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**End invariants for  $SL(2, \mathbb{C})$  characters of the one-holed torus. (English summary)**

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A geodesic lamination on a hyperbolic surface is a foliation of a closed subset of the surface whose leaves are geodesics. A measured lamination is a pair  $(\lambda, \mu)$ , where  $\lambda$  is a geodesic lamination and  $\mu$  is a transverse measure defined on each local leaf-space that is invariant under coordinate changes. Let  $\Sigma_{1,1}$  be the (closed) 1-holed torus and let  $\mathcal{PL}$  be the space of projective laminations on  $\Sigma_{1,1}$ , that is, the space of equivalence classes of measured laminations where  $(\lambda, \mu_1) \sim (\lambda, \mu_2)$  if and only if  $\mu_1/\mu_2 \in (0, \infty)$ .

Let  $\mathcal{C} \cong \mathbb{QP}^1$  be the set of free homotopy classes of unoriented, nontrivial, non-boundary, simple, closed curves in  $\Sigma_{1,1}$ . Then  $\mathcal{PL} \cong \mathbb{RP}^1$  is the completion of  $\mathcal{C}$ .

Let  $\mathfrak{X}$  be the moduli space of  $SL_2(\mathbb{C})$ -valued representations of  $\pi_1(\Sigma_{1,1})$ . It is a classical result of Fricke and independently Vogt that  $\mathfrak{X} \cong \mathbb{C}^3$ . For any  $[\rho]$  in  $\mathfrak{X}$  and  $k > 0$  define  $\mathcal{E}_k(\rho) = \{\lambda \in \mathcal{PL} \mid \exists X_n \in \mathcal{C} \text{ such that } X_n \rightarrow \lambda \text{ and } |\text{tr } \rho(X_n)| \leq k\}$ , and let the set  $\mathcal{E}(\rho)$  of end invariants be the union over  $k$  of  $\mathcal{E}_k(\rho)$ .

Let  $X$  and  $Y$  generate the fundamental group of  $\Sigma_{1,1}$ . Then the space of representations of the fundamental group into  $SL_2(\mathbb{C})$  is isomorphic to  $SL_2(\mathbb{C}) \times SL_2(\mathbb{C})$  by the evaluation mapping. As it is a variety, it has a coordinate ring and  $\mathfrak{X}$  parametrizes the maximal ideals of the subring of conjugation invariant polynomials. In these terms the Fricke-Vogt isomorphism is given by

$$[\rho] \mapsto (\text{tr } \rho(X), \text{tr } \rho(Y), \text{tr } \rho(XY)) \in \mathbb{C}^3.$$

A class of representations  $[\rho] = (x, y, z)$  in  $\mathfrak{X} \cong \mathbb{C}^3$  is called dihedral if two of its coordinates are zero, real if all its coordinates are real, and imaginary if it is not dihedral and two of its coordinates are purely imaginary and the third is real.

The first result in the paper under review establishes  $\mathcal{E}(\rho) = \mathcal{PL}$  if and only if  $[\rho]$  is dihedral or  $[\rho]$  corresponds to an  $SU(2)$ -valued representation. In all other cases the set of end invariants have empty interior. At the other extreme the set of end invariants is empty if and only if they satisfy the extended Bowditch conditions: (1)  $\text{tr } \rho(X) \notin (-2, 2)$  for all  $X \in \mathcal{C}$  and (2)  $|\text{tr } \rho(X)| \leq 2$  for at most finitely many  $X \in \mathcal{C}$ .

Next the authors specialize to representations that satisfy certain conditions. It is shown that with respect to a reducible representation  $[\rho]$ , the set of end invariants is either  $\mathcal{PL}$  or a singleton set  $\{X\}$ . Also, the authors provide explicit algebraic conditions which completely classify  $\mathcal{E}(\rho)$  for irreducible real representations and also for imaginary representations. In all cases,  $\mathcal{E}(\rho)$  is either  $\emptyset$ ,  $\mathcal{PL}$ ,  $\{X\}$ , or a Cantor set.

The last result of this paper states that if  $\{\text{tr } \rho(X) \mid X \in \mathcal{C}\}$  is discrete and  $\mathcal{E}(\rho) \neq \mathcal{PL}$  and has at least 3 elements, then the end invariants form a Cantor set.

This work builds on past work of the authors [S. P. Tan, Y. L. Wong and Y. Zhang, *Geom.*

Dedicata **119** (2006), 199–217; [MR2247658 \(2007e:57017\)](#); Adv. Math. **217** (2008), no. 2, 761–813; [MR2370281 \(2008k:57035\)](#)] and is strongly motivated, both in method and content, by the previous work of B. H. Bowditch [Proc. London Math. Soc. (3) **77** (1998), no. 3, 697–736; [MR1643429 \(99f:57014\)](#)].

As such, the proofs rely on realizing  $\mathcal{C}$  as the connected components of the complement of the dual tree to the Farey tessellation (by ideal triangles) of the Poincaré disk. Limits along sub-trees correspond to elements in  $\mathcal{PL}$ . The authors use an intimate knowledge of the flows (determined by the traces of generating triples of the representation) along the dual tree and the subsequent connectivity of sub-trees determined by the flow.

The main idea is that the complementary regions bound subtrees which limit to laminations. Since the regions correspond to closed laminations this gives the authors control over the boundedness of the limits of the flows which themselves are determined by the values of the traces of the representations along the tree.

The explicit classification results obtained rely on understanding the algebraic structure of the relative character varieties, that is, affine hypersurfaces in  $\mathcal{X} \cong \mathbb{C}^3$  obtained by fixing the value of  $[\rho]$  on the boundary  $\partial\Sigma_{1,1} \cong S^1$ . The boundary of the 1-holed torus corresponds to the commutator of  $X$  and  $Y$ . The explicit form of this polynomial, obtainable via trace reduction formulas, is  $x^2 + y^2 + z^2 - xyz - 2$ . Along these lines results of W. Goldman and G. Stantchev [“Dynamics of the automorphism group of the  $GL(2, \mathbb{R})$ -characters of a once-punctured torus”, preprint, arxiv.org/abs/math/0309072] concerning mapping class group dynamics on the real and imaginary representations are utilized. In fact, this work may be considered as part of a larger project to understand the finer aspects (properness and non-properness) of the mapping class group action at various classes  $[\rho]$ .

Reviewed by *Sean Lawton*

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*Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.*

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