a left transitive reactive frame:
  Let $\lambda w w' w'' \in I$ then there are some $\gamma \in W^*$ and $s, s', s'' \in W$ such that $\gamma s s' s''$ is an I-initial path and $l_I(\gamma s s' s'') = \lambda w w' w''$. By (LT) there exists some $t \in W$ such that $sRtPs''Pw$ and so γst is an I-initial path. Hence $l_I(\gamma st) = \lambda w w'' \in \Delta^I$.

2. Given a X-reactive model $\mathfrak{M} = (W, \Delta, \nu)$ over a middle transitive reactive frame and $\lambda w \in \Delta$ such that

$$\mathfrak{M}, \lambda w \vDash \Diamond_R \Diamond_R \varphi,$$

i.e. there are w', w'' such that $\lambda w w' w'' \in \Delta$ and $\mathfrak{M}, \lambda w w' w'' \vDash \varphi$. By middle transitivity, there is λ' such that $\lambda' w w'' \in \Delta$. Since $t(\lambda w w' w'') = t(\lambda' w w'')$ and $t(\lambda w) = t(\lambda' w)$ we have

$$\mathfrak{M}, \lambda w \vDash \Diamond_P \Diamond_R \Diamond_P \varphi.$$

If a reactive frame (W, Δ) is not middle transitive then there exists $\lambda w w' w'' \in \Delta$ and no λ' such that $\lambda' w w'' \in \Delta$. So we pick a $p \in \Pi$ and choose a ν such that $\gamma \in \nu(p)$ iff $t(\gamma) = w''$. Obtaining

$$(W, \Delta, \nu), \lambda w \vDash \Diamond_R \Diamond_R p \wedge \neg \Diamond_P \Diamond_R \Diamond_P p.$$

Soundness of $L_X^{4_l}$ follows from the soundness of the new axiom which has just been established.

We prove completeness as before:

- $\Diamond_R \Diamond_R p \to \Diamond_P \Diamond_R \Diamond_P p$ is a Sahlqvist formula and so the canonical model (W, R, P) for $L_X^{4_m}$ satisfies:

$$\forall t v w \exists t' w' \; tRvRw \to tPt'Rw'Pw \qquad \text{(MT)}$$

  since it is its first-order correspondent.

- $\mathcal{B}_4$ preserves property (MT):
  Given $(x_k, i_k) \in W_4$ for $k = 1, 2, 3$ such that $(x_1, i_1)R_4(x_2, i_2)R_4(x_3, i_3)$. So $i_1 \leq i_2 < i_3$, $x_k = (x_{k,1}, x_{k,2})$ for $k = 2, 3$. If $x_1 \in W$ we have that

$$(x_1, i_1)P_4(x_1, i_1)R_4((x_{3,2}, x_{3,2}), i_3)P_4(x_3, i_3).$$

  If $x_1 = (x_{1,1}, x_{1,2})$ then $x_{1,1}Rx_{2,1}Rx_{3,1}$ hence, by property (MT), there are x, x' such that

$$x_{1,1}PxRx'Px_{3,1}(Px_{3,2})$$

  and so $t'(x, x_{3,2})$ is defined and

$$(x_1, i_1)P_4((x, x_{1,2}), i_1)R_4((t'(x, x_{3,2})), i_3)P_4(x_3, i_3).$$

- let (W, R, P) be a cs-frame satisfying property (MT) and I an initial family. (I, Δ^I) is a middle transitive reactive frame:
  Let $\lambda ww'w'' \in I$ then there are some $\gamma \in W^*$ and $s, s', s'' \in W$ such that $\gamma ss's''$ is an initial path and $l_I(\gamma ss's'') = \lambda ww'w''$. From property (MT) follows that there are some $t, t'' \in W$ such that $sPtRt''Ps''$. By initiality of I there is an I-initial path $\gamma't$, making $\gamma'tt''$ also an I-initial thus $l_I(\gamma'tt'') = l_I(\gamma')ww'' = \lambda'ww'' \in \Delta^I$.

So $L_X^{4_m} = \{\varphi : \varphi \text{ is } X\text{-valid in every middle transitive reactive frame}\}$.

3. To estabilish soundness of $L_\Pi^{4_r}$ it is enough to verify soundness of the added axiom: given a Π-reactive model $\mathfrak{M} = (W, \Delta, v)$ over a right transitive reactive frame and $\lambda w \in \Delta$ such that

$$\mathfrak{M}, \lambda w \models \Diamond_R \Diamond_R \varphi,$$

so there are w', w'' such that $\lambda ww'w'' \in \Delta$ and $\mathfrak{M}, \lambda ww'w'' \models \varphi$. By right transitivity there is $\lambda'ww'' \in \Delta$ such that $\lambda ww'w'' \sim \lambda'ww''$ and so, by Proposition 4.1.5, $\mathfrak{M}, \lambda'ww'' \models \varphi$. Thus

$$\mathfrak{M}, \lambda w \models \Diamond_P \Diamond_R \varphi.$$

For completeness:

- $\Diamond_R \Diamond_R p \to \Diamond_P \Diamond_R p$ is a Sahlqvist formula and so the canonical model (W, R, P) for $L_\Pi^{4_r}$ satisfies:
  $$\forall tvw \exists t' \; tRvRw \to tPt'Rw \tag{RT}$$
  since it is its first-order correspondent.

- $\mathcal{B}_3$ preserves property (RT):
  Given $(x_k, i_k) \in W_3$ for $k = 1, 2, 3$ such that $(x_1, i_1)R_3(x_2, i_2)R_3(x_3, i_3)$. So $i_1 \leq i_2 < i_3$, $x_k = (x_{k,1}, x_{k,2})$ for $k = 2, 3$ and $x_{3,1} = x_{3,2}$. If $x_1 \in W$ then immediately we conclude that $(x_1, i_1)P_3(x_1, i_1)R_3(x_3, i_3)$. If $x_1 = (x_{1,1}, x_{1,2})$ then $x_{1,1}Rx_{2,1}Rx_{3,1} = x_{3,2}$ so by property (RT) there is a x such that $x_{1,1}PxRx_{3,1}$ and so $(x_1, i_1)P_3((x, x_{1,2}), i_1)R_3(x_3, i_3)$.

- let (W, R, P) be a cs-frame satisfying property (RT) and I an initial family. (I, Δ^I) is a right transitive reactive frame:
  Let $\lambda ww'w'' \in \Delta^I$ then there is an I-initial path $\gamma s_1 s_2 s_3$ ($s_i \in W$) such that $l_I(\gamma s_1 s_2 s_3) = \lambda ww'w''$. By property (RT) there is $s \in W$ such that $s_1 PsRs_3$. From the initiality of

I we know that there is some γ' such that $\gamma' s$ is an I-initial path and so $\gamma' s s_3$ is also an I-initial. Applying Lemma 4.2.7 we conclude that $\Delta^I \ni l_I(\gamma' st) = l_I(\gamma')ww'' = \lambda' ww'' \sim \lambda ww'w''$.

Hence,

$$L_{\Pi}^{4_r} = \{\varphi : \varphi \text{ is } \Pi\text{-valid in every right transitive reactive frame}\}.$$

Given $X \subsetneq \Pi$ and a (non-empty) class of reactive frames F:

- if there is a reactive frame $(W, \Delta) \in \mathsf{F}$ with a path $\lambda w_0 w_1 w_2$, where $w_0 \neq w_1$, we pick $p \in \Pi - X$ and ν such that $\gamma \in \nu(p)$ iff $\gamma = \lambda w_0 w_1 w_2$. Thus

$$(W, \Delta, \nu), \lambda w_0 \vDash \Diamond_R \Diamond_R p \wedge \neg \Diamond_P \Diamond_R p.$$

- if F contains only reactive frames with paths of length bigger 2 of the form $\lambda www'$ then it validates $\Diamond_R(\varphi \wedge \Diamond_R \top) \to \Diamond_P \varphi$. Consider that $L_X^{4_r} \subseteq L_{\Pi}^{4_r}$ and the right transitive Π-reactive model: $\mathfrak{M} = (\{0,1\}, \{0,1,01,011\}, \nu)$ such that $\lambda \in \nu(p)$ iff $t(\lambda) = 1$. It is clear that $\mathfrak{M}, 0 \vDash \Diamond_R(p \wedge \Diamond_R \top) \wedge \neg \Diamond_P p$ and so:

$$L_X^{4_r} \subsetneq \{\varphi : \varphi \text{ is valid in all } F \in \mathsf{F}\}.$$

- if F contains only reactive frames with paths of length smaller then 3, following the same reasoning as in proposition (4.2.9.3), together with the fact that the strong transitive reactive frame used there - $(\{a\}, \{a\}^*)$ - is also a right transitive reactive frame, we conclude that $L_X^{4_r}$ is not complete with respect to F.

4. Given a X-reactive model $\mathfrak{M} = (W, \Delta, \nu)$ over a globally left transitive reactive frame and $\lambda w \in \Delta$ such that

$$\mathfrak{M}, \lambda w \vDash \Diamond_R \Diamond_P \Diamond_R \varphi,$$

i.e. there are $w', w'' \in W$ and λ' such that $\lambda ww', \lambda' w'w'' \in \Delta$ and $\mathfrak{M}, \lambda' w'w'' \vDash \varphi$. By light left transitivity, there is $\lambda ww'' \in \Delta$ and since $t(\lambda ww'') = t(\lambda' w'w'')$ we have

$$\mathfrak{M}, \lambda w \vDash \Diamond_R \Diamond_P \varphi.$$

If a reactive frame (W, Δ) is not globally left transitive then there is $\lambda ww', \lambda' w'w'' \in \Delta$ such that $\lambda ww'' \notin \Delta$. So we pick a $p \in \Pi$ and choose a ν such that $\gamma \in \nu(p)$ iff $t(\gamma) = w''$. Hence

$$(W, \Delta, \nu), \lambda w \vDash \Diamond_R \Diamond_P \Diamond_R p \wedge \neg \Diamond_R \Diamond_P p.$$

Soundness of $L_X^{4_l}$ comes easily from the soundness of the new axiom which has just been established.

We establish completeness as before:

- $\Diamond_R \Diamond_P \Diamond_R p \to \Diamond_R \Diamond_P p$ is a Sahlqvist formula and so the canonical model (W, R, P) for $L_X^{4_{gl}}$ satisfies:
$$\forall tvv'w \exists w' \; tRvPv'Rw \to tRw'Pw, \tag{LLT}$$
since it is its first-order correspondent.

- $\mathcal{B}_4$ preserves property (LLT):
Given $(x_k, i_k) \in W_4$ for $k = 1, 2, 3, 4$ such that $(x_1, i_1)R_4(x_2, i_2)P_4(x_3, i_3)R_4(x_4, i_4)$. So $i_1 \leq i_2 = i_3 < i_4$, $x_k = (x_{k,1}, x_{k,2})$ for $k = 2, 3, 4$. If $x_1 \in W$ we have that
$$(x_1, i_1)R_4((x_{4,2}, x_{4,2}), i_4)P_4(x_4, i_4).$$
If $x_1 = (x_{1,1}, x_{1,2})$ then $x_{1,1}Rx_{2,1}Px_{3,1}Rx_{4,1}$ hence, by property (LLT), there is x such that $x_{1,1}RxP\; x_{4,1}(Px_{4,1})$ and so $t'(x_{1,1}, x_{4,2})$ is defined and
$$(x_1, i_1)R_4((t'(x_{1,1}, x_{4,2})), i_3)P_4(x_4, i_4).$$

- given a cs-frame (W, R, P) satisfying property (LLT) with initial family I, (I, Δ^I) is a light left transitive reactive frame: Let $\lambda ww', \lambda'w'w'' \in \Delta^I$ then there are I-initial paths $\gamma s_1 s_2$ and $\gamma' s_3 s_4$ ($s_i \in W$) such that $l_I(\gamma s_1 s_2) = \lambda ww'$ and $l_I(\gamma' s_3 s_4) = \lambda' w' w''$ (so $s_1 R s_2 P s_3 R s_4$). By property (LLT) there exists $s \in W$ such that $s_1 R s P s_4$. So $\gamma s_1 s$ is also an I-initial path and $l_I(\gamma s_1 s) = \lambda ww'' \in \Delta^I$.

Hence
$$L_X^{4_{gl}} = \{\varphi : \varphi \text{ is } X\text{-valid in every light left transitive reactive frame}\}.$$

5. Given a X-reactive model $\mathfrak{M} = (W, \Delta, \nu)$ over a light transitive reactive frame and $\lambda w \in \Delta$ such that
$$\mathfrak{M}, \lambda w \vDash \Diamond_R \Diamond_P \Diamond_R \varphi,$$
i.e. there are $w', w'' \in W$ and λ' such that $\lambda ww', \lambda'w'w'' \in \Delta$ and $\mathfrak{M}, \lambda'w'w'' \vDash \varphi$. By globally middle transitivity, there is λ'' such that $\lambda''ww'' \in \Delta$, thus
$$\mathfrak{M}, \lambda w \vDash \Diamond_P \Diamond_R \Diamond_P \varphi.$$

If a reactive frame (W, Δ) is not globally middle transitive then there exists some $\lambda ww'$, $\lambda' w' w'' \in \Delta$ such that for all λ'' we have $\lambda'' ww'' \notin \Delta$. So we pick $p \in \Pi$ and ν such that $\gamma \in \nu(p)$ iff $t(\gamma) = w''$. Hence

$$(W, \Delta, \nu), \lambda w \models \Diamond_R \Diamond_P \Diamond_R p \wedge \neg \Diamond_P \Diamond_R \Diamond_P p.$$

Soundness of $L_X^{4_l}$ comes easily from the soundness of the new axiom which has just been established.

Completeness:

- $\Diamond_R \Diamond_P \Diamond_R p \to \Diamond_P \Diamond_R \Diamond_P p$ is a Sahlqvist formula and so the canonical model (W, R, P) for $L_X^{4_l}$ satisfies:

$$\forall t, v, v', w \exists t', w' \; tRvPv'Rw \to tPt'Rw'Pw \tag{GM}$$

  since it is the first-order correspondent to the new axiom.

- $\mathcal{B}_4$ preserves property (GM):
  Given $(x_k, i_k) \in W_4$ for $k = 1, 2, 3, 4$ such that $(x_1, i_1)R_4(x_2, i_2)P_4(x_3, i_3)R_4(x_4, i_4)$. So $i_1 \leq i_2 = i_3 < i_4$, $x_k = (x_{k,1}, x_{k,2})$ for $k = 2, 3, 4$. If $x_1 \in W$ we have that

$$(x_1, i_1)P_4(x_1, i_1)R_4((x_{4,2}, x_{4,2}), i_4)P_4(x_4, i_4).$$

  If $x_1 = (x_{1,1}, x_{1,2})$ then $x_{1,1}Rx_{2,1}Px_{3,1}Rx_{4,1}$ hence, by property (GM), there is x, x' such that $x_{1,1}PxRx'Px_{4,1}$ and so $t'(x, x_{4,2})$ is defined and

$$(x_1, i_1)P_4((x, x_{1,2}), i_1)R_4((t'(x, x_{4,2})), i_3)P_4(x_4, i_4).$$

- given a cs-frame (W, R, P) satisfying property (GM) with initial family I, (I, Δ^I) is a globally middle transitive reactive frame:
  Let $\lambda ww', \lambda' w' w'' \in \Delta^I$ then there are I-initial paths $\gamma s_1 s_2$ and $\gamma' s_3 s_4$ ($s_i \in W$) such that $l_I(\gamma s_1 s_2) = \lambda ww'$ and $l_I(\gamma' s_3 s_4) = \lambda' w' w''$ (in particular $s_1 R s_2 P s_3 R s_4$). By property (GM) there exists $t, t' \in W$ such that $s_1 P t R t' R s_4$. By initiality of I, there is some γ'' such that $\gamma'' t$ is an I-initial path and so $\gamma'' tt'$ is also an I-initial path and $l_I(\gamma'' tt') = \lambda'' ww'' \in \Delta^I$.

Hence $L_X^{4_{gm}} = \{\varphi : \varphi$ is X-valid in every globally middle transitive reactive frame$\}$.

6. To establish soundness of L^{4gr}_{Π} it is enough to verify soundness of the added axiom: given a Π-reactive model $\mathfrak{M} = (W, \Delta, v)$ over a globally right transitive reactive frame and $\lambda w \in \Delta$ such that

$$\mathfrak{M}, \lambda w \vDash \Diamond_R \Diamond_P \Diamond_R \varphi,$$

i.e. there are $w', w'' \in W$ and λ' such that $\lambda w w', \lambda' w' w'' \in \Delta$ and $\mathfrak{M}, \lambda' w' w'' \vDash \varphi$. By globally right transitivity we get that there is λ'' such that $\lambda'' w w'' \sim \lambda' w' w''$, so by Proposition 4.1.5 we get $\mathfrak{M}, \lambda'' w w'' \vDash \varphi$ and so

$$\mathfrak{M}, \lambda w \vDash \Diamond_P \Diamond_R \varphi.$$

For completeness:

- $\Diamond_R \Diamond_P \Diamond_R p \to \Diamond_P \Diamond_R p$ is a Sahlqvist formula and so the canonical model (W, R, P) for L^{4gr}_{X} satisfies:

$$\forall t v v' w \exists t' \; tRvPv'Rw \to tPt'Rw \tag{GR}$$

  since it is its first-order correspondent.

- $\mathcal{B}_3$ preserves property (GR):
  Given $(x_k, i_k) \in W_3$ for $k = 1, 2, 3, 4$ such that $(x_1, i_1) R_3 (x_2, i_2) P_3 (x_3, i_3) R_3 (x_4, i_4)$. So $i_1 \le i_2 = i_3 < i_4$, $x_k = (x_{k,1}, x_{k,2})$ for $k = 2, 3, 4$ and $x_{4,1} = x_{4,2}$. If $x_1 \in W$ then immediately we conclude $(x_1, i_1) P_3 (x_1, i_1) R_3 (x_4, i_4)$. If $x_1 = (x_{1,1}, x_{1,2})$ then $x_{1,1} R x_{2,1} P x_{3,1} R x_{4,1} = x_{4,2}$ so by property (GR) exists a x such that $x_{1,1} P x R x_{4,1}$ and so $(x_1, i_1) P_3 ((x, x_{1,2}), i_1) R_3 (x_4, i_4)$.

- given a cs-frame (W, R, P) satisfying property (GR) with initial family I, (I, Δ^I) is a globally right transitive reactive frame:
  Let $\lambda w w', \lambda' w' w'' \in \Delta^I$ then there are I-initial paths $\gamma s_1 s_2$ and $\gamma' s_3 s_4$ ($s_i \in W$) such that $l_I(\gamma s_1 s_2) = \lambda w w'$ and $l_I(\gamma' s_3 s_4) = \lambda' w' w''$ (so $s_1 R s_2 P s_3 R s_4$). By property (GR) there exists $s \in W$ such that $s_1 P s R s_4$. By initiality of I, there is some γ'' such that $\gamma'' s$ is an I-initial path and so $\gamma'' s s_4$, thus $l_I(\gamma'' s s_4) = \lambda'' w w'' \in \Delta$. Applying Lemma 4.2.7 we conclude that $\lambda' w' w'' \sim \lambda'' w w''$.

So

$$L^{4gr}_{\Pi} = \{\varphi : \varphi \text{ is valid in every globally right transitive reactive frame}\}.$$

Given $X \subsetneq \Pi$, L_X^{4gr} is not sound and complete for any class of reactive frames:
It follows from the proof in 3. for L_X^{4r}. In the first case we can also conclude that the frame does not validate $\Diamond_R \Diamond_P \Diamond_R p \rightarrow \Diamond_P \Diamond_R p$, thus $L_X^{4rl} \subset L_X^{4r}$ is not sound in relation any class of frames containing a reactive frame with a path of the form $\lambda w_0 w_1 w_2$ with $w_0 \neq w_1$. Nor it is complete with respect to the other two cases considered since both reactive frames used in the proof are also globally right transitive.

□

Static and quasi-static

Proposition 4.2.12. *We have that* $(\Diamond_P \Diamond_R p \rightarrow \Diamond_R \Diamond_P p) \leftrightarrow (\Diamond_R \Box_P p \rightarrow \Box_P \Diamond_R p) \in L_X$, *and* $L_X^S = L_X \oplus \Diamond_P \Diamond_R p \rightarrow \Diamond_R \Diamond_P p = L_X \oplus \Diamond_R \Box_P p \rightarrow \Box_P \Diamond_R p$ *is sound and complete with respect to the class of X-reactive models over* static *reactive frames, i.e. satisfying:*

$$\lambda w, \lambda' w \in \Delta \rightarrow \lambda w \sim \lambda' w.$$

Notice that if $X = \Pi$, the new axioms are equivalent to $\Diamond_P p \rightarrow p$ and that if we impose this axiom instead we get a result limited to this case. This tells us that in the other cases where we have this kind of restriction there may be better axiomatisations valid for all X.

Proof. Let $com^r_{PR} = \Diamond_P \Diamond_R p \rightarrow \Diamond_R \Diamond_P p$ and $chr_{PR} = \Diamond_R \Box_P p \rightarrow \Box_P \Diamond_R p$.

- Using the equality $L_{r,X} = L_X$:
  - chr_{PR} implies com^r_{PR}
    $\mathfrak{M}, \lambda w \vDash \Diamond_P \Diamond_R \varphi$, since $\varphi \rightarrow \Box_P \Diamond_P \varphi$ of P we have $\mathfrak{M}, \lambda w \vDash \Diamond_P \Diamond_R \Box_P \Diamond_P \varphi$. Applying chr_{PR} we get $\mathfrak{M}, \lambda w \vDash \Diamond_P \Box_P \Diamond_R \Diamond_P \varphi$ and, again from $\psi \rightarrow \Box_P \Diamond_P \psi$ (equivalent to $\Diamond_P \Box_P \psi \rightarrow \psi$), we obtain $\mathfrak{M}, \lambda w \vDash \Diamond_R \Diamond_P \varphi$.
  - com^r_{PR} implies chr_{PR}
    $\mathfrak{M}, \lambda w \vDash \Diamond_P \Box_R \varphi$, since $\varphi \rightarrow \Box_P \Diamond_P \varphi$, we have $\mathfrak{M}, \lambda w \vDash \Diamond_P \Box_R \Box_P \Diamond_P \varphi$. Applying com^r_{PR} we get $\mathfrak{M}, \lambda w \vDash \Diamond_P \Box_P \Box_R \Diamond_P \varphi$ and, again from $\Diamond_P \Box_P \psi \rightarrow \psi$, we obtain $\mathfrak{M}, \lambda w \vDash \Box_R \Diamond_P \varphi$.
- Soundness of L_X^S comes easily from the soundness of the new axiom:

Given a X-reactive model $\mathfrak{M} = (W, \Delta, v)$ over a static frame and $\lambda w \in \Delta$ such that $\mathfrak{M}, \lambda w \models \Diamond_P \Diamond_R \varphi$ then there exists $\lambda' w w' \in \Delta$ such that $\mathfrak{M}, \lambda' w w' \models \varphi$ thus $\lambda w w' \in \Delta$ (since $\lambda w \sim \lambda' w$) and $\mathfrak{M}, \lambda w \models \Diamond_R \Diamond_P \varphi$ (since $t(\lambda w w') = t(\lambda' w w')$).

- For completeness we proceed as before:
  - $\Diamond_P \Diamond_R p \to \Diamond_R \Diamond_P p$ is a Sahlqvist formula and so the canonical model $\mathfrak{F} = (W, R, P)$ for L_X^S satisfies PR-commutativity:

    $$\forall xyz \exists y' \; xPyRz \to xRy'Pz$$

    since it is the first-order correspondent to the new axiom.
  - PR-commutativity is preserved by $\mathcal{B}_1$:

    Let $(x, i), (y, j), (z, k) \in W'$ such that $(x, i)P'(y, j)R'(z, k)$. If $i = *$ then $(x, i) = (y, j)$ and $(x, i)R'(z, k)P'(z, k)$. If $i \neq *$ then $xPyRz$, $i = j = (x', a)$ and $k = (z', b)$. So exists y' such that $xRy'Pz$, and so $t'(x, v)$ is defined for any v such that vPz. Or $i = j = k$, so $t'(x, x)$ is defined and $(x, i)R'(t'(x, x), k)P'(z, k)$, or $i = j \neq k$, so $t'(x, z')$ is defined and $(x, i)R'(t'(x, z'), k)P'(z, k)$.

- Given a PR-commutative cs-frame with initial family I then (I, Δ^I) is a static reactive frame:

  Let $\lambda w, \lambda' w \in \Delta^I$ so there is βb_1 and $\beta' b_1'$ I-initial paths such that $l_I(\beta b_1) = \lambda w$ and $l_I(\beta' b_1') = \lambda' w$. Let us see that for every $\gamma = w_1 \dots w_n$ we have $\beta b_1 \gamma$ iff exist $\gamma' = v_1 \dots v_n$ such that $\beta' b_1 \gamma'$ and $v_i P w_i$ for $i = 1, \dots, n$. By induction on n (we do it only in one direction, the other is the same):

  - $n = 1$

    $\beta b_1 w_1$ is an I-initial path so by PR-commutativity exists v_1 such that $b_2 R v_1 P w_1$ and so $\beta' b_2 v_1$ is an I-initial path.
  - $n + 1$

    $\beta b_1 w_1 \dots w_n w_{n+1}$ is an I-initial path so by PR-commutativity exists v_{n+1} such that $v_n R v_{n+1} \; P w_{n+1}$ and so $\beta' b_2 v_1$ is an I-initial path.

  Hence we conclude that $\lambda w \sim \lambda' w$.

□

A static frame is a reactive frame that does not react, that is, the accessible worlds depend only on the current world and not on how you get there. It is clear that from such a reactive frame (W, Δ) we can obtain a classic Kripke frame (W, R) where

$$R = \{(w, v) : wv \in \Delta\} = \{(w, v) : \lambda wv \in \Delta\}.$$

It is straightforward to see that in such a reactive frame all the variants of transitivity and reflexivity on reactive frames coincide with the usual notions on the correspondent Kripke frame.

It is easy to see that a Kripke model over a general frame (W, R, P, A), where P is an equivalence class, is a Π-shattered model iff $A \subseteq 2^W$ - the boolean algebra generated by the P equivalence classes - is closed for the operators: $m_R(X) = \{x \in W : \exists y \in W \text{ such that } xRy\}$ and $m_P(X) = \{x \in W : \exists y \in W \text{ such that } xPy\}$. Being P an equivalence relation, A is trivially closed under m_P. It is easy to see that A being closed under m_R, means that if a world is in $m_R(X)$ so it must be all its P-class, which corresponds to PR-commutativity. From the point of view of reactivity, the use of (shattered) general frames to deal with the restrictions over the valuations (even if only in the case of $X = \Pi$) does not help, it demands a very strict interaction between R and P, it corresponds, in the reactive level to ask it not to react! See [60] for an equivalent presentation of Π-shattered models (with only $\Diamond_R$) and its relation with general frames.

Proposition 4.2.13. *Let* $L_X^{qS} = L_X \oplus \Diamond_R \Diamond_P p \to \Diamond_P \Diamond_R p$. L_Π^{qS} *is sound and complete with respect to the class of (*Π*-shattered models over)* quasi-static *reactive frames, i.e. that satisfy:*

$$\lambda ww', \lambda' w' \in \Delta \to \exists \lambda'' \; \lambda' w' \sim \lambda'' ww' \in \Delta$$

If $X \subsetneq \Pi$, L_X^{qS} *is not sound and complete with respect to (all X-models over) any class of reactive frames.*

Proof.

- Soundness:

  Given a Π-reactive model $\mathfrak{M} = (W, \Delta, v)$ over a quasi-static reactive frame and $\lambda w \in \Delta$ such that $\mathfrak{M}, \lambda w \models \Diamond_R \Diamond_P \varphi$ then there exists $\lambda ww', \lambda' w' \in \Delta$ such that $\mathfrak{M}, \lambda' w' \models \varphi$ thus $\lambda' w' \sim \lambda'' ww' \in \Delta$ and $\mathfrak{M}, \lambda'' ww' \models \varphi$ (by Lemma 4.1.5). Hence $\mathfrak{M}, \lambda w \models \Diamond_P \Diamond_R \varphi$ (since $t(\lambda ww') = t(\lambda' ww')$).

- Completeness:

$\Diamond_R \Diamond_P\, p \to \Diamond_P \Diamond_R\, p$ is a Sahlqvist formula and so the canonical frame $\mathfrak{F} = (W, R, P)$ for L^{qS}_X satisfies RP-commutativity:

$$\forall xyz \exists y'\ xRyPz \to xPy'Rz,$$

since it is the first-order correspondent to the new axiom.

Given $\mathfrak{M} = (W, R, P, V)$ a RP-commutative shattered frame, let $(\mathcal{B}_5(\mathfrak{M}) =)\mathfrak{M}' = (W', R', P', V')$ be defined by:

- $W' = (W^+)^3 \cup (W^+)^2 \times \{*\}$,
- $(a, b, c)R'(a', b', c')$ iff $c = * \ \&\ a = a'$ or
  $c \neq * \neq c' \ \&\ o(a)Ro(a') \ \&$
  $\big(a \in W \ \&\ a' = b' = c'$ or $a \notin W \ \&$
  $(b' \neq a' \neq c' \ \&\ r(a) = a'$ or $r(a) = b' \ \&\ a' = c'$ or $r(a) = c' \ \&\ a' = b')\big)$,
- $(a, b, c)P'(a', b', c')$ iff $(b, c) = (b', c') \ \&\ o(a)Po(a')$,
- $(a, b, c) \in V'(p)$ iff $o(a) \in V(p)$.

Where $o(w_1 \ldots w_n) = w_1$, $t(w_1 \ldots w_n) = w_n$ and $r(w_1 \ldots w_n) = w_2 \ldots w_n$.

We have that (W', R', P') is a RP-commutative ($\mathcal{B}_5$ preserves RP-commutativity) cs-frame and the M' is a bounded morphic image of M:

- (W', R', P') is a RP-commutative Let $(a, b, c)R'(a', b', c')P'(a'', b'', c'')$. If $c = *$ then $a = a'$ and $o(a) = o(a')Po(a'')$ thus

  $$(a, b, c)P'(a'', b, c)R'(a'', b'', c'').$$

  If $c \neq * \neq c'$ then $o(a)Ro(a')Po(a'')$ so there is w such that $o(a)PwRo(a'')$. We have three possibilities:
  either $b' = b'' \neq a'' \neq c'' = c'$ and so $(a, b, c)P'(wa'', b, c)R'(a'', b'', c'')$;
  or $a'' = b''$ thus

  $$(a, b, c)P'(wc'', b, c)R'(a'', b'', c'');$$

  or else $a'' = c''$ and

  $$(a, b, c)P'(wb'', b, c)R'(a'', b'', c'').$$

- Clearly V' is admissible and any choice of I picking one element from each P-class and containing $\{(a,a,*) : a \in W^+\}$ works as an initial family.
- Let us check that it is coherent. Let $(a,b,c)R'(a',b',c')$, $(a,b,c)R'(a'',b'',c'')$ and $(a',b',c')P'(a'',b'',c'')$.

  From $(a',b',c')P'(a'',b'',c'')$ we conclude that $(b',c') = (b'',c'')$. If $c = *$ then $a' = a = a''$. If $c \neq * \neq c'$, then or $a \in W$ and $a' = b' = c' = c'' = b'' = a''$; or $a \notin W$, in which case we have three subcases: or $b' \neq a' \neq c'$ and so $r(a) = a' = a''$; or $a' = b'$, thus $r(a) = c' = c''$ and $a'' = b'' = b' = a'$; or $a' = c'$ and $r(a) = b' = b''$ so $a'' = c'' = c' = a'$. In any case, $(a',b',c') = (a'',b'',c'')$.
- Let $f : (W^+)^3 \to W$ be defined by $f((a,b,c)) = o(a)$.

  It is straightforward to see that f is surjective, let us see that it is actually a p-morphism:

  The condition in the valuation is trivial as before;

  * f is a p-morphism in R
    · $(a,b,c)R'(a',b',c')$ then $f((a,b,c)) = o(a)Ro(a') = f((a',b',c'))$,
    · $f((a,b,c))Rw$,
      if $a \in W$ then $(a,b,c)R'(w,w,w)$ otherwise $(a,b,c)R'(w,r(a),w)$;
  * f is a p-morphism in P
    · $(a,b,c)P'(a',b',c')$ then $f((a,b,c)) = o(a)Po(a') = f((a',b',c'))$,
    · $f((a,b,c))Pw$ then $(a,b,c)P'(w,b,c)$;

- Given a PR-commutative cs-frame with initial family I then (I, Δ^I) is a such a reactive frame:

  Let $\lambda ww', \lambda' w' \in \Delta^I$, so there are $\gamma, \gamma' \in W^*$ and $s, s', s'' \in W$ such that $\gamma ss'$ and $\gamma' s''$ are I-initial paths, $l_I(\gamma ss') = \lambda ww'$ and $l_I(\gamma' s'') = \lambda' w'$. So $sRs'Ps''$, thus there is t such that $sPtRs''$. Let γ'' be such that $\gamma'' t$ is an I-initial path, $\gamma'' ts''$ is I-initial too. Furthermore, using Lemma 4.2.7, we conclude that $\lambda w' \sim \lambda'' ww' = l_I(\gamma'' ts'')$.

□

It is clear that if a reactive frame is static then it is also quasi-static. Furthermore it is interesting to notice that if a frame is static all the variants of transitivity and reflexivity coincide.

Open problem - symmetry

Our method does not seem to be so fruitful with the notions of generalised symmetry. We are unable to prove reactive frame completeness. To prove that it is sound in relation to a certain class of reactive frames and that it is complete to the correspondent class of shattered frames is straightforward as before. However, completeness cannot be proved in the same way. We just cannot simply pass from the shattered to cs-frames. Let us look just to the case of strong symmetry.

Question 4.2.14. *Let $L_X^B = L_X \oplus p \to \Box_R \Diamond_R p$. Is L_Π^B complete with respect to the class of reactive frames that are strongly symmetric, i.e. satisfying:*

$$\lambda ww' \in \Delta \to \lambda w \sim \lambda ww'w \in \Delta \text{ ?}$$

We leave the question and add a brief explanation on how our attempts failed to prove it.

We present a R-reflexive shattered frame that cannot be transformed into a cs-frame using the classic backward truth preserving transformations.

If a shattered frame (W, R, P) with $R \neq \varnothing$ satisfies for all w there is w' such that wPw', there is no w'' satisfying $w'Rw''$ or $w''Rw'$, then it does not admit an initial family. Let us assume the contrary, let I be its initial family and

$$A = \{w : \text{ there is no } w' \text{ such that } wRw' \text{ or } w'Rw\}.$$

Since I R-generates W, the isolated points must be in I, so $A \subseteq I$. From the fact that I picks only one element from each P-class and for every w there is $w' \in A$ such that wPw' we have that $I = A$. As there is an element in $W - A$ I does not R-generate all W. It is also clear that a frame that has such a frame as generated subframe cannot have an initial family. Or else the elements of the initial family of the bigger frame, present in the smaller one, would be an initial family to the latest. Furthermore, the operation of taking pre-images, ultrafilter extensions and ultrapowers preserve this (bad) property.

The following shattered frame $(\{0,1\}, \{(1,1)\}, \{0,1\}^2)$ is R-symmetric and satisfies the bad property. Hence classical ways of generating new models from old ones, preserving modal satisfaction do not allow us to find a general recipe to convert the relevant shattered frames into the correspondent cs-frames. In particular no variation of blow up will work.

This may not be due to the method's limitations, instead it may be that the answer to Question 4.2.14 is negative and that the missing axiom would restrict us to a class of shattered frames that do not have this property. We have not been able to prove either way.

In the presence of (strong) reflexivity this problem disappears and the blow up method works:

Proposition 4.2.15. *Let $L_X^{TB} = L_X^T \oplus p \to \Box_R \Diamond_R p = L_X^B \oplus p \to \Diamond_R p$. L_Π^{TB} is sound and complete with respect to the class of strongly symmetric reactive frames that are also strongly reflexive.*

Proof. Soundness is just as before obtained by checking that $p \to \Box_R \Diamond_R p$ is sound:

Given a strongly symmetric reactive model $\mathfrak{M} = (W, \Delta, v)$, $\lambda w \in \Delta$ such that $\mathfrak{M}, \lambda w \models \varphi$, and v such that $\lambda wv \in \Delta$ then $\lambda wv \sim \lambda wvw \in \Delta$. So, by Proposition 4.1.5, $\mathfrak{M}, \lambda wvw \models \varphi$, thus $\mathfrak{M}, \lambda wv \models \Diamond_R\varphi$. Hence and $\mathfrak{M}, \lambda w \models \Box_R \Diamond_R \varphi$.

In this case we are able to prove completeness by applying the blow up method:

Given $\mathfrak{M} = (W, R, P, V)$ a R-symmetric and R-reflexive shattered model, let us define $(\mathcal{B}_6 =)\mathfrak{M}' = (W', R', P', V)$ by:

- $W' = \bigcup_{n<\omega} W^{2n+1} \times \bigcup_{n<\omega} W^{2n+1}$,
- $(x,y)R'(x',y')$ iff $x = x'$ or
  $\big(y = y'$ & wRw' & $(x = z$ & $x' = z'$ or $x' = z$ & $x = z')$ &
  $z = (w, \overline{v})$ & $z' = (w', w', w, \overline{v})\big)$,
- $(x,y)P'(x',y')$ iff $y = y'$ & $x = (w, \overline{v})$ & $x' = (w', \overline{v})$ & wPw',
- $((x_1, \ldots, x_k), w) \in V'(p)$ iff $x_1 \in V(p)$,

  where $\overline{v} = v_1, \ldots, v_k$.

We have that (W', R', P') is a cs-frame (where R' is clearly symmetric and reflexive) and the $\mathfrak{M}'$ is a bounded morphic image of $\mathfrak{M}$:

- clearly V' is admissible and any choice of I picking one element from each P-class and containing $\{(x,y) \in W' : x = y\}$ works as an initial family.
- let us check that it is coherent. Given $(x,y), (x',y'), (x'',y'') \in W'$. If $(x',y')P'(x'',y'')$ then $x' = (v, \overline{v})$, $x'' = (v', \overline{v})$ (so $|x'| = |x''|$) with wPw' and $y' = y''$. Let us assume that we also have $(x,y)R'(x',y')$ and $(x,y)R'(x'',y'')$.
  If $x = x'$ and $x \neq x''$ then $|x'| = |x| \neq |x''|$ which contradicts $(x',y')P'(x'',y'')$, so $x = x''$. The same applies if $x = x''$ and $x \neq x'$.

If $x \neq x'$ and $x' \neq x''$, then either $x = (w, \overline{v})$ and $x' = (w', w', w, \overline{v}) = x''$, or $x = (w', w', w, \overline{v})$ and $x' = (w, \overline{v}) = x''$.
In any case: $(x', y') = (x'', y'')$.

- let $f : W' \to W$ be defined by $f(((w, \overline{x}), y)) = w$.

  It is straightforward to see that f is surjective, let us see that it is actually a p-morphism.

  The condition in the valuation is trivial as before;

  - f is a p-morphism in R
    * $((w, \overline{x}), y)R'((w', \overline{x}), y')$ then or $w = w'$ or wRw', in any case $f(((w', \overline{x}), y)) = wRw' = f(((w', \overline{x}), y'))$,
    * $f(((w, \overline{x}), v))Rw'$ then $(((w, \overline{x}), v))R'(((w', w', w, \overline{x}), v))$;
  - f is a p-morphism in P
    * $((w, \overline{x}), y)P'((w', \overline{x}), y')$ then $f(((w', \overline{x}), y)) = wPw' = f(((w', \overline{x}), y'))$,
    * $f(((w, \overline{x}), v))Pw'$ then $(((w, \overline{x}), v))P'(((w', w, \overline{x}), v))$.

    In Proposition 4.2.8 we have checked that given a R-reflexive cs-frame with initial family I then (I, Δ^I) is a strongly reflexive reactive frame. Let us see now that if the cs-frame is also R-symmetric then (I, Δ^I) is also strongly symmetric: Let $\lambda ww' \in \Delta^I$, so there is $\gamma \in W^*$ and $s, s' \in W$ such that $\gamma ss'$ is an I-initial path and $l_I(\gamma ss') = \lambda ww'$. Since $\mathfrak{F}$ is R-symmetric we have that $\gamma ss's$ is also an I-initial path and clearly $l_I(\gamma ss's) = \lambda ww'w \in \Delta$. Applying Lemma 4.2.7 we conclude that $\lambda w \sim \lambda ww'w$.

□

This result tells us that if the answer to Question 4.2.14 is negative, the formula valid in all strong symmetric reactive frames that is not in L_X^B must be a L_X-consequence of $p \to \Diamond_R p$.

4.3 Decidability

All the completeness results we presented are based in the equality between the logics obtained from certain classes of reactive frame semantics and some classes of X-shattered models, which are models over bimodal frames where one of the relations is an equivalence relation.

Next we show that in some of these cases the filtration method is successful in proving the strong finite model property. Since all of these logics are also finitely axiomatisable we conclude that they are also decidable.

Proposition 4.3.1. *Given a shattered model* $\mathfrak{M} = (W, R, P, V)$ *and* $\Gamma \subseteq \mathcal{L}_r$ *closed under taking subformulas then let:*

$$\Gamma_v = \{\varphi : \varphi \in \Gamma \ \& \ \mathfrak{M}, v \vDash \varphi\},$$
$$v \sim_\Gamma w \text{ iff } \Gamma_v = \Gamma_w,$$
$$|v| = \{v' : v \sim_\Gamma v'\},$$
$$W_\Gamma = \{|v| : w \in W\},$$
$$|v| R^\sigma |w| \text{ iff } \exists v', w' \ v' \in |v| \ \& \ w' \in |w| \ \& \ v' R w',$$
$$|v| R^\tau |w| \text{ iff for all } \Diamond_R \varphi \in \Gamma \text{ if } \mathfrak{M}, v \vDash \Diamond_R \varphi \text{ then } \mathfrak{M}, w \vDash \varphi \wedge \Diamond_R \varphi,$$
$$|v| P^{eq} |w| \text{ iff for all } \Diamond_P \varphi \in \Gamma, (\mathfrak{M}, v \vDash \Diamond_P \varphi \text{ iff } \mathfrak{M}, w \vDash \Diamond_P \varphi).$$

Notice that R^σ *and* R^τ *are the known smallest and transitive* Γ*-filtrations, see [38]. Then:*

1. *If* R *is reflexive then* R^σ *and* R^τ *are also reflexive.*
2. *If* R *is transitive then* R^τ *is also transitive.*
3. *If* R *is symmetric then* R^σ *is also symmetric.*
4. P^{eq} *is a* Γ*-filtration and an equivalence relation.*
5. *If* $\forall w \exists w' \ wRw'Pw$ *then* $\forall |w| \exists |w'| \ |w| R' |w'| P' |w|$.
6. *If* $\forall w \exists w' \ wPw'Rw$ *then* $\forall |w| \exists |w'| \ |w| P' |w'| R' |w|$.
7. *If* $\forall w \exists w' w'' \ wPw'Rw''Pw$ *then* $\forall |w| \exists |w'| |w''| \ |w| P' |w'| R' |w''| P' |w|$. *where* R' *and* P' *are* Γ*-filtrations of* R *and* P *respectively.*

Notice that if Γ *is finite then*

$$|W_\Gamma| \leq 2^{|\Gamma|} \times 2^{|\Gamma|}.$$

Proof.

1. All filtrations preserve reflexivity, see [38].

2. Transitive filtrations preserve transitivity, see [38].

3. $|v|R^{\sigma}|w|$ iff $\exists v', w'\ v' \in |v|\ \&\ w' \in |w|\ \&\ v'Rw'$ iff (by symmetry) $\exists v', w'\ v' \in |v|\ \&\ w' \in |w|\ \&\ w'Rv'$ iff $|w|R^{\sigma}|v|$.

4. P^{eq} is a Γ-filtration: vPw clearly implies that $|v|P^{eq}|w|$ and if $|v|P^{eq}|w|$ then from $\mathfrak{M}, v \vDash \Diamond_P\varphi$ we get that $\mathfrak{M}, w \vDash \Diamond_P\varphi$, thus from P-reflexivity we obtain $\mathfrak{M}, v \vDash \varphi$.

   So P^{eq} is clearly an equivalence relation.

5. Given $|w|$, there exists w' such that $wRw'Pw$ so $|w|R'|w'|P'|w|$. The last two claims are proved similarly.

□

Corollary 4.3.2. *L_X, L_Π^T, $L_X^{T_o}$, $L_\Pi^{T_i}$, $L_X^{T_l}$, L_Π^4, L_X^{TB} have the exponential finite model property and thus its decidability problem is co-NEXPTIME.*

Proposition 4.3.3. *Given a shattered model $\mathfrak{M} = (W, R, P, V)$ and $\Gamma \subseteq \mathcal{L}_r$ closed under taking subformulas then let:*

$$\begin{aligned}
\Gamma_v &= \{\varphi : \varphi \in \Gamma\ \&\ \mathfrak{M}, v \vDash \varphi\},\\
v \sim_\Gamma w \textit{ iff } \Gamma_v &= \Gamma_w\ \&\ \{\Gamma_z : vPz\} = \{\Gamma_z : wPz\},\\
|v| &= \{v' : v \sim_\Gamma v'\},\\
W_\Gamma &= \{|v| : w \in W\},\\
|v|R^{\sigma}|w| &\textit{ iff } \exists v', w'\ v' \in |v|\ \&\ w' \in |w|\ \&\ v'Rw',\\
|v|P^{\sigma}|w| &\textit{ iff } \exists v', w'\ v' \in |v|\ \&\ w' \in |w|\ \&\ v'Pw'.
\end{aligned}$$

Notice that R^{σ} and R^{σ} are the known smallest and transitive, see [38]. Then:

1. *P^{σ} is a Γ-filtration and an equivalence relation.*

2. *If $\forall xyz \exists x'\ xRyPz \to xPx'Ry$ then $\forall |x||y||z| \exists |x'|\ |x|R|y|P|z| \to |x|P|x'|R|y$.*

3. *If* $\forall xyz \exists x'\ xPyRz \to xRx'Py$ *then* $\forall |x||y||z| \exists |x'|\ |x|R|y|R|z| \to |x|R|x'|P|y$.

Remark 4.3.4. This kind of modification on the filtration method was introduced in [26] and was used in [25] to filtrate product frames where one of the components is an equivalence relation. Indeed the logic $L^S_\varnothing = [\mathbf{K}, \mathbf{S5}]$. It is interesting that imposing the commutativity on these operators coincides with not having reactivity.

Notice that if Γ *is finite then*

$$|W_\Gamma| \leq 2^{|\Gamma|} \times 2^{2^{|\Gamma|}}.$$

Proof. From the definition of P^σ it is clear that its reflexivity and symmetry follow from P's reflexivity and symmetry. All the other claims follow from the fact that the equivalence relations $\sim_\Gamma$ and P commute.

Check in [23, Theorem .5.15] for the proof that

$$\exists z\ x \sim_\Gamma zPy \qquad \text{iff} \qquad \exists u\ xPu \sim_\Gamma y. \tag{4.1}$$

and that it implies that P^σ is also transitive (so (1)) and (2), in there the notation is different $R = R_h$, $P = R_v$, $R^\bullet = R^\sim_h$ and $P^\sigma = R^\sim_v$.

We will proof that it also implies (3). Since it is not present there:

If $|x|R|y|R|z|$ then there are x', y', y'', z' such that

$$x \sim_\Gamma x'Py \sim_\Gamma y' \sim_\Gamma y \sim_\Gamma y''Rz' \sim_\Gamma z.$$

Using transitivity and 4.1 we get that there exists u such that

$$x \sim_\Gamma x'PyRz' \sim_\Gamma u \sim_\Gamma z$$

and using our hypothesis we obtain

$$x \sim_\Gamma x'RwPz' \sim_\Gamma u \sim_\Gamma z$$

and thus $|x|P|w|R|z|$. □

Corollary 4.3.5. L^S_X, L^{qS}_Π *have the* 2*-exponential finite model property and thus its decidability problem is in co-N2EXPTIME.*

4.4 Discussion

Results Table 4.1 summarises this section main results. One can read along its lines the correspondence between logic, reactive frame property (including its dependence on X) and the shattered frame unfolded property.

For example, in the first line we see that the logic L_X corresponds to all shattered frames and to the whole class of (X-models over) reactive frames, hence being the smallest (minimal) reactive logic. In general there is a dependence on X. For instance in the sixth line we see that the completeness of $L_X^T = L \oplus p \to \Diamond p$ with respect to the class of (all X-models over) strong reflexive reactive frames requires that $X = \Pi$, i.e. that all variables have to be fixed.

Question 4.4.1. *Can this dependence be avoided by a more inspired choice of axiom as in the case of static reactive frames (Proposition 4.2.12) ?*

Filtration: failed cases We have failed to find working filtration methods for $L_X^{4_l}$, $L_X^{4_m}$, $L_\Pi^{4_r}$, $L_X^{4_{gl}}$, $L_X^{4_{gm}}$, $L_\Pi^{4_{gr}}$, so we ask:

Question 4.4.2. *Does some variation of the filtration method work in these cases as well? Do these logics have the f.m.p.?*

Accumulation of properties. It is not clear whether the combination of the axioms corresponding to certain reactive properties, corresponds to the combination of those properties. Even though it is so in the classical (unimodal) case when we add reflexivity to transitivity or symmetry. It may be that the reactive properties interact and those classes satisfy formulas not captured by the axioms as it happens in the case of product logics, see Chapter 3. It certainly does not follow from the results we presented above. For that to happen the (blow up) transformation would have to preserve the combination of all properties in question. The fact is that we have been unable to find a transformation that would work for all cases, in particular when considering generalisations of different properties.

Question 4.4.3. *Is there such transformation?*

By contrast, in the strong symmetry plus strong reflexivity case we see how gathering properties may also be helpful. The strong symmetry case alone is worse behaved than the combined one.

Stronger tools required. We have used classical tools to study properties of the logics coming from this new semantics. We have been only partially successful (see for example Section 4.2). Other techniques should be developed for completeness and decidability, as in the case of product logics a suitable notion of quasimodel seems essential (see [23]). Quasimodels retain the needed information by finitising some model features.

Question 4.4.4. *What is the appropriate notion of a reactive quasimodel?*

It would also be interesting to find some new methods that would allow a more direct way of studying this and other reactive semantics. Ones that capture its dynamic flavour, leading to a reactivisation of modal techniques.

logic	X	reactive property		unfolded property	where?	f.m.p.
$L_X = \mathbf{K}_R \oplus \mathbf{S5}_P +_X p \leftrightarrow \Box_P p$	any		minimal		4.2.4	**Yes**
$L^S_\Pi = L_X \oplus \Diamond_P p \to p$	Π					
$L^S_X = L_X \oplus \Diamond_P \Diamond_R p \to \Diamond_R \Diamond_P p$	any	$\lambda w, \lambda w' \in \Delta \to \lambda w \sim \lambda' w$	static	$\forall xyz \exists y'\ xPyRz \to xRy'Pz$	4.2.12	**Yes**
$L^S_X = L_X \oplus \Diamond_R \Box_P p \to \Box_P \Diamond_R p$	any					
$L^{qS}_X = L_X \oplus \Diamond_R \Diamond_P p \to \Diamond_P \Diamond_R p$	Π	$\lambda ww', \lambda' w' \in \Delta \to \exists \lambda''\ \lambda' w' \sim \lambda'' ww' \in \Delta$	q-static	$\forall xyz \exists x'\ xRyPz \to xPx'Ry$	4.2.13	**Yes**
$L^T_X = L \oplus p \to \Diamond p$	Π	$\lambda w \in \Delta \to \lambda w \sim \lambda ww \in \Delta$	gener-	R-reflexive	4.2.8	**Yes**
$L^{T_o}_X = L \oplus p \to \Diamond_R \Diamond_P p$	any	$\lambda w \in \Delta \to \lambda ww \in \Delta$	alising	$\forall w \exists w'\ wRw'Pw$	4.2.101	**Yes**
$L^{T_i}_X = L \oplus p \to \Diamond_P \Diamond_R p$	Π	$\lambda w \in \Delta \to \exists \lambda'\ \lambda w \sim \lambda' ww \in \Delta$	reflxe-	$\forall w \exists w'\ wPw'Rw$	4.2.102	**Yes**
$L^{T_l}_X = L \oplus \Box_P p \to \Diamond_P \Diamond_R p$	any	$w \in W \to \exists \lambda\ \lambda ww$	tivity	$\forall w \exists w' w''\ wPw'Rw''Pw$	4.2.103	**Yes**
$L^4_X = L \oplus \Diamond_R \Diamond_R p \to \Diamond_R p$	Π	$\lambda ww'w'' \in \Delta \to \lambda ww'w'' \sim \lambda ww'' \in \Delta$		R-transitive	4.2.9	**Yes**
$L^{4_l}_X = L \oplus \Diamond_R \Diamond_R p \to \Diamond_R \Diamond_P p$	any	$\lambda ww'w'' \in \Delta \to \lambda ww'' \in \Delta$	general-	$\forall tvw \exists w'\ tRvRw \to tRw'Pw$	4.2.11.1	**?**
$L^{4_m}_X = L \oplus \Diamond_R \Diamond_R p \to \Diamond_P \Diamond_R \Diamond_P p$	any	$\lambda ww'w'' \in \Delta \to \exists \lambda'\ \lambda' ww'' \in \Delta$	ising	$\forall tvw \exists t' w'\ tRvRw \to tPt'Rw'Pw$	4.2.11.2	**?**
$L^{4_r}_X = L \oplus \Diamond_R \Diamond_R p \to \Diamond_P \Diamond_R p$	Π	$\lambda ww'w'' \in \Delta \to \exists \lambda'\ \lambda ww'w'' \sim \lambda' ww'' \in \Delta$		$\forall tvw \exists t'\ tRvRw \to tPt'Rw$	4.2.11.3	**?**
$L^{4_{gl}}_X = L \oplus \Diamond_R \Diamond_P \Diamond_R p \to \Diamond_R \Diamond_P p$	any	$\lambda ww', \lambda' w'w'' \in \Delta \to \lambda ww'' \in \Delta$	transi-	$\forall tvv'w \exists w'\ tRvPv'Rw \to tRw'Pw$	4.2.11.5	**?**
$L^{4_{gm}}_X = L \oplus \Diamond_R \Diamond_P \Diamond_R p \to \Diamond_P \Diamond_R \Diamond_P p$	any	$\lambda ww', \lambda' w'w'' \in \Delta \to \exists \lambda''\ \lambda'' ww'' \in \Delta$	tivity	$\forall tvv'w \exists t'w'\ tRvPv'Rw \to tPt'Rw'Pw$	4.2.11.4	**?**
$L^{4_{gr}}_X = L \oplus \Diamond_R \Diamond_P \Diamond_R p \to \Diamond_P \Diamond_R p$	Π	$\lambda ww', \lambda' w'w'' \in \Delta \to \exists \lambda''\ \lambda' w'w'' \sim \lambda'' ww'' \in \Delta$		$\forall tvv'w \exists t'\ tRvPv'Rw \to tPt'Rw$	4.2.11.6	**?**
$L^{TB}_X = L^T_X \oplus p \to \Box_R \Diamond_R p = L^B_X \oplus p \to \Diamond_R p$	Π	$\lambda ww' \in \Delta \to \lambda w \sim \lambda ww \in \Delta\ \&\ \lambda w \sim \lambda ww'w \in \Delta$	ref.+trans.	R-reflexive and R-symmetric	4.2.15	**Yes**

Table 4.1: The table of results.

5

Switch Graphs: the global view

A graph is the abstract representation of a binary relation between objects. There are many graph-based structures, and often, to read (some of) the information represented in them, one needs to travel across their vertices following their edges, e.g. to check if two points are connected or to interpret modal formulas in Kripke model. In the usual notion of graph the vertices that are accessible from a vertex are fixed.

When we modify the notion of graph allowing that the accessible vertices depend on the sequence of edges we have crossed we obtain a reactive graph. So, to each sequence of crossed edges corresponds a (relational) state of the graph, where the edges that are available are the ones that can prolong the current sequence. The information in such a graph boils down to the sequences of edges (or actions) over a certain set.

A direct way of representing such a graph would be to draw the tree given by the admissible sequences of edges, and, perhaps, to draw at each of its nodes the state of the graph at that point. Although this idea of having different points representing the relational state of the same point is useful (in Chapter 4 we used it to obtain a relation with classical modal logic), it does not offer an easy reading of the changes that crossing an edge implies. As stated in the introduction (Section 1.1), the concept of reactivity was initially introduced using an enrichment of graphs with new kinds of arrows representing the local effects of traversing an edge in the global relation. The representation offered by these structures seems to be much more interesting as it truly grasps the reactivity flavour. Furthermore, in Section 5.1 we show that these new multi-level-arrowed structures are enough to code all the relational dynamics.

The simplest effect crossing an edge can have over the accessibility relation is to turn on or off a connection. These elementary changes can be represented by drawing an arrow from the crossed edge to the edge representing the connection being altered. Crossing an edge does not have necessarily always the same effects. To represent these changes we use arrows from the edges to the arrows representing the elementary effects, and so on, obtaining infinite levels of arrows.

In this context we refer to the arrows as switches. The edges connecting points are 0-level switches and the switches connecting 0-level and n-level switches are $n+1$-level switches. The switches of level greater than 0 can be of two kinds, the connecting and disconnecting ones. A set equipped with a set of switches is called switch graph.

The state of the relation after a certain sequence of actions in a switch graph takes into account its levelled structure. The switches of level bigger than 0 do not represent the accessibility relation between points but how this relation changes after each action, corresponding to the 0-level arrows. When we cross an edge we turn on/off the switches that are the targets of the connecting/disconnecting switches coming out of it.

In Figure 5.1 we can see the switch graph representing the reactive graph with set of points $\{a, b\}$, and admitting as set of actions the set given by the following regular expression (being $\{a, b\}^2$ the alphabet):

$$((a, a) + (a, b))^*(b, b)((a, b) + (b, b))^*(a, a)((a, a) + (b, b))^*.$$

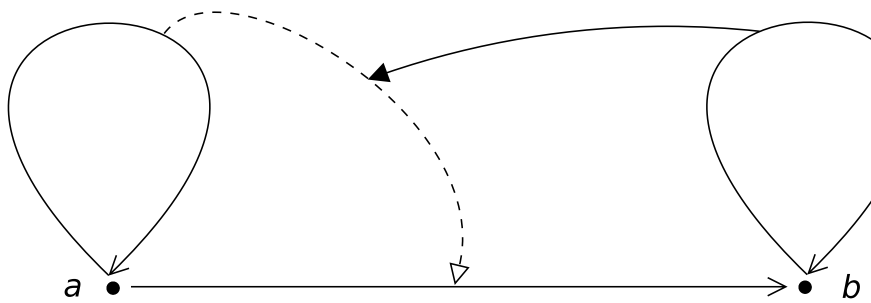

Figure 5.1: The white pointed arrows represent the connecting switches, the black arrows the disconnecting ones and the dashed line the fact that the switch is initially disconnected.

We see that once (b, b) is crossed, the disconnecting switch coming from (a, a) to (a, b) becomes connected. Thus, after crossing (b, b) and (a, a), (a, b) can no longer be crossed.

Many other kinds of switches, having different effects, could be considered. These seem to be the most primitive ones, since they incorporate the basic actions in graph reactivity, connecting and disconnecting edges. We shall see that this is enough to represent all reactive graphs. That is: any reactive behaviour can be decomposed into this kind of local actions.

We conclude with some examples and suggestions for future investigation.

5.1 Reactive by Switch

Definition 5.1.1. A *reactive graph* is a pair (W, Δ), where

- W is a non-empty set, the set of *worlds*, and
- Δ, the *behaviour*, is the set of admissible sequences of edges, i.e., a subset of $(W \times W)^*$ closed under prefixes containing ϵ (the empty sequence is always admissible).

The state of the accessible relation after an admissible sequence of edges λ being covered is given by

$$R_\lambda = \{(w, w') : \lambda(w, w') \in \Delta\}.$$

We now introduce the enriched notion of graph, the switch graphs, and formalise in which sense they can be used to represent (or generate) reactive graphs highlighting the effects that moving around the base graph has on the accessibility relation.

Definition 5.1.2. For a non-empty set W and $n < \omega$, the set $\mathcal{A}_n(W)$ of *switches* over W of level n is defined as:

- $\mathcal{A}_0(W) = W \times W$ (of level 0),
- $\mathcal{A}_{n+1}(W) = (W \times W) \times \mathcal{A}_n(W) \times \{\bullet, \circ\}$ (of level n).

A switch with $\bullet$ ($\circ$) as its third component is called a *connecting (disconnecting) switch*.

The set of all switches on W is defined by taking $\mathcal{A}(W) = \bigcup_{n<\omega} \mathcal{A}_n(W)$.

Definition 5.1.3. A *switch graph* is a pair (W, R), where

- W is a non-empty set, the set of *worlds*, and
- $R \subseteq \mathcal{A}(W)$ is the set of *switches*.

We say that (W, R) has *reactivity of level n* if $R \subseteq \bigcup_{i \leq n} \mathcal{A}_i(W)$.

Remark 5.1.4. In the graphical representation we use normal arrows for the 0-level switches, a black (white) pointed arrow for the connecting (disconnecting) switches and switches that are off are drawn with a dashed line.

Here we introduce some notation to refer to switches in an easier fashion:

Definition 5.1.5. $(v_1 v_2, \ldots, v_{2n-1} v_{2n}, a, \sigma)$ for $\sigma = s_1 \ldots s_n \in \{\bullet, \circ\}^n$, $n < \omega$ and $a \in \mathcal{A}(W)$ is defined as:

- $(a, \epsilon) = a$,
- $(v_1 v_2, \ldots, v_{2n+1} v_{2n+2}, a, s_1 \ldots s_{n+1}) = ((v_1, v_2), (v_3 v_4, \ldots, v_{2n+1} v_{2n+2}, a, s_1 \ldots s_n), s_{n+1})$.

We say that $(v_1 v_2, \ldots, v_{2n-1} v_{2n}, \sigma)$ is a switch of type σ.
For example,

$$(w_1 w_2, w_3 w_4, w_5, w_6, \circ \bullet \bullet)_p = \Big((w_1, w_2), \big((w_2, w_3), ((w_3, w_4), (w_4, w_5), \circ), \bullet\big), \bullet\Big)$$

is of type $\circ \bullet \bullet$.

Definition 5.1.6. Given a switch graph $S = (W, R)$, the *behaviour of S* is the smallest set, Δ_S, such that:

- $\epsilon \in \Delta_S$,
- If $\alpha \in \Delta_S^n$ then $\alpha(w, w') \in \Delta_S$ for all $(w, w') \in R_\alpha$.

  Where $R_\alpha \subseteq \mathcal{A}(W)$ is the *switch state* of our switch graph after crossing the sequence of edges α:

  - $R_\epsilon = R$, the initial state,
  - $R_{\alpha(w,w')} = (R_\alpha - \{a : ((w, w'), a, \circ) \in R_\alpha\}) \cup \{a : ((w, w'), a, \bullet) \in R_\alpha\}$.

(W, Δ_S) is the reactive graph generated by S and

$$\Delta_S = \{R_\alpha : \alpha \in \Delta_S\}$$

of *switch states* of S.

Notice that the reactive level of R_α is the same for all α. Neither the highest order arrows can ever be turned off, nor higher order ones can be introduced, since a switch that acts over a switch of order n has order $n+1$.

Remark 5.1.7. When we informally introduced the switches dynamics we did not specify what should happen when a connecting and a disconnecting switch act simultaneously over the same switch. In the previous definition the convention is that the connecting action prevails. We could have opted instead for:

- the disconnecting switch would prevail

  $R_{\alpha(w,w')} = (R_\alpha \cup \{a : ((w,w'),a,\bullet) \in R_\alpha\}) - \{a : ((w,w'),a,\circ) \in R_\alpha\}$,

- or that it would depend on the state of the target, by, for example, always changing its state

  $R_{\alpha(w,w')} = (R_\alpha - \{a : ((w,w'),a,\circ) \in R_\alpha\}) \cup \{a : ((w,w'),a,\bullet) \in R_\alpha \ \&\ a \notin R_\alpha\}$.

We are interested in studying how expressive these structures are regarding the generation of reactive graphs. It is easy to see that given a reactive graph $\mathfrak{F} = (W,\Delta)$ the switch graph $\mathfrak{G} = (W, R^\circ \cup R^\bullet)$ where

$$\begin{aligned} R^\bullet &= \{(w_1w_2,\ldots,w_{2|\sigma|+1}w_{2|\sigma|+2},\sigma) : \sigma \in \{\bullet\}^*,\ w_1\ldots w_{2|\sigma|+2} \in \Delta\}, \\ R^\circ &= \{(vv',w_1w_2,\ldots,w_{2|\sigma|+1}w_{2|\sigma|+2},\sigma\circ) : \sigma \in \{\bullet\}^*,\ v,v',w_i \in W\}, \end{aligned}$$

generates $\mathfrak{F}$ *if the connecting switches prevail*. For the other options there does not seem to be such a direct way of coding the reactive behaviour. We can then ask whether these options are relevant for this goal. Next theorem shows that if one allows unbounded levels, these options do not limit the switch graphs expressivity. Furthermore, it presents a general construction such that given any reactive graph we obtain a switch graph that, regardless of the chosen option, generates the given reactive graph.

Theorem 5.1.8. *Any reactive graph can be generated by a switch graph, furthermore this switch graph can be chosen such that no connecting and disconnecting switches ever act simultaneously over the same switch.*

Proof. Given a reactive graph (W,Δ), we define a relation $C \subseteq \{\bullet,\circ\}^* \times (W\times W)^*$ by taking

$$C(\sigma,\alpha) \quad \text{iff} \quad \begin{array}{l} \text{either the number of } \circ\text{s in } \sigma \text{ is even and } \alpha \in \Delta, \\ \text{or the number of } \circ\text{s in } \sigma \text{ is odd and } \alpha \notin \Delta. \end{array}$$

This definition clearly implies the following:

Lemma 5.1.9. $C(\sigma\bullet,\alpha) \leftrightarrow \neg C(\sigma\circ,\alpha)$.

Now let

$$R = \{(w_1w_2,\ldots,w_{2|\sigma|+1}w_{2|\sigma|+2},\sigma) : C(\sigma,(w_1,w_2)\ldots(w_{2|\sigma|+1},w_{2|\sigma|+2})),\ \sigma \in \{\bullet,\circ\}^*,\ w_i \in W \text{ and } w_1\ldots w_{2|\sigma|} \in \Delta\}.$$

We claim that, for every $\alpha \in \Delta_S$, the following hold:

Lemma 5.1.10. *For every* $(\beta,\sigma) \in \mathcal{A}(W)$, $(\beta,\sigma) \in R_\alpha \iff C(\sigma,\alpha\beta)$.

We prove the lemma by induction on α. For $\alpha = \epsilon$ this is just the definition of R. Now suppose that $\alpha(w,w') \in \Delta_S$ and Lemma 5.1.10 holds for α. Then, by Lemma 5.1.9, we have that

$$((w,w')\beta,\sigma\bullet) \in R_\alpha \iff ((w,w')\beta,\sigma\circ) \notin R_\alpha. \tag{5.1}$$

Now by (5.1) we have:

$$\text{for every } (\beta,\sigma) \in \mathcal{A}(W),\quad (\beta,\sigma) \in R_{\alpha(w,w')} \iff ((w,w')\beta,\sigma\bullet) \in R_\alpha. \tag{5.2}$$

Now we can show Lemma 5.1.10 for $\alpha(w,w')$:

$$(\beta,\sigma) \in R_{\alpha(w,w')}$$

iff (by (5.2))

$$((w,w')\beta,\sigma\bullet) \in R_\alpha$$

iff (by the IH)

$$C(\sigma\bullet,\alpha(w,w')\beta)$$

iff

$$C(\sigma,\alpha(w,w')\beta).$$

Now we can complete the proof of the theorem as follows. First, $\epsilon \in \Delta \cap \Delta_S$. Otherwise, $\alpha(w,w') \in \Delta_S$ iff $\alpha \in \Delta_S$ and $(w,w') \in R_\alpha$ iff (by Lemma 5.1.10) $\alpha \in \Delta_S$ and $C(\epsilon,\alpha(w,w'))$ iff $\alpha \in \Delta_S$ and $\alpha(w,w') \in \Delta$ iff $\alpha(w,w') \in \Delta$. □

Remark 5.1.11. *Notice that the condition that* $w_1\ldots w_{2|\sigma|} \in \Delta$ *in the definition of* R *is not necessary but avoids the inclusion of switches that will have no part in the dynamics. Indeed it is easy to see that in what respects to the behaviour of a switch graph only such switches matter.*

5.2 Modelling multiple agents or processes

Reactive/Switch graphs can model situations where the accessibility relations change when an edge is crossed. Without any limitation on the number of individuals going through its edges. In fact a reactive graph may be used to design a particular interaction between many agents going around in a graph.

In a k-agents (processes, individuals, etc.) setting, the switch configuration ceases to be the only relevant information, we need to keep track of each agent's position. The k-behaviour of a switch graph with set of initial configurations $C \subseteq W^k$ is the set of allowed sequences of moves when the k agents are located they start in positions in C elements are allowed to do. Let us formalise this notion.

Definition 5.2.1. *Given a reactive graph $\mathcal{R} = (W, \Delta)$ and a number k of agents wandering about in the graph we define:*

- $(W^k)^\%$ *is the subset of* $(W^k)^+$ *formed by the sequences where each element differs from its successor of only one component, that is:*
  - $W^k \in (W^k)^\%$,
  - $\alpha(w_1, \ldots, w_i, \ldots, w_k) \in (W^k)^\%$ *then* $\alpha(w_1, \ldots, w_i, \ldots, w_k)[i \to w] \in (W^k)^\%$ *for any* $1 \leq i \leq n$*, where*

$$\alpha(w_1, \ldots, w_i, \ldots, w_k)[i \to w] = \alpha(w_1, \ldots, w_i, \ldots, w_k)(w_1, \ldots, w, \ldots, w_k).$$

- $\mathcal{E} : (W^k)^\% \to (W \times W)^+$ *is defined as:*
  - $\alpha \in W^k$ *then* $\mathcal{E}(\alpha) = \epsilon$,
  - $\alpha = \alpha'(w_1, \ldots, w_i, \ldots, w_k)[i \to w] \in (W^k)^\%$ *then*

$$\mathcal{E}(\alpha) = \mathcal{E}(\alpha(w_1, \ldots, w_i, \ldots, w_k))(w_i, w).$$

Let $C \subseteq W^k$ be a set of initial allowed configurations for the k-agents, C generated k-behaviour is:

$$\Delta_C^k = \{\alpha \in (W^k)^\% : \mathcal{E}(\alpha) \in \Delta^k \text{ and exists } \gamma \in C \text{ that is a prefix of } \alpha\}.$$

Switch graphs can be useful in modelling programs or protocols, capturing the intended interaction between the entities involved. The design of the constraints can be directly imposed by strategically locating the appropriate switches. Ideally, this double perspective on the dynamics, would allow that the verification of such properties over a switch graph S, departing from a given configuration γ, could either be extracted from the shape of S switches or exhaustively checked over Δ_C^k.

Example 5.2.2. Let us consider the mutual exclusion problem, taken from [50]. There we can find a model-based approach to the verification of the required properties in a given system, presented as the solution to this problem. And, the idea is to code the intended properties in CTL and verify if the transition system associated to the solution satisfies them. Here we lay the basis for a different approach.

As we referred in the introduction, in [16] by adding higher order arrows to the structure of an automata, and thus allowing its transition table to change while it is reading a sentence, the authors achieved an exponential reduction in the minimal number of states needed to accept a language. Therefore, this way of representing systems may have impact in model checking, where the state explosion problem is a serious drawback. The switches configuration at each point determines all the future dynamics, coding the interdependence of actions. Of course that if one obtains less states when considering the switch graph corresponding to a certain protocol instead of considering its associated transition system, is not because some information was thrown away. The fact is that this information is coded in a different form, hopefully in a more intuitive and accessible way. It is the extra expressivity, given by the higher order arrows, that allows us to identify different states with the same point by associating them to different switches configuration. Moreover, if one finds an appropriate way to extract the corresponding switch graph of a program or protocol, one may expect that the verification of some properties can be reduced (by means of some intermediary reasoning) to a simpler verification over the switches. Although we do not develop the verification part, in the next section we introduce a language to reason about switches and their effects. We believe that this language can be extended to a point where we can reason about both the switch view and the usual transition system view, and derive enough knowledge about their interaction in such a way that in order to verify some properties at one level it is enough to guarantee some related properties in the other and vice-versa. Nevertheless, here, we concentrate on showing how switches can be integrated in transition systems to improve their modelling expressivity. We understand that in this case the switches are associated to conditional commands and changes on the auxiliar variables used in these conditions. However, no systematic knowledge about that connection was obtained yet. We will limit ourselves to present examples of switch graphs ‘associated’ (in an informal and intuitive way) to protocols that constitute possible solutions to the problem and discuss them.

The mutual exclusion problem as presented in Chapter 3 of [50] is:

> *When concurrent processes share a resource (such as a file on a disk or a database entry), it may be necessary to ensure that they do not have access to it at the same time. Several processes simultaneously editing the same file would not be desirable. We therefore identify certain critical sections of each process’s code and arrange that only one process can be in its critical section at a time. The critical section should include all the access to the shared resource (though it should be as small as possible so that no unnecessary exclusion takes place). The problem we are faced with is to find a protocol for determining which process is allowed to enter its critical section at which time. Once we have found one which we think it works, we verify our solution by checking that it has some expected properties, such as the following ones:*
>
> ***Safety**: The protocol allows only one process to be in its critical section at any time.*
>
> *This safety property is not enough, since a protocol which permanently excludes every process from its critical section would be safe, but not very useful. Therefore, we should also require:*
>
> ***Liveness**: Whenever any process wants to enter its critical section, it will eventually be permitted to do so.*
>
> ***Non-blocking:** A process can always request to enter its critical section.*
>
> *Some rather crude protocols might work on the basis that they cycle through the processes, making each one in turn enter its critical section. Since it might be naturally the case that some of them request accesses to the shared resource more than others, we should make sure our protocol has the property:*
>
> ***No strict sequencing:** Processes need not enter their critical section in strict sequence.*

We model N processes, each of which is in its non-critical state (n), or trying to enter its critical state (t), or in its critical state (c). Each individual process undergoes transitions in the cycle $n \rightarrow t \rightarrow c \rightarrow n \rightarrow ...$, see in Figure 5.2[1] the case $N = 2$, but the two processes interleave with each other.

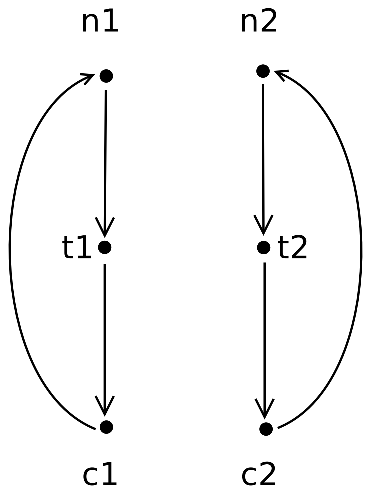

Figure 5.2: The mutual exclusion problem: combining processes sharing resources, two processes scenario.

One immediate solution (for $N = 2$) one can think of, is to use switches to impose directly the condition that one process gets to its critical area the other cannot – using $(t_1c_1, t_2c_2, \circ)$ and $(t_2c_2, t_1c_1, \circ)$ – and to remove this restriction when it returns to the non-critical state – using $(c_1n_1, t_2c_2, \bullet)$ and $(c_2n_2, t_1c_1, \circ)$, see Figure 5.3. The two processes start off in their non-critical states as indicated by the incoming edges with no source. Either of them may now move to its trying state, but only one of them can ever make a transition at a time (asynchronous interleaving). The problem here is that nothing prevents one of the processes from getting stuck at the trying state while the other one accesses continuously to the critical area, so liveness fails.

A way to guarantee liveness is, when a process requires permission to move to the critical state (moves to state t), to allow the other process to require access the critical area only

[1]In the graphical representation one finds numbers juxtaposed to the letters, this is due a limitation of the application used, thus they should be seen has being underscripts. Outside the figures we will use the proper form to avoid that with their intense use the text becomes unreadable. E.g. $t1$ in the figure corresponds to t_1 in the text.

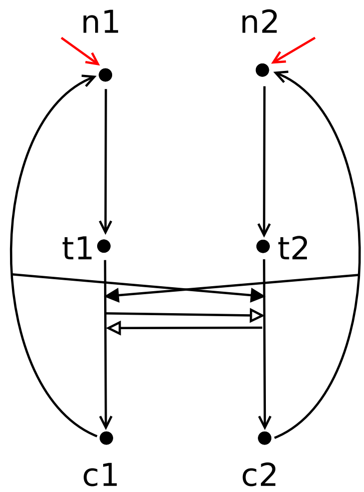

Figure 5.3: $N = 2$: liveness fails.

once until the first process accesses to it. In Figure 5.4 we can see that the extra switches do exactly this. The situation is symmetric so lets check liveness for the first process, that is, if it moves to t_1, it will eventually be able to move to c_1. If process 1 moves to t_1, it turns on $(c_2n_2, n_2t_2, \circ)$ (by the action of $(n_1t_1, c_2n_2, n_2t_2, \circ\bullet)$), which guarantees that if process 2 does the whole cycle once, then (since $(c_2n_2, n_2t_2, \circ)$ is on) it can only try again if process 2 passes by c_2, $(t_1c_1, n_2t_2, \bullet)$ and $(t_1c_1, c_2n_2, n_2t_2, \circ\circ)$ remove that restriction. But this is done by restricting process's 2 ability to *require* access to its critical section, thus failing the non-blocking constraint.

This is easily solved in Figure 5.5, where the limitation is forced only at the last stage. One can see that the two switch graphs are really similar, the difference is that the disconnected edge is (t_2, c_2) instead of (n_2, t_2). That is, instead of using the following set of switches

$$\{(n_it_i, c_jn_j, n_jt_j, \circ\bullet), (t_ic_i, n_jt_j, \bullet), (t_ic_i, c_jn_j, n_jt_j, \circ\circ) : i, j \in \{1,2\}, i \neq j\},$$

we use

$$\{(n_it_i, c_jn_j, t_jc_j, \circ\bullet), (t_ic_i, t_jc_j, \bullet), (t_ic_i, c_jn_j, t_jc_j, \circ\circ) : i, j \in \{1,2\}, i \neq j\}.$$

So when one process requires to access to the critical state, the other may access only once to it, though being still able require access to it, until the first's access is granted (other levels could be as easily set). Thus this solution complies with all the problem requirements.

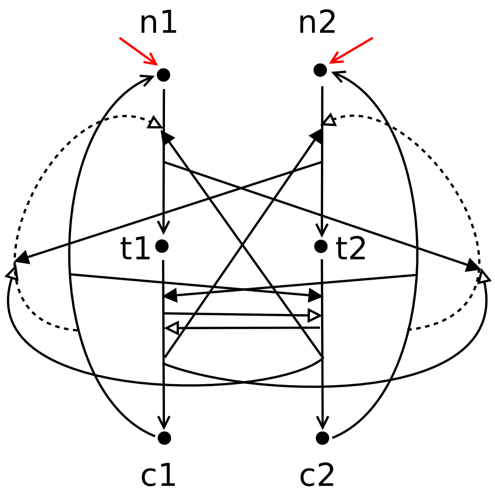

Figure 5.4: $N = 2$: non-blocking fails.

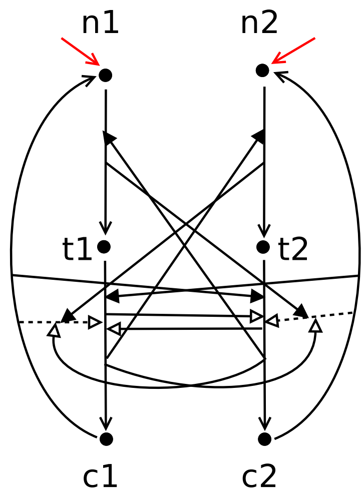

Figure 5.5: $N = 2$: a complete solution.

Hence, the advantage of the switch graph representation is that we can force/check the properties at the meta-level. We directly observe the effect that each transition has on the global accessibility relation. This strongly suggests how these solutions could be implemented by programming. Though the other direction (from programs to switch graphs) is not approached here, it seems a fundamental step in evaluating these structures potential.

Till now we tried solutions where each process runs around on different connected components of the presented switch graph, but there is no reason to be so. If we allow more process to run around the solutions in 5.4 and 5.5, then both liveliness and non-blocking properties would clearly be lost. Let us look to some solutions with only one connected component where all process run, thus saving in the number of required points in the graph.

In Figure 5.6 we have a general straightforward solution for the N processes case, but again liveness is not guaranteed. Again nothing forbids a process (or a group of $N' < N$ processes) to keep accessing the critical area making it impossible for some processes to do it. There does not seem to be a way of avoiding this with this base graph (without

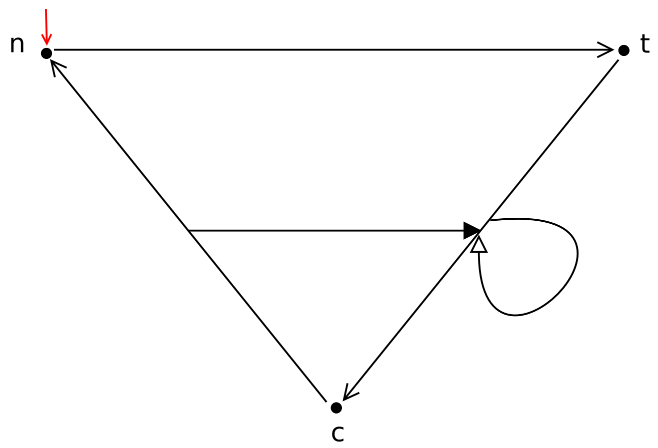

Figure 5.6: N arbitrary: safety and no strict sequence only.

losing the Non-blocking property), at least if we do not use multiple connection between each point, which are not allowed in our definition of switch graphs. Of course it would be easy to consider labelled switch graphs where this was allowed but it would be closer to considering different connected components.

Instead, in order to guarantee the other properties we will consider a slightly more complex base graph. We use a number of 'trying' states equal to the number of processes. We learned the kind of restrictions needed from Figure 5.5 and we can see in Figure 5.7 a switch graph using the same idea for avoiding the loss of liveness. Since here both processes share the same graph point for non-critical and critical sections the way of imposing it changes slightly. First we need to guarantee we do not have more than one process in the same point representing 'trying to get to critical section', by having $(nt_i, nt_i, \circ)$ and $(t_i c, nt_i, \bullet)$, for $i = 1, 2$, controlling that. Initially we have both $(cn, t_1 c, \bullet)$ and $(cn, t_2 c, \bullet)$ on. One way to avoid the loss of liveness is to have that when a process moves to t_i then $(cn, t_j c, \bullet)$ (for $j = 3 - i$) becomes off, that is we have the switch $(nt_i, cn, t_j c, \bullet\circ)$. Clearly when that process moves out of t_i the restriction is not needed anymore, and so we have $(t_i c, cn, t_j c, \bullet\bullet)$. This

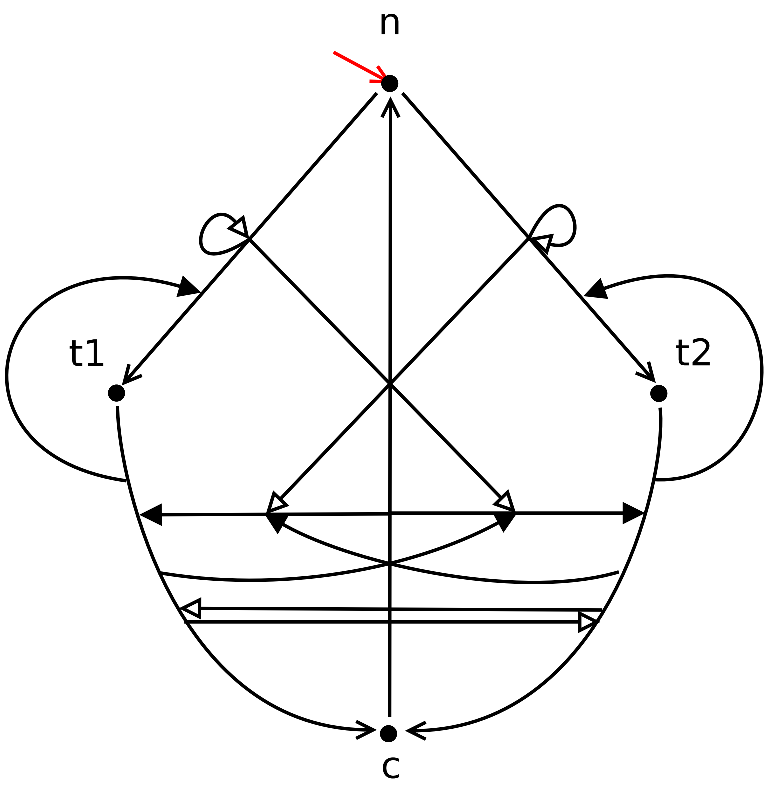

Figure 5.7: N arbitrary: safety, liveness, no strict sequencing and non-blocking for $N = 2$.

constitutes a complete solution for $N = 2$ but for $N > 2$ it loses liveness and non-blocking. If we want to accommodate more processes we can simply add more trying states, as many as the number of processes being considered and have the same switch structure between every pair t_i, t_j for $i \neq j$. Meaning that a solution for arbitrary N is given by $S_N = (W_N, R_N)$ with $W_N = \{n, c, t_1, \ldots, t_N\}$ and

$$\begin{aligned} R_N = \{ & (n, t_i), (t_i, c), (c, n), \\ & (nt_i, nt_i, \circ), (t_i c, nt_i, \bullet), (cn, t_i c, \bullet), (cn, t_j c, \bullet), (t_i c, t_j c, \circ), \\ & (nt_i, cn, t_j c, \bullet\circ), (t_i c, cn, t_j c, \bullet\bullet) : 1 \leq i, j \leq N, i \neq j\} \end{aligned}$$

Clearly for $N > 2$, S_N is not as easily visualised, but this is also a cost to pay also when drawing the transition system associated to the protocol for the mutual exclusion with various processes. We believe that it is still impressive the fact that the switch structure is easily definable for all N, being clear from the $N = 2$ case analysis why it works in the general case.

5.3 Discussion

Level of Reactivity Being true that all reactive graphs can be represented by switch graphs, the level of reactivity of the representation is of obvious practical importance.

Given a reactive graph, it is easy to see that the minimal reactive level for a switch graph representing it, depends on the choice of dynamics we discussed above. Consider $R = (\{a, b\}, \Delta)$, where $\Delta = \{\alpha : \lambda = (a, a)^n$ or $\alpha = (a, a)^{2n}(a, b), n < \omega\}$. Using the definition where the connecting switches win we cannot find a finite reactive level switch graph that represents it. Whereas using the alternating one it can be represented by a switch graph of reactivity level 1:

$$(\{a, b\}, \{(a, a), (a, b), ((a, a), (a, b), \circ), ((a, a), (a, b), \bullet)\}).$$

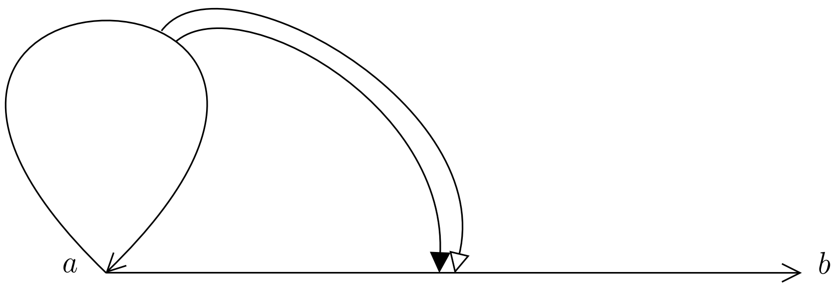

Figure 5.8: Alternating dynamics example with the following behaviour: $\Delta = \{\alpha : \lambda = (a, a)^n$ or $\alpha = (a, a)^{2n}(a, b), n < \omega\}$.

It is also easy to cook a reactive graph that cannot be represented by a finite level graph for any of the options we mentioned above, nor with any switches with 'recursive behaviour'. Let f be a non recursive binary sequence and

$R = (\{a, b\}, \Delta)$, where $\Delta = \{\alpha : \alpha = (a, a)^n$ or $\alpha = (a, a)^k(a, b), n < \omega, f(k) = 1\}$.

Question 5.3.1. *For each of the considered options which is the set of reactive graphs generated by switch graphs of reactivity level $n \in \omega$?*

Further extensions We have seen how arrows can represent more than the basic transitions, they can represent the transitions between states of the accessibility relation itself. These new arrows are a natural extension of the concept of graph corresponding to a kind of meta-transitions in the above sense. They provide an explicit way of expressing the meta-level graph's notion of reactivity, in a way that allows an immediate (and complete) reading of the effects of crossing an edge.

In Section 5.2 we considered multiple entities going through the graph, the effects of each action were independent of whom was doing it and no synchronous movements were allowed. Other types of dependences can be considered:

- Dependence on the identity of the individual entities could be modelled by having different relations for each entity, or groups of entities.

- Synchronous movements could also be represented by special relations. Of course switches can connect arrows regardless of the relation they represent.

Thus leading to the question:

Question 5.3.2. *Is the switches formalism enough to represent these more general behaviours?*

6

Reactive Hybrid Switch Logics

In the previous section we have shown that switch graphs are suitable structures to embody and represent the reactive paradigm. In this section we introduce a logic that allows us to reason about switches and their effects and prove completeness. A basic feature of such a logic would be to be able to express that if a certain switch is on, then after such move the state of certain switch will be such.

In the introduction we gave some motivation for the used language. So, on one hand, given a switch graph (W,R), for each switch type $\sigma \in \{\circ,\bullet\}^*$, the set of σ-switches forms $|\sigma|+2$-ary relation over W and there is the $|\sigma|+1$-ary modal correspondent, $\diamondplus_\sigma$. On the other hand, similarly to $\Diamond_R$ in Chapter 4, there is the operator corresponding to its behaviour, $\diamonddiv$. The goal is to relate both, expressing how the switches determine the behaviour.

The logics we considered in Chapter 4 were suitable to talk about local effects of reactivity, but here we are dealing with its global effects and at the same time we feel the need to to explicitly refer to specific states. Ordinary modal logic's lack of mechanisms for dealing with states explicitly is a recognised weakness. In fact the idea of adding variables that are used to name worlds dates back to the pioneering work of Prior [65, 64] and especially of Bull [14](see [37]). Recently the study of this idea gained popularity and its development became an autonomous subfield of modal logic, called *hybrid logic*[1].

Hybrid languages are a very simple extension of modal ones. A special set of propositional variables, called nominals, is added and it is used to name worlds. Nominals are true at exactly one world in any model. Many operators related to the nominals were studied. A particularly simple one is @ that allows one to jump to particular worlds, in the sense that,

[1]For more details see [3, 2, 9].

for each nominal i, $@_i\varphi$ is true if φ is true in the world named i, which suits our need to think globally.

We consider models constructed from switch graphs in such a way that for each admissible sequence of edges we get an usual hybrid model based on the Kripke frame given by the switches' state at that point. Of course each nominal must be true in the same world in all these models but all the other variables may change.

This language allows us to express some strong reactive assertions. Consider the formula

$$@_i \diamond (j \to @_i \neg \diamond j).$$

This says that if we cross the edge from the world named i to the world named j, that connection will be turned off. Another way to say this is to say that there is a disconnecting switch from that edge to itself:

$$@_i \diamond_{\circ} (j, i, j).$$

We obtain completeness for the introduced logic adapting the usual Henkin style proof of completeness for hybrid logics, see [10, 12]. This further justifies the choice of language by showing that it is appropriate to capture the switches' dynamics.

6.1 Switch Models

Definition 6.1.1. We consider the *reactive switch similarity type*

$$s = (\{\diamond\} \cup \{\diamond_\sigma : \sigma \in \{\circ, \bullet\}^*\}, \rho),$$

where $\rho(\diamond) = 1$ and $\rho(\diamond_\sigma) = 2|\sigma| + 1$. Thus we define the hybrid modal language $\mathcal{H}_s(@)$ is defined by

$$\varphi \ ::= \ i \mid p \mid \neg\varphi \mid \varphi_1 \wedge \varphi_2 \mid \diamond\varphi \mid \diamond_\sigma(\varphi_1, \ldots, \varphi_{2|\sigma|+1}) \mid @_i\varphi\,,$$

where $p \in \Pi$, $i \in NOM$ and $\sigma \in \{\circ, \bullet\}^*$. The other connectives: $\top$, $\bot$, $\vee$, $\to$, $\leftrightarrow$, $\boxminus$ and $\boxminus_\sigma$ are introduced by the usual abbreviations.

Given $\lambda \in (NOM \times NOM)^*$ we define the abbreviation $\diamond^\lambda\varphi$ by recursion:

- $\diamond^\epsilon\varphi = \varphi$,
- $\diamond^{(i,j)\lambda}\varphi = @_i \diamond (j \wedge \diamond^\lambda\varphi)$ (clearly $\diamond^{(i,j)\lambda}\varphi = \diamond^{(i,j)} \diamond^\lambda \varphi$),

and $\boxslash^{\lambda}\varphi = \neg \diamondslash^{\lambda} \neg\varphi$.

Definition 6.1.2. Given a switch graph $S = (W, R)$ and using the dynamics defined in 5.1.6 we generate, for each $\lambda \in \Delta_S$, a reactive graph: $S_\lambda = (W, R_\lambda)$.

A switch frame is the Kripke frame given by the disjoint union of all these switch graphs, where σ switches in R_λ give origin to local $(2|\sigma| + 2)$-ary relations and with a global accessibility relation connecting w in S_λ and w' in $S_{\lambda(w,w')}$, see Figure 6.1.

Formally, the **switch frame** over S is $\mathfrak{F}_S = (W \times \Delta_S, \check{R}_S)$ where

- $\check{R}_S = \{\check{R}^{\sigma}_{\lambda}\}_{\sigma\in\{\circ,\bullet\}^*,\lambda\in\Delta_S}$ and
- $((w_1, \lambda), \ldots, (w_{2n+2}, \lambda)) \in \check{R}^{\sigma}_{\lambda}$ iff $(w_1 w_2, \ldots, w_{2n+1} w_{2n+2}, \sigma) \in R_\lambda$.

A *switch model* over $\mathfrak{F}_S$ is a pair $\mathfrak{M} = (\mathfrak{F}, \nu)$, where ν is a function

$$\nu : \Pi \cup NOM \to 2^{\Delta_S \times W}$$

such that for $s \in NOM$ we have $V(s) = \{w\} \times \Delta_S$ for some $w \in W$.

Given a switch model $\mathfrak{M} = (W \times \Delta_S, R_S, \nu)$, for every $(w, \lambda) \in W \times \Delta_S$ and every L-formula φ, we define the notion φ *is true at* (w, λ) *in* $\mathfrak{M}$ ($\mathfrak{M}, (w, \lambda) \models \varphi$) inductively as follows:

- $\mathfrak{M}, (w, \lambda) \models p$ iff $(w, \lambda) \in \nu(p)$ for variables p,
- $\mathfrak{M}, (w, \lambda) \models s$ iff $(w, \lambda) \in \nu(s)$ for nominals s,
- $\mathfrak{M}, (w, \lambda) \models \neg\varphi$ iff $\mathfrak{M}, (w, \lambda) \not\models \varphi$,
- $\mathfrak{M}, (w, \lambda) \models \varphi_1 \wedge \varphi_2$ iff $\mathfrak{M}, (w, \lambda) \models \varphi_1$ and $\mathfrak{M}, (w, \lambda) \models \varphi_2$,
- $\mathfrak{M}, (w, \lambda) \models \diamondslash\varphi$ iff there is $w' \in W$ such that $\lambda(w, w') \in \Delta$ and $\mathfrak{M}, (w', \lambda(w, w')) \models \varphi$,

  Notice that $\lambda(w, w')$ iff $(w, w') = (w, w', \epsilon) \in R_\lambda$ iff $(w, w') \in \check{R}^{\epsilon}_{\lambda}$,
- $\mathfrak{M}, (w, \lambda) \models @_i\varphi$ iff $\mathfrak{M}, (w', \lambda) \models \varphi$ for $\nu(i) = \Delta_S \times \{w'\}$,
- $\mathfrak{M}, (w, \lambda) \models \diamondminus_{\sigma}(\varphi_1, \ldots, \varphi_{2n+1})$ iff there are $v_1 \ldots v_{2n+1} \in W$ such that $(w, v_1, \ldots, v_{2n}, v_{2n+1}) \in \check{R}^{\sigma}_{\lambda}$ and $\mathfrak{M}, (\lambda, v_i) \models \varphi_i$.

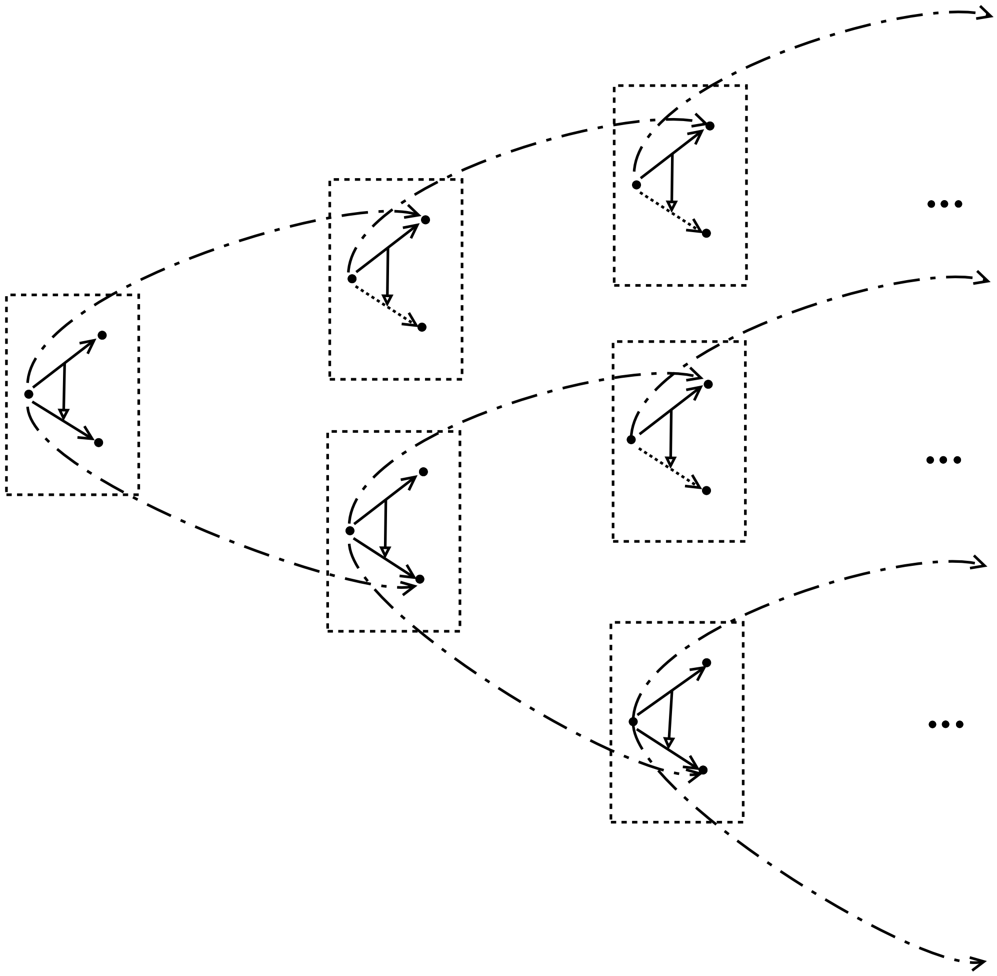

Figure 6.1: A representation of a switch frame. The various components are limited by a dashed line, the switches of that component represented by the arrows as before and the (reactive) transitions by the line formed by lines and dots.

We say that φ *is true in* $\mathfrak{M}$ iff $\mathfrak{M},(w,\lambda) \models \varphi$ for every $(w,\lambda) \in \Delta_S \times W$. We say that φ *is valid in* a switch frame if it is true in every switch model over it.

Remark 6.1.3. The idea of allowing the valuation to be updated at each move leads us to define the switch frames in such a way that makes them to have an infinite number of components (if the base switch graph has an infinite behaviour) in order to have a classical semantics over this frame. Clearly to have a representation of the switch dynamics it would be enough to have one component for each reachable switch configuration. One can then consider the Kripke frame $\mathfrak{F}'_S = (W \times \Delta_S, \check{R}'_S)$ (see Figure 6.4) where

- $\check{R}'_S = \{\hat{R}^\sigma\}_{\sigma\in\{\circ,\bullet\}^*}$ and
- $((w_1, R'), \ldots, (w_{2n+2}, R')) \in \check{R}^\sigma_\lambda$ iff $(w_1w_2, \ldots, w_{2n+1}w_{2n+2}, \sigma) \in R'$,

which is always finite if S is finite.

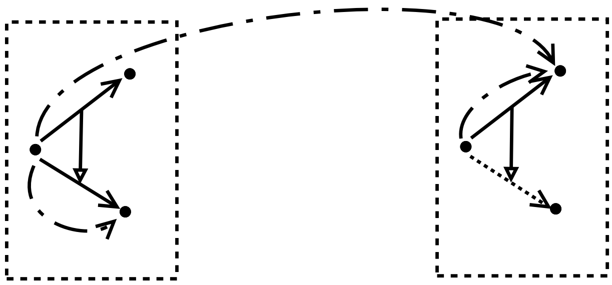

Figure 6.2: A finite representation of a switch frame: small switch frame.

Let us call them small switch frames and consider the semantics of $\mathcal{H}_s(@)$ over them trivially adapted from Definition 6.1.2, and call them small switch models. If for example $\Pi = \emptyset$, then these two classes of frames would clearly generate the same logic. Or for modal checking purposes, where everything in the model is finite, including the number of propositional variables involved, one would consider similarly a finite representation by having one component for each configuration of switches and distribution of propositional variables.

Still, it is interesting to notice that even if the classes of small and the original switch frames are not immediately reducible to one another they yield the same logic.

It is trivial to define a model over $\mathfrak{F}_S$ from a model over $\mathfrak{F}'_S$ preserving modal truths, since we just have to copy the valuation of each configuration and paste it where it appears, so

$$\{\varphi \in \mathcal{H}_s(@) : \varphi \text{ is valid over switch frames}\} \subseteq \{\varphi \in \mathcal{H}_s(@) : \varphi \text{ is valid over small switch frames}\}.$$

The converse is not so immediate since if we cannot fit two different configurations in the same component of a small switch model. Still, it is not hard to see that these two classes define the same logic. The idea of the proof is the following:

Given φ and a switch model $(W \times \Delta_S, R_S, \nu), (w, \epsilon) \vDash \varphi$ (we can assume $\lambda = \epsilon$ since the past clearly does not interfere). We can clearly consider a switch graph S' which is a copy of S plus some switches of a higher enough level so that they do not interfere with the valuation of φ (also fresh points forming lower level switches to which the former switches point). This is done in such a way that when an edge is crossed, during the valuation of φ, the new switches' configuration changes and we can accommodate the ν.

6.2 Axiomatising

In this section we prove that the axiomatisation presented in Figure 6.3, generates all the $\mathcal{H}_s(@)$-formulas that are valid in all switch frames. Furthermore, we obtain the usual hybrid automatic completeness for pure axioms. The following axiomatisation is the natural adaptation of the ones for the standard hybrid logics given in [11, 12].

Remark 6.2.1. The fact that the propositional valuation is allowed to change at each move implies that switch frames based on switch graphs with infinite behaviour have infinite components. Nevertheless, to have a complete view on the switch dynamics it would be enough to have one component for each reachable switch configuration. Let $A_S = \{R_\lambda : \lambda \in \Delta_S\}$ be the set of switches' configurations that are obtained in S's dynamics. One can then consider the Kripke frame $\mathfrak{F}'_S = (W \times A, \check{R}'_S)$ (the small switch frame over S) where

- $\check{R}'_S = \{\hat{R}^\sigma\}_{\sigma \in \{\circ, \bullet\}^*}$,
- $((w_1, R), \ldots, (w_{2n+2}, R)) \in \check{R}^\sigma_\lambda$ iff $(w_1 w_2, \ldots, w_{2n+1} w_{2n+2}, \sigma) \in R$ and
- (w, R_λ) connects (reactively) with $(w', R_{\lambda w w'})$.

So everything is the same apart from the fact that we do not always get a new component from a move, the tree structure disappears. It is easy to see that $\mathfrak{F}'_S$ is finite if and only if S is finite. See Figure 6.4 for an example.

Let us consider a semantics of $\mathcal{H}_s(@)$ over the small switch frames trivially adapted from Definition 6.1.2 (only the interpretation of $\diamondsuit$ changes), and call them small switch models. If the set of propositional symbols is empty ($\Pi = \emptyset$) these two classes of frames trivially generate the same logic. Also, when, e.g. for modal checking purposes, everything in the model is finite, including the number of propositional variables involved, one would consider similarly a finite representation by having one component for each configuration of the switches and distribution of the relevant propositional variables.

L_s	
Axioms:	
CT	All classical tautologies
$K^{l}_{\boxbar_\sigma}$	$\boxbar_\sigma(\varphi_1,\ldots,\varphi\to\psi,\ldots,\varphi_{2n+1})\to$ $(\boxbar_\sigma(\varphi_1,\ldots,\varphi,\ldots,\varphi_{2n+1})\to\boxbar_\sigma(\varphi_1,\ldots,\psi,\ldots,\varphi_{2n+1}))$ where l is the component where $\varphi\to\psi$ is.
$K_{@}$	$@_i(\varphi\to\psi)\to(@_i\varphi\to @_i\psi)$
$Selfdual_{@}$	$@_i\varphi\leftrightarrow\neg @_i\neg\varphi$
$Ref_{@}$	$@_i i$
$Agree$	$@_i @_j\varphi\leftrightarrow @_j\varphi$
$Intro$	$i\to(\varphi\leftrightarrow @_i\varphi)$
$Sync$	$\diamondslash i\leftrightarrow\diamondbar_\epsilon i$
$Det_{\diamondslash}$	$\diamondslash(i\wedge\varphi)\to\boxslash(i\to\varphi)$
$K_{\boxslash}$	$\boxslash(\varphi\to\psi)\to(\boxslash\varphi\to\boxslash\psi)$
Dyn	$\diamondslash^{(i,j)}@_{i_1}\diamondbar_\sigma(i_2,\ldots,i_{2n+2})\leftrightarrow$ $(@_{i_1}\diamondbar_\sigma(i_2,\ldots,i_{2n+2})\wedge\neg @_i\diamondbar_{\sigma\circ}(j,i_1\ldots,i_{2n+2})\vee @_i\diamondbar_{\sigma\bullet}(j,i_1\ldots,i_{2n+2}))$
Rules:	
MP	If $\vdash\varphi$ and $\vdash\varphi\to\psi$ then $\vdash\psi$
$Subst$	If $\vdash\varphi$ then $\vdash\varphi^\theta$
$Gen_{\boxbar_\sigma}$	If $\vdash\varphi$ then $\vdash\boxbar_\sigma(\bot,\ldots,\bot,\varphi,\bot,\ldots,\bot)$
$Gen_{@}$	If $\vdash\varphi$ then $\vdash @_i\varphi$
$Gen_{\boxslash}$	If $\vdash\varphi$ then $\vdash\boxslash\varphi$
$Name$	If $\vdash @_i\varphi$ and i does not occur in φ then $\vdash\varphi$
$Paste_{\diamondslash^\lambda\diamondbar_\sigma}$	If $\vdash\diamondslash^\lambda @_i\diamondbar_\sigma(j_1,\ldots,j_{2n+1})\wedge\diamondslash^\lambda(\bigwedge_{1\geq l\geq 2n+1}@_{j_l}\varphi_l)\to\psi$ and $j_m\neq i$ does not occur in φ_l or ψ then $\vdash\diamondslash^\lambda @_i\diamondbar_\sigma(\varphi_1,\ldots,\varphi_{2n+1})\to\psi$
$Paste'_{\diamondslash^\lambda\diamondslash}$	If $\vdash\diamondslash^\lambda @_i\diamondslash j\wedge\diamondslash^\lambda\diamondslash @_j\varphi\to\psi$ and $j\neq i$ does not occur in φ or ψ then $\vdash\diamondslash^\lambda @_i\diamondslash\varphi\to\psi$

Figure 6.3: The L_s axiomatisation.

Moreover, it is interesting to notice that even if the classes of small and the original switch frames are not immediately reducible to one another they yield the same logic.

It is trivial to define a model over $\mathfrak{F}_S$ from a model over $\mathfrak{F}'_S$ preserving modal truths,

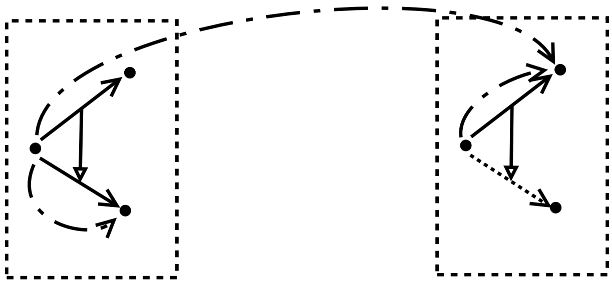

Figure 6.4: Small switch frame of $\{\{a, b, c\}, \{(a, b), (a, c), (ab, cd, \circ)\}\}$.

since we just have to copy the valuation of each configuration and paste it where it appears, so

$$\{\varphi \in \mathcal{H}_s(@) : \varphi \text{ is valid over switch frames}\} \subseteq \{\varphi \in \mathcal{H}_s(@) : \varphi \text{ is valid over small switch frames}\}.$$

The converse is not so immediate since if we cannot fit two different configurations in the same component of a small switch model. However, it is not hard to see that these two classes define the same logic. The idea of the proof is the following:

Given φ and a switch model $(W \times \Delta_S, R_S, \nu), (w, \epsilon) \models \varphi$ (we can assume $\lambda = \epsilon$ since the past clearly does not interfere with the evaluation of φ). We then consider a switch graph S' which is a copy of S plus some switches of a high enough level so that they do not interfere with the evaluation of φ and that during the evaluation of φ the new switches' configuration changes and so that we can accommodate the ν. Let k be the higher level of the switches referred in φ, it is clear we can add new points and switches such that at each crossing of an edge a different switch, of level higher than k, changes its state. For example one could add for each admissible sequence of edges covered in the interpretation of $\diamondsuit$, $\lambda = (w_1 w_2) \ldots (w_{2n-1} w_{2n})$, a point w_λ and the switch

$$(w_1 w_2, \ldots, w_{2n-1} w_{2n}, [w_\lambda w_\lambda]^k, \bullet^{n-1+k})$$

where $[w_\lambda w_\lambda]^k$ stands for $k+1$-occurrences of $w_\lambda w_\lambda$ with a comma dividing each occurrence. Thus guaranteeing a different component for each sequence of edges and therefore being able to define the appropriate valuation to each of them.

Here is the theorem we shall prove:

Theorem 6.2.2. *Let Λ be a set of pure $\mathcal{H}_s(@)$ formulas. A set of $\mathcal{H}_s(@)$ formulas Σ is $L_s + \Lambda$-consistent iff Σ is satisfiable in a model satisfying the frame properties defined by Γ. Where $L_s + \Lambda$ is the above axiomatisation extended with the axioms of Λ.*

Lemma 6.2.3. *The following are derivable*

- $K_{@}^{-1}$: $\vdash (@_i\varphi \to @_i\psi) \to @_i(\varphi \to \psi)$,
- *Nom:* $\vdash @_i j \to (@_i\varphi \to @_j\varphi)$,
- *Sym:* $\vdash @_i j \to @_j i$,
- *Name′: If* $\vdash i \to \varphi$ *then* $\vdash \varphi$ *where* i *does not occur in* φ,
- $Back_{\Diamond_\sigma}$: $\Diamond_\sigma(\varphi_1, \dots, @_i\varphi_k, \dots, \varphi_{2n+1}) \to @_i\varphi_k$,
- $Bridge_{\Diamond_\sigma}$: $@_i \Diamond_\sigma (j_1, \dots, j_{2n+1}) \wedge (\bigwedge_{1\geq l\geq 2n+1} @_{j_l}\varphi_l) \to @_i \Diamond_\sigma (\varphi_1, \dots, \varphi_{2n+1})$.

Proof. The first 4 are proved in [12].

- $Back_{\Diamond}$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{@_i@_j\varphi \leftrightarrow @_u@_j\varphi}\;Agree, Agree}{@_u \Diamond_\sigma (\varphi_1, \dots, j, \dots, \varphi_{2n+1}) \wedge @_i@_j\varphi \to @_u@_j\varphi}}{@_u \Diamond_\sigma (\varphi_1, \dots, @_j\varphi, \dots, \varphi_{2n+1}) \to @_u@_j\varphi}\;Paste_{\Diamond_\sigma}}{\Diamond_\sigma(\varphi_1, \dots, @_j\varphi, \dots, \varphi_{2n+1}) \to @_j\varphi}\;K_{@}^{-1}, Name$$

- $Bridge_{\Diamond^\sigma}$ by induction on σ, $\sigma = \epsilon$ is trivial and

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\boxbar_\epsilon(\neg\varphi \to \neg j) \to (\boxbar_\epsilon\neg\varphi \to \boxbar_\epsilon\neg j)}\;K_{\boxbar_\epsilon}}{\boxbar_\epsilon\neg\varphi \wedge \Diamond_\epsilon j \to \Diamond_\epsilon(j \wedge \neg\varphi)}\;CT \qquad \dfrac{}{j \wedge \neg\varphi \to @_j\neg\varphi}\;Int_@, CT}{\boxbar_\epsilon\neg\varphi \wedge \Diamond_\epsilon j \to \Diamond_\epsilon @_j\neg\varphi}\;CT \qquad \dfrac{}{\Diamond_\epsilon @_j\neg\varphi \to @_j\neg\varphi}\;Back}{@_i(\boxbar_\epsilon\neg\varphi \wedge \Diamond_\epsilon j) \to @_i@_j\neg\varphi}\;Gen_@, K_@}{@_i \boxbar_\epsilon \neg\varphi \wedge @_i \Diamond_\epsilon j \to @_j\neg\varphi}\;K_@, CT, Agree, Selfdual_@}{@_i \Diamond_\epsilon j \wedge @_j\varphi \to @_i \Diamond_\epsilon \varphi}\;CT}{}$$

The proof for arbitrary σ's consists in the repetition of this derivation applied to each of the $2|\sigma| - 1$ coordinates, where in each time an argument eats a conjunct.

$\square$

Lemma 6.2.4. *The following are derivable:*

- $\boxslash^{(i,j)\lambda}\varphi \leftrightarrow @_i \boxslash (j \to \boxslash^{\lambda}\varphi)$,
- $Gen_{\boxslash^\lambda}$: *If* $\vdash \varphi$ *then* $\vdash \boxslash^{\lambda}\varphi$,
- $K_{\boxslash^\lambda}$: $\boxslash^{\lambda}(\varphi \to \psi) \to (\boxslash^{\lambda}\varphi \to \boxslash^{\lambda}\psi)$,
- $Det_{\diamondslash^\lambda}$: $\diamondslash^{\lambda}\varphi \leftrightarrow \boxslash^{\lambda}\varphi \wedge \diamondslash^{\lambda}\top$,
- $Gen'_{\diamondslash^\lambda}$: $\vdash \varphi \to \psi$ *then* $\vdash \diamondslash^{\lambda}\varphi \to \diamondslash^{\lambda}\varphi$.

Furthermore from $Gen_{\boxslash^\lambda}$ *and* $K_{\boxslash^\lambda}$ *we easily get* $\diamondslash^{\lambda}(\varphi \wedge \psi) \leftrightarrow \diamondslash^{\lambda}\varphi \wedge \diamondslash^{\lambda}\psi$ *and* $\diamondslash^{\lambda}(\varphi \vee \psi) \leftrightarrow \diamondslash^{\lambda}\varphi \vee \diamondslash^{\lambda}\psi$.

Proof.

- $\boxslash^{(i,j)\lambda}\varphi = \neg @_i \diamondslash (j \wedge \diamondslash^{\lambda}\neg\varphi)$ that is equivalent to $@_i \boxslash (j \to \neg\diamondslash^{\lambda}\neg\varphi) = @_i \boxslash (j \to \boxslash^{\lambda}\varphi)$.

- $Gen_{\boxslash^\lambda}$ by induction on λ, ϵ is trivial and

$$\cfrac{\cfrac{\cfrac{\cfrac{\varphi}{\boxslash^{\lambda}\varphi}\ IH}{j \to \boxslash^{\lambda}\varphi}\ CT}{\boxslash(j \to \boxslash^{\lambda}\varphi)}\ Gen_{\boxslash}}{@_i \boxslash (j \to \boxslash^{\lambda}\varphi)}\ Gen_{@}$$

- $K_{\boxslash^\lambda}$ by induction on λ, $\lambda = \epsilon$ is trivial and

$$\cfrac{\cfrac{\cfrac{\cfrac{\varphi}{\boxslash^{(i,j)\lambda}(\varphi \to \psi)}\ IH}{@_i \boxslash (j \to (\boxslash^{\lambda}\varphi \to \boxslash^{\lambda}\psi))}\ CT}{@_i \boxslash (j \to \boxslash^{\lambda}\varphi) \to @_i \boxslash (j \to \boxslash^{\lambda}\psi)}\ K_{\boxslash}, K_{@}}{\boxslash^{(i,j)\lambda}\varphi \to \boxslash^{(i,j)\lambda}\psi}\ def$$

- $Det_{\diamondslash^{(i,j)\lambda}}$ by induction on λ, $\lambda = \epsilon$ is trivial and

$$\cfrac{\cfrac{}{\boxslash^{(i,j)\lambda}\varphi \wedge \diamondslash^{(i,j)\lambda}\top \to \diamondslash^{(i,j)\lambda}\varphi}\ K_{\boxslash}^{(i,j)\lambda}, CT \qquad \cfrac{\cfrac{\cfrac{}{\diamondslash^{(i,j)\lambda}\varphi \to @_i \diamondslash (j \wedge [\boxslash^{\lambda}\varphi \wedge \diamondslash^{\lambda}\top])}\ Def, IH}{\diamondslash^{(i,j)\lambda}\varphi \to @_i \diamondslash (j \wedge \boxslash^{\lambda}\varphi)}\ IH, CT}{\diamondslash^{(i,j)\lambda}\varphi \to \boxslash^{(i,j)\lambda}\varphi \wedge \diamondslash^{(i,j)\lambda}\top}\ CT}{\diamondslash^{(i,j)\lambda}\varphi \leftrightarrow \boxslash^{(i,j)\lambda}\varphi \wedge \diamondslash^{(i,j)\lambda}\top}\ CT$$

- $Gen'_{\diamondslash^\lambda}$ by induction on λ, $\lambda = \epsilon$ is trivial and

$$\dfrac{\dfrac{\dfrac{\varphi \to \psi}{\boxslash^\lambda(\varphi \to \psi)}\ Gen_{\boxslash^\lambda}}{\diamondslash^\lambda \top \wedge \boxslash^\lambda(\varphi \to \psi)}\ CT}{\dfrac{\boxslash^\lambda \varphi \wedge \diamondslash^\lambda \top \to \boxslash^\lambda \psi \wedge \diamondslash^\lambda \top}{\diamondslash^\lambda \varphi \to \diamondslash^\lambda \psi}\ Det_{\diamondslash^\lambda}}\ K_{\boxslash^\lambda}, CT$$

□

Definition 6.2.5. *Let Σ be a set of $\mathcal{H}_s(@)$-formulas:*

- *Σ is named if one of its elements is a nominal.*
- *Σ is $\diamondvert_\sigma$-saturated if for all σ' and $\diamondslash^\lambda @_i \diamondvert_{\sigma'} (\varphi_1, \ldots, \varphi_{2n+1}) \in \Sigma$ there are nominals $j_1, \ldots, j_{2n+1}$ such that $\diamondslash^\lambda @_i \diamondvert_{\sigma'} (j_1, \ldots, j_{2n+1}) \in \Sigma$ and $\diamondslash^\lambda @_{j_k} \varphi_k \in \Sigma$, $k = 1, \ldots, 2n+1$.*
- *Σ is $\diamondslash$-saturated if for all $\diamondslash^\lambda @_i \diamondslash \varphi \in \Sigma$ there is a nominal j such that $\diamondslash^\lambda @_i \diamondslash (j \wedge \varphi) = \diamondslash^{\lambda(i,j)} \varphi \in \Sigma$.*

Lemma 6.2.6. *(Lindenbaum Lemma). Every $L_s + \Gamma$-consistent set of formulas can be extended to a named, $\diamondslash$-saturated and $\diamondvert_\sigma$-saturated MCS, by adding countably many new nominals to the language.*

Proof. Let $(i_n)_{n<\omega}$ be an enumeration of the new nominals and $(\varphi_n)_{n<\omega}$ an enumeration of the formulas in the extended language.

We define $\Sigma^0 = \Sigma \cup i_0$, *Name′* guarantees that it is consistent.

If $\Sigma^n \cup \{\varphi_n\}$ is inconsistent then $\Sigma^{n+1} = \Sigma^n$. Otherwise:

1. $\Sigma^{n+1} = \Sigma^n \cup \{\varphi_n\}$ if φ_n is not of the form $\diamondslash^\lambda @_i \diamondvert_\sigma (\psi_1, \ldots, \psi_{2k+1})$ or $\diamondslash^\lambda @_i \diamondslash \psi$,
2. $\Sigma^{n+1} = \Sigma^n \cup \{\varphi_n\} \cup \{\diamondslash^\lambda @_i \diamondvert_\sigma (i_m, \ldots, i_{m+2n+1})\} \cup \{\diamondslash^\lambda @_{i_{m+l}} \psi_l : 1 \leq l \leq 2k+1\}$ if it is of the form $\diamondslash^\lambda @_i \diamondvert_\sigma (\psi_1, \ldots, \psi_{2k+1})$,
3. $\Sigma^{n+1} = \Sigma^n \cup \{\varphi_n\} \cup \{\diamondslash^\lambda @_i \diamondslash (i_m \wedge \psi)\}$ if it is of the form $\diamondslash^\lambda @_i \diamondslash \psi$.

Where i_m is the first new nominal that does not occur in Σ^n or φ_n.
Let $\Sigma^\omega = \bigcup_{n<\omega} \Sigma^n$. Then $\Sigma \subseteq \Sigma^\omega$ and Σ^ω is named, $\diamondvert_\sigma$-saturated, $\diamondslash$-saturated, maximal and consistence. The only non-trivial step is in 2, and here consistency is guaranteed by $Paste_{\diamondslash^\lambda \diamondvert_\sigma}$ and $Paste'_{\diamondslash^\lambda \diamondslash}$. □

Definition 6.2.7. *(Henkin model from* Γ*). Let* Γ *be a maximal consistent set of* $\mathcal{H}_s(@)$ *formulas. For all nominals i, let* $|i| = \{ j : @_i j \in \Gamma\}$*. Then the* $\mathfrak{M}_\Gamma = (W, \Delta, T, \nu)$ *is given by:*

$$W = \{|i| : i \text{ is a nominal}\},$$
$$\Delta = \{\langle\lambda\rangle : \diamondminus^{\lambda}\top \in \Gamma\},$$
$$\check{R} = \{T^{\sigma}_{\langle\lambda\rangle}\}_{\sigma,\langle\lambda\rangle},$$
$$\check{R}^{\sigma}_{\langle\lambda\rangle}(|i_1|, \ldots, |i_{2n+2}|) \text{ iff } \diamondminus^{\lambda} @_{i_1} \Diamond_{\sigma} (i_2, \ldots, i_{2n+2}) \in \Gamma,$$
$$\nu(p) = \{(|i|, \langle\lambda\rangle) : \diamondminus^{\lambda} @_i p \in \Gamma\},$$
$$\nu(i) = \{(|i|, \langle\lambda\rangle) : \langle\lambda\rangle \in \Delta\},$$

where $\langle\epsilon\rangle = \epsilon$ *and* $\langle\lambda(i, j)\rangle = \langle\lambda\rangle\,(|i|, |j|)$*. That* $\mathfrak{M}_\Gamma$ *is well-defined follows from* Ref*,* Sym*,* Nom *and* $Gen'_{\Diamond_\lambda}$*.*

Lemma 6.2.8. $\diamondminus^{\lambda(i,j)}\varphi = \diamondminus^{\lambda}@_i \diamondminus (j \wedge \varphi)$.

Proof. By induction on λ:

For $\lambda = \epsilon$ the two formulas coincide trivially and for $\lambda = (s, t)\gamma$

$$\diamondminus^{(s,t)\gamma(i,j)}\varphi = \diamondminus^{(s,t)} \diamondminus^{\gamma(i,j)} \varphi =^{IH} \diamondminus^{(s,t)} \diamondminus^{\gamma} @_i \diamondminus (j \wedge \varphi) = \diamondminus^{(s,t)\gamma}@_i \diamondminus (j \wedge \varphi).$$

□

Lemma 6.2.9. $\mathfrak{M}_\Gamma$ *is a switch model where* $(W \times \Delta, \check{R})$ *is the switch frame generated by* $S = (W, R)$ *such that*

$$R = \{(|i_1||i_2|, \ldots, |i_{2n+1}||i_{2n+2}|, \sigma) : (|i_1|, |i_2|, \ldots, |i_{2n+1}|, |i_{2n+2}|) \in \check{R}^{\sigma}_{\epsilon}\}.$$

Proof. Let Δ_S and $R_S = \{R^{\sigma}_{\lambda}\}$ as defined in 5.1.6. The proof that $\Delta = \Delta_S$ and $\check{R}^{\sigma}_{\lambda} = R^{\sigma}_{\lambda}$ is done by induction on the length of $\langle\lambda\rangle$.

The case of $\langle\lambda\rangle = \epsilon$ is trivial since it is in both Δ ($\top \in \Gamma$ by CT) and Δ_S, furthermore $\check{R}^{\sigma}_{\epsilon} = R^{\sigma}_{\epsilon}$ by definition.

The induction step ($\langle\lambda\rangle \to \langle\lambda(i, j)\rangle$):

$\langle\lambda(|i|, |j|)\rangle \in \Delta_S$ iff

$\langle\lambda\rangle \in \Delta_S$ and $(|i|, |j|) \in R_\lambda$ iff $\langle\lambda\rangle \in \Delta_S$ and $R^{\epsilon}_{\langle\lambda\rangle}(|i|, |j|)$ iff (IH)

$\diamondminus^{\lambda}\top \in \Gamma$ and $\check{R}^{\epsilon}_{\langle\lambda\rangle}(|i|, |j|)$ iff $\diamondminus^{\lambda}@_i \diamondminus (j \wedge \top) = \diamondminus^{\lambda(i,j)}\top \in \Gamma$ iff

$\langle\lambda\rangle\,(|i|, |j|) \in \Delta$.

And,

$(|i_1||i_2|, \ldots, |i_{2n+1}||i_{2n+2}|, \sigma) \in R_{\langle \lambda(i,j) \rangle}$ iff

$(|i_1||i_2|, \ldots, |i_{2n+1}||i_{2n+2}|, \sigma) \in R_{\langle \lambda \rangle}$ and $(|i||j|, |i_1||i_2|, \ldots, |i_{2n+1}||i_{2n+2}|, \sigma\circ) \notin R_{\langle \lambda \rangle}$

or

$(|i||j|, |i_1||i_2|, \ldots, |i_{2n+1}||i_{2n+2}|, \sigma\bullet) \in R_{\langle \lambda \rangle}$

iff (IH)

$\check{R}^{\sigma}_{\lambda}(|i_1|, |i_2|, \ldots, |i_{2n+1}|, |i_{2n+2}|, \sigma)$ and $\check{R}^{\sigma\circ}_{\lambda}(|i|, |j|, |i_1|, |i_2|, \ldots, |i_{2n+1}|, |i_{2n+2}|)$

or

$\check{R}^{\sigma\bullet}_{\lambda}(|i|, |j|, |i_1|, |i_2|, \ldots, |i_{2n+1}|, |i_{2n+2}|)$

iff

$\Diamond^{\lambda} @_{i_1} \Diamond_{\sigma} (i_2, \ldots, i_{2n+2}) \in \Gamma$ and $\Diamond^{\lambda} @_{i} \Diamond_{\sigma\circ} (j, i_1 \ldots, i_{2n+2}) \notin \Gamma$

or

$\Diamond^{\lambda} @_{i} \Diamond_{\sigma\bullet} (j, i_1 \ldots, i_{2n+2}) \in \Gamma$

iff

$\Diamond^{\lambda} @_{i_1} \Diamond_{\sigma} (i_2, \ldots, i_{2n+2}) \wedge \Diamond^{\lambda} \neg @_{i} \Diamond_{\sigma\circ} (j, i_1 \ldots, i_{2n+2}) \vee \Diamond^{\lambda} @_{i} \Diamond_{\sigma\bullet} (j, i_1 \ldots, i_{2n+2}) \in \Gamma$

iff (*Dyn*, $Gen'_{\Diamond^{\lambda}}$ and Lemma 6.2.4)

$\Diamond^{\lambda} \Diamond^{(i,j)} @_{i_1} \Diamond_{\sigma} (i_2, \ldots, i_{2n+2}) \in \Gamma$

iff (Lemma 6.2.8)

$\Diamond^{\lambda(i,j)} @_{i_1} \Diamond_{\sigma} (i_2, \ldots, i_{2n+2}) \in \Gamma$.

□

Lemma 6.2.10 (Truth lemma)**.** *For all $\Diamond$-saturated and $\Diamond_{\sigma}$-saturated $L_s + \Lambda$-MCS's Γ, nominals i and formulas φ,*

$$\mathfrak{M}, (|i|, \langle \lambda \rangle) \models \varphi \textit{ iff } \Diamond^{\lambda} @_i \varphi \in \Gamma.$$

Proof. By induction on the length of φ.

- propositional symbols and nominals by definition;

- $\varphi = \neg\psi$

  $\mathfrak{M}_{\Gamma}, (|i|, \langle \lambda \rangle) \models \neg\psi$ iff $\mathfrak{M}_{\Gamma}, (|i|, \langle \lambda \rangle) \not\models \psi$ and $\mathfrak{M}_{\Gamma}, (|i|, \langle \lambda \rangle) \models \top$

  iff (IH)

  $\Diamond^{\lambda} @_i \psi \notin \Gamma$ and $\Diamond^{\lambda} @_i \top \in \Gamma$ iff $\neg \Diamond^{\lambda} @_i \psi \wedge \Diamond^{\lambda} \top \in \Gamma$

iff ($Selfdual_@$, $Gen'_{\diamondminus^\lambda}$ and $Det_{\diamondminus^\lambda}$)

$\diamondminus^\lambda @_i \neg\psi$;

- $\varphi = \psi_1 \to \psi_2$ as in the previous case we apply $K_@$, $K^{-1}_@$, $Gen'_{\diamondminus^\lambda}$ and $Det_{\diamondminus^\lambda}$;
- $\varphi = @_j\psi$ apply *Agree* and $Gen'_{\diamondminus^\lambda}$;
- $\varphi = \Phi_\sigma\psi$;

  If $\mathfrak{M}_\Gamma, (|i|, \langle\lambda\rangle) \models \Phi_\sigma(\psi_1, \ldots, \psi_{2n+1})$ then there are $|j_1|, \ldots, |j_{2n+1}|$ such that

  $$\check{R}^\sigma_{\langle\lambda\rangle}(|i|, |j_1|, \ldots, |j_{2n+1}|)$$

  and $\mathfrak{M}_\Gamma, (|j_l|, \langle\lambda\rangle) \models \psi_l$. By definition $\diamondminus^\lambda @_i \Phi_\sigma\ (j_1, \ldots, i_{2n+1}) \in \Gamma$ and (IH) $@_{j_l}\psi_l \in \Gamma$ for $1 \leq l \leq 2n+1$. Thus, using $Bridge_{\Phi_\sigma}$, $Gen'_{\diamondminus^\lambda}$ and Lemma 6.2.4, we have that

  $$\diamondminus^\lambda @_i \Phi_\sigma\ (\varphi_1, \ldots, \varphi_{2n+1}) \in \Gamma.$$

  Conversely, suppose $\diamondminus^\lambda @_i \Phi_\sigma\ (\varphi_1, \ldots, \varphi_{2n+1}) \in \Gamma$ then by $\diamondminus^\lambda$-saturation there are $j_1, \ldots, j_{2n+1}$ such that $\diamondminus^\lambda @_i \Phi_\sigma\ (j_1, \ldots, j_{2n+1}) \in \Gamma$ and $\diamondminus^\lambda @_{j_l}\psi_l \in \Gamma$ for $1 \leq l \leq 2n+1$. By IH $\mathfrak{M}_\Gamma, (|j_l|, \langle\lambda\rangle) \models \psi_l$ and by definition $\check{R}^\sigma_{\langle\lambda\rangle}(|i|, |j_1|, \ldots, |j_{2n+1}|)$. Hence,

  $$\mathfrak{M}_\Gamma, (|i|, \langle\lambda\rangle) \models \Phi_\sigma(\psi_1, \ldots, \psi_{2n+1}).$$

- $\varphi = \diamondminus\psi$

  $\mathfrak{M}_\Gamma, (|i|, \langle\lambda\rangle) \models \diamondminus\varphi$ iff

  exists $|j|$ such that $\langle\lambda\rangle\ (|i|, |j|) \in \Delta$ such that $\mathfrak{M}_\Gamma, (|j|, \langle\lambda\rangle\ (|i|, |j|)) \models \diamondminus\varphi$ iff (I.H.)

  exists $|j|$ such that $\diamondminus^{\lambda(i,j)} @_j\varphi = \diamondminus^\lambda @_i \diamondminus (j \wedge @_j\varphi) \in \Gamma$ iff (using $\diamondminus$-saturation and CT)

  $\diamondminus^\lambda @_i \diamondminus \varphi \in \Gamma$.

□

Lemma 6.2.11 (Frame lemma)**.** *For all $\diamondminus$-saturated and Φ_σ-saturated $L_s + \Lambda$-MCS's, $\mathfrak{M}_\Gamma$ satisfies the switch frame properties defined by Λ.*

Proof. It follows from the fact that $\Lambda \subseteq \Gamma$ contains $\boxminus^\lambda @_i\varphi$ where φ is an instance of an element of Λ from $Subst$, $Gen_@$ and $Gen_{\boxminus^\lambda}$. □

We can now prove Theorem 6.2.2.

Proof. (Of Theorem 6.2.2) Suppose Σ is $L_s + \Lambda$-consistent. By Lemma 6.2.6, Σ can be extended to a named, $\diamondsuit$-saturated and $\diamondsuit_\sigma$-saturated $L_s + \Lambda$-MCS's Γ. Let $i \in \Sigma$. By Lemma 6.2.10 we have $\mathfrak{M}_\Gamma, (|i|, \langle\epsilon\rangle) \models \Sigma$. By Lemma 6.2.11, $\mathfrak{M}_\Gamma$ satisfies all required frame properties. □

6.3 Discussion

Decidability and f.m.p. Given a switch graph in $S = (S, R)$ with an infinite behaviour Δ_S, the corresponding switch frame is infinite and so are the models over it. Moreover, unless there is a point where we turn off all the edges, regardless of the edges we cross, the models over that switch graph will be infinite.

Nevertheless, when we consider a satisfiable formula φ, that is, for some $(w, \lambda) \in W \times \Delta_S$ and valuation μ, we have:

$$(W \times \Delta_S, R_S, \mu), (w, \lambda) \models \varphi.$$

Clearly this is determined by just a part of the whole model. We can start by throwing out the past. Let $S' = (W, R_\lambda)$, and $\mu'(p) = \{(w, \lambda') : (w, \lambda\lambda') \in \mu(p)\}$, then it is obvious that

$$(W \times \Delta_{S'}, R_{S'}, \mu'), (w, \epsilon) \models \varphi.$$

So we can assume that $\lambda = \epsilon$. It is also clear that only the fragment corresponding to the paths bounded by the $\diamondsuit$-modal nesting-depth of φ, $md_\diamondsuit(\varphi)$, matters. Furthermore, the level of the relevant switches is bounded by $n + md_\diamondsuit(\varphi)$, where n is the level of the highest switch in the formula. Thus, the satisfability of φ is reduced to checking the satisfability over certain Kripke models corresponding to that kind of fragment. Of course, if W is infinite these fragments are still infinite. The easiest way to finitise such a fragment would be to adapt the filtration method by identifying only the points that satisfy the same relevant formulas in all of the fragment components.

We start by noticing that this method fails over general switch models. Consider the following switch graph

$$S = (\omega, \{(0, n), (0n, nn, \circ) : n < \omega\}),$$

see Figure 6.5, and $\diamondsuit\diamondsuit\top \in \epsilon \subseteq \mathcal{H}_s(@)$.

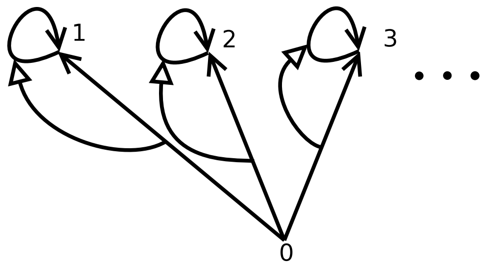

Figure 6.5: A switch graph originating a filtration problem.

Given a model over the switch frame generated by S it is clear that no two points will be identified since $(n, 0n)' \models' \neg \diamondsuit \diamondsuit \top$ and $(n, 0m)' \models' \diamondsuit \diamondsuit \top$ for $n \neq m$.

We were also unable to adapt the game-based argument to establish the PSPACE upper bound for the satisfability problem of $\mathcal{H}(@)$, see [4, 10]. The problem is, when adding a new world to a particular component, how to update the other components without falling in a infinite loop of verifications.

And again we ask:

Question 6.3.1. *What is the appropriate notion of quasimodel in the context of reactive hybrid switch logics?*

Usage of non-orthodox rules The use of non-structural rules is not completely consensual. Indeed in Chapter 3, the considered axiomatising problem allowed only the use of the classic modal logic inference rules: generalisation and modus ponens. Actually, if one allows the use of Gabbay's irreflexivity rule then the commutator is enough to axiomatise **K4.3** × **K**, see [68]. We refer to [75] for an introduction on the use of these rules to define more classes of Kripke frames, and to [49] for a characterisation of its power in the pure modal logic context.

In hybrid logics, completeness results are usually obtained using these non-orthodox inference rules and we did not attempt to do it otherwise, see [46] for an approach to the axiomatisation of hybrid logics using only orthodox rules.

Relation with Reactive Logics Unfortunately there are no obvious connections (e.g. translations) between these switch logics and the ones studied in Chapter 4. Even if we can extract a reactive frame from a switch graph by considering its local behaviour (its paths without jumps), the $\Diamond_P$ operator allows a global access over all the reactive states of a point that cannot be mimicked by the operators considered here. The two approaches are two transversal extensions of classical modal logic with different expressivity. Applications may assert which should be developed or even determine the necessity of joining both views.

Further Extensions Many other hybrid operators have been studied and it would be interesting to investigate how they would play in this context. A particularly natural extension of these logics would be to enrich $\mathcal{H}_s(@)$ with computational tree logic's (CTL) operators ([13]) or even μ-calculus ([69]). This would greatly reinforce our ability to reason about the behaviour of a switch graph and increase its usability. The way this would be done is not completely clear and it would depend on one's particular interests.

If we want to reason about k agents acting in the graph (in Section 5.3 there are such examples) and they are not allowed to jump, we are not interested in the whole set of sequences of edges, but only in the ones that the agents may cross, which are determined by their position at each moment. To express these restrictions, it could be necessary to include in the language the possibility of referring explicitly to the distribution of the agents in the graph. In this direction, a possibility would be to model each agent's location by variables, maybe even introduce another sort of variables to deal with it like we did with nominals. In the used language we would be able to express for example that each agent can be only in one point at a time:

$$@_i(a \wedge \neg j) \rightarrow @_j \neg a,$$

considering a to be an agent variable. Also, to express their movement we could say that

$$@_i(a \wedge \diamondminus j) \rightarrow \boxslash(j \rightarrow (a \wedge @_i \neg a)).$$

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