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Rationality problem of three-dimensional purely monomial group actions: the last case. (English summary)

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Let k be a field and $K = k(x_1, x_2, \ldots, x_n)$ be the field of rational functions over k in n variables. Let G be a finite subgroup of $\mathrm{GL}_n(\mathbb{Z})$ acting trivially on k. Let $(g_{ij}) \in G$. Then the action of G on K given by $x_j \mapsto \prod_{i=1}^{n} x_i^{g_{ij}}$ is k-automorphic (such actions are called purely monomial). The "rationality problem" is to determine if the field of invariants K^G is isomorphic to a rational function field over k. In other words, is $K^G \subset K$ a purely transcendental extension field of k? If the answer is yes, then the field of invariants is said to be k-rational.

For dimension one (i.e., $n = 1$), $k(x)^G$ is k-rational since the only subgroups G are $\langle 1 \rangle$ and Z_2. In the first case the invariant field is $k(x)$ itself and in the second case the invariant field is $k(x + x^{-1})$. Either way $k(x)^G$ is k-rational. For dimension two, in [M. Hajja, J. Algebra **109** (1987), no. 1, 46–51; MR0898335 (88j:12002)] it was shown that for any finite $G < \mathrm{GL}_2(\mathbb{Z})$ the field of invariants $k(x_1, x_2)^G$ is k-rational.

Now let G_0 be the finite subgroup of $\mathrm{GL}_3(\mathbb{Z})$ generated by

$$\begin{pmatrix} 1 & 1 & 0 \\ -2 & -1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} -1 & -1 & -1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Then in [M. Hajja and M. C. Kang, J. Algebra **149** (1992), no. 1, 139–154; MR1165204 (93d:12009); J. Algebra **170** (1994), no. 3, 805–860; MR1305266 (95k:12008)] it was established that for any finite $G < \mathrm{GL}_3(\mathbb{Z})$ that is not conjugate to G_0 (there exist exactly 72 conjugacy classes of such groups), the field of invariants $k(x_1, x_2, x_3)^G$ is again k-rational. This solves the k-rationality problem in dimensions one, two and three except in only one case.

The main result in the paper under review furthers the above work by proving: for any field k and any group G conjugate to G_0 the field of invariants $k(x_1, x_2, x_3)^G$ is also k-rational. In the proof, the authors first observe that G_0 is isomorphic to the symmetric group on 4 letters, and then the G_0 action is decomposed into subactions given by normal subgroups of G_0 (or more specifically a conjugate of G_0, but this is not important since all conjugates give the same invariant field up to isomorphism). The authors then explicitly compute the field of invariants of these normal subgroups (smaller groups give bigger invariant fields) working their way to the full action. At the last step in the process they construct generators of the invariant field that satisfy a set of general conditions that, together with the dimension two result, imply k-rationality.

It is then natural to wonder if there are any counterexamples at all. But according to Hajja and Kang [op. cit., 1994], k-rationality fails in dimension four for a certain finite abelian purely monomial action.

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