Poisson structures on moduli of surface group representations into $\mathrm{SL}(3, \mathbb{C})$

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Outline of Presentation

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- Moduli of surface group representations: character varieties

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- Symplectic foliation and Goldman's bracket

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- Poisson structure relative to a one-holed torus

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- Let $\mathfrak{G} = \mathrm{SL}(3,\mathbb{C})$. For much of what follows, any reductive linear algebraic group will suffice.
- $\mathfrak{R} = \mathrm{Hom}(\mathtt{F}_r,\mathfrak{G})$ is bijectively equivalent to $\mathfrak{G}^{\times r}$:

$$\rho \mapsto (\rho(\mathtt{x}_1),...,\rho(\mathtt{x}_r)),$$

and so inherits the structure of an affine variety from $\mathfrak{G}$.

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- Thus, $\mathbb{C}[\mathfrak{R}]^{\mathfrak{G}}$ is a finitely generated domain, and so its geometric points are an irreducible algebraic set, $\mathfrak{X}$, called the character variety.

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- Bottom line: either throw out points or make further identifications to get a quotient that is an algebraic set and parameterizes subvarieties of representations.
- Why? Because varieties have interesting structure and the usual orbit space isn't even Hausdorff.

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- So $\mathfrak{G}/\!\!/\mathfrak{G} = \mathbb{C}^2$ which we parameterize by coordinates $(T^{(1)}, T^{(-1)})$.
- We then define the boundary map

$$\mathfrak{b}_i : \mathfrak{X} = \mathfrak{R}/\!\!/\mathfrak{G} = \mathrm{Hom}(\pi_1(S_{n,g}, *), \mathfrak{G})/\!\!/\mathfrak{G} \longrightarrow \mathfrak{G}/\!\!/\mathfrak{G}$$

by sending a representation class $[\rho] \mapsto [\rho_{|_{\mathsf{b}_i}}] = (T_i^{(1)}, T_i^{(-1)})$, to the class corresponding to the restriction of ρ to the boundary b_i.

- Subsequently we define

$$\mathfrak{b} = (\mathfrak{b}_1, ..., \mathfrak{b}_n) : \mathfrak{X} = \mathfrak{G}^{\times r} /\!\!/ \mathfrak{G} \longrightarrow (\mathfrak{G} /\!\!/ \mathfrak{G})^{\times n}.$$

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- The map $\mathfrak{b}$ depends on the surface, not only its fundamental group.
- We refer to it as a peripheral structure, and the pair $(\mathfrak{X}, \mathfrak{b})$ as the relative character variety.

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- So $\mathcal{X}$ is a complex manifold that is dense in $\mathfrak{X}$.
- At regular values of $\mathfrak{b}$, $\mathfrak{F} \cap \mathcal{X}$ is a submanifold of dimension $8r - 8 - 2n = 16(g-1) + 6n$.
- The union of these leaves, $\mathcal{F} = \mathfrak{F} \cap \mathcal{X}$, foliate $\mathcal{X}$ by symplectic submanifolds, making $\mathcal{X}$ a Poisson manifold.

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- The tangent space to a representation in $\mathfrak{X}$ is generically $H^1(S; \mathfrak{g}_{\mathrm{Ad}})$.
- The subset of cocycles that are zero on ∂S in cohomology form a distribution corresponding to $\mathfrak{F}$: $H^1_{\mathrm{par}}(S; \mathfrak{g}_{\mathrm{Ad}})$.

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H^1(S, \partial S; \mathfrak{g}_{\mathrm{Ad}}) \times H^1(S; \mathfrak{g}_{\mathrm{Ad}}) & \xrightarrow{\cup} & H^2(S, \partial S; \mathfrak{g}_{\mathrm{Ad}} \otimes \mathfrak{g}_{\mathrm{Ad}}) \\
\uparrow & & \downarrow \mathrm{tr}_* \\
 & & H^2(S, \partial S; \mathbb{C}) \\
 & & \downarrow \cap [Z] \\
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With respect to this 2-form, Goldman's proof of the bracket formula generalizes directly to relative cohomology.

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Then Goldman showed in 1986 that

$$\{\mathrm{tr}(\rho(\alpha)), \mathrm{tr}(\rho(\beta))\} = \sum_{p \in \alpha \cap \beta} \epsilon(p, \alpha, \beta)\big((\mathrm{tr}(\rho(\alpha_p \beta_p)) - (1/3)\mathrm{tr}(\rho(\alpha))\mathrm{tr}(\rho(\beta))\big)$$

defines a Poisson Bracket on $\mathbb{C}[\mathfrak{X}]$.

Euler Characteristic -1 Surfaces

There are exactly two oriented surfaces with fundamental group free of rank 2.

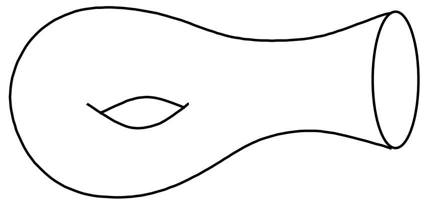

Figure 1: One-Holed Torus

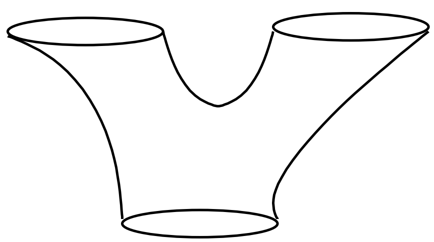

Figure 2: Three-Holed Sphere

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$$R = \mathbb{C}[t_{(1)}, t_{(-1)}, t_{(2)}, t_{(-2)}, t_{(3)}, t_{(-3)}, t_{(4)}, t_{(-4)}]$$

be a subring.

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$$t_{(3)} \mapsto \mathrm{tr}(\mathbf{X}_1\mathbf{X}_2) \qquad t_{(-3)} \mapsto \mathrm{tr}(\mathbf{X}_1^{-1}\mathbf{X}_2^{-1})$$
$$t_{(4)} \mapsto \mathrm{tr}(\mathbf{X}_1\mathbf{X}_2^{-1}) \qquad t_{(-4)} \mapsto \mathrm{tr}(\mathbf{X}_1^{-1}\mathbf{X}_2)$$
$$t_{(5)} \mapsto \mathrm{tr}(\mathbf{X}_1\mathbf{X}_2\mathbf{X}_1^{-1}\mathbf{X}_2^{-1})$$

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- Hence, $\ker(\Pi)$ is non-zero and principal.

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- The action is induced by the following elements of the $\mathrm{Out}(\mathtt{F}_2)$:

$$\mathfrak{t} = \begin{cases} \mathtt{x}_1 \mapsto \mathtt{x}_2 \\ \mathtt{x}_2 \mapsto \mathtt{x}_1 \end{cases} \qquad \mathfrak{i}_1 = \begin{cases} \mathtt{x}_1 \mapsto \mathtt{x}_1^{-1} \\ \mathtt{x}_2 \mapsto \mathtt{x}_2 \end{cases}$$

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- The group generated has order 8 and is isomorphic to the dihedral group D_4.

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3. *There is a* D_4*-equivariant surjection* $\mathfrak{X} \to \mathbb{C}^8$*, generically* 2*-to-*1*.*
4. P *and* Q *are given by:*

$$P = \mathbb{S}\Big(\frac{1}{8}\big(t_{(1)}t_{(-1)}t_{(2)}t_{(-2)} - 4t_{(1)}t_{(-2)}t_{(-4)} + 2t_{(1)}t_{(-1)} + 2t_{(3)}t_{(-3)}\big)\Big) - 3$$

$$\begin{aligned} Q = \mathbb{S}\Big(\frac{1}{8}\big(&2t_{(-2)}t_{(-1)}^2t_{(1)}^2t_{(2)} + 4t_{(1)}^2t_{(2)}^2t_{(3)} - 4t_{(1)}^3t_{(-2)}t_{(2)} - \\ &8t_{(-4)}t_{(-2)}t_{(-1)}t_{(1)}^2 - 4t_{(4)}t_{(3)}t_{(2)}t_{(1)}t_{(-2)} + 8t_{(1)}t_{(3)}t_{(-4)}^2 + \\ &8t_{(-4)}t_{(1)}t_{(2)}^2 - 8t_{(3)}^2t_{(2)}t_{(1)} + 4t_{(4)}t_{(-3)}t_{(2)}^2 + t_{(-2)}t_{(-1)}t_{(2)}t_{(1)} + \\ &t_{(-3)}t_{(-4)}t_{(3)}t_{(4)} + 4t_{(-3)}t_{(-1)}t_{(3)}t_{(1)} + 4t_{(1)}^3 + 4t_{(3)}^3 + \\ &12t_{(-4)}t_{(-2)}t_{(1)} - 12t_{(-4)}t_{(2)}t_{(3)} - 12t_{(1)}t_{(-1)} - 12t_{(3)}t_{(-3)}\big)\Big) + 9 \end{aligned}$$

Three-Holed Sphere

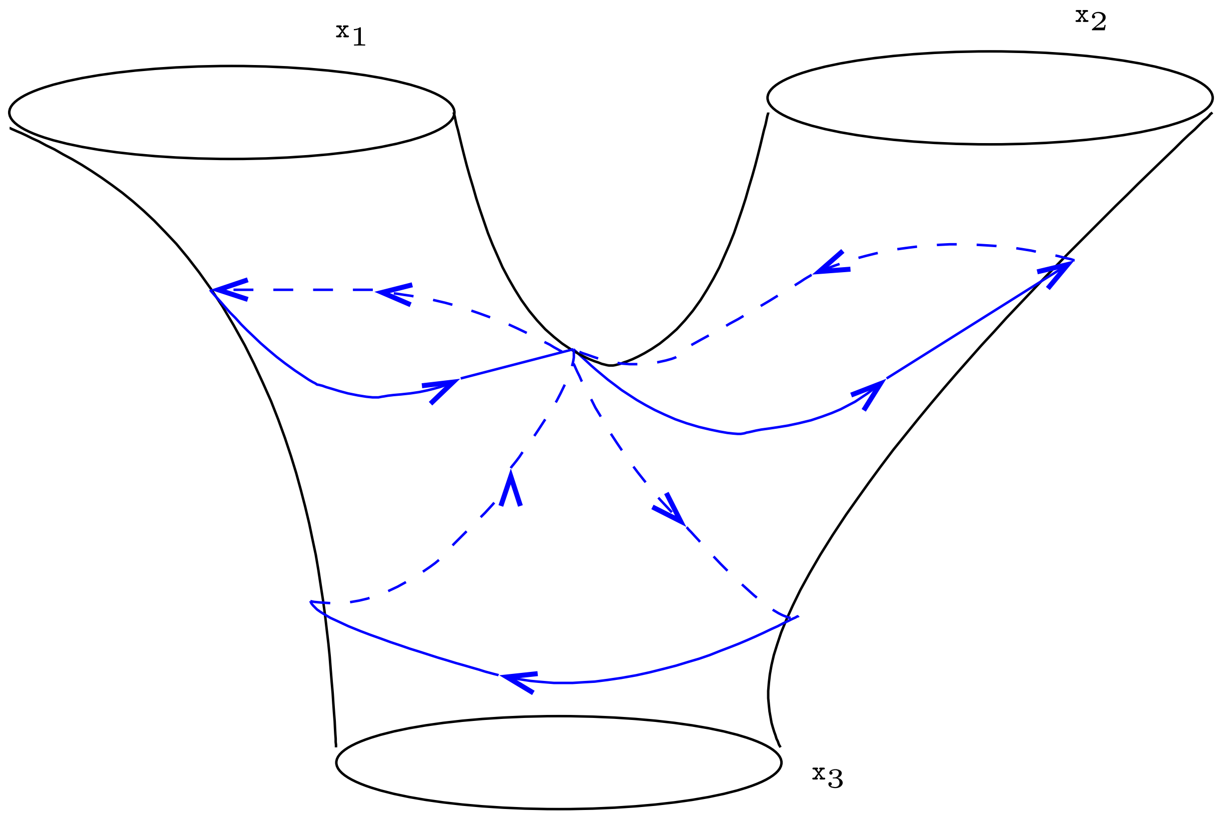

Figure 3: Presentation of $\pi_1(S_{3,0}, *)$

Theorem 0.2 (L, 2006)**.** *Let* $\mathfrak{X}$ *be the relative character variety of* $S = S_{3,0}$*. Then the Poisson bracket on* $\mathbb{C}[\mathfrak{X}]$ *is determined by the following:*

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Therefore, the Poisson bivector field is:

$$(P - 2t_{(5)})\frac{\partial}{\partial t_{(4)}} \wedge \frac{\partial}{\partial t_{(-4)}} + (1 - \mathfrak{i})\left(\mathfrak{a}_{4,5}\frac{\partial}{\partial t_{(4)}} \wedge \frac{\partial}{\partial t_{(5)}}\right)$$

Sketch of Proof

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The fundamental group of $S = S_{3,0}$ is geometrically presented as

$$\{\mathtt{x}_1, \mathtt{x}_2, \mathtt{x}_3 \ : \ \mathtt{x}_3\mathtt{x}_2\mathtt{x}_1 = 1\},$$

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- Since a Poisson bracket is a bilinear, anti-commutative derivation, it is completely determined once it is formulated on the generators of $\mathbb{C}[\mathfrak{X}]$.
- $t_{(\pm 1)}, t_{(\pm 2)}, t_{(\pm 3)}$ are Casimirs since they correspond to disjoint boundary curves and the bracket sums over intersections.

Using the derivation property and the identity

$$t_{(5)}^2 - Pt_{(5)} + Q = 0,$$

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$$= 2t_{(5)}\{t_{(\pm 4)}, t_{(5)}\}.$$

Hence

$$\{t_{(\pm 4)}, t_{(5)}\} = \frac{t_{(5)}\{t_{(\pm 4)}, P\} - \{t_{(\pm 4)}, Q\}}{(2t_{(5)} - P)}.$$

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- Consider immersed closed curves freely homotopic to $\alpha = \mathtt{x}_1\mathtt{x}_2^{-1}$ and $\beta = \mathtt{x}_2\mathtt{x}_1^{-1}$ only intersecting at transverse double points.
- Since $S_{3,0}$ is homotopic to a closed rectangle with two open disks removed, we depict all curves as in the following figure.

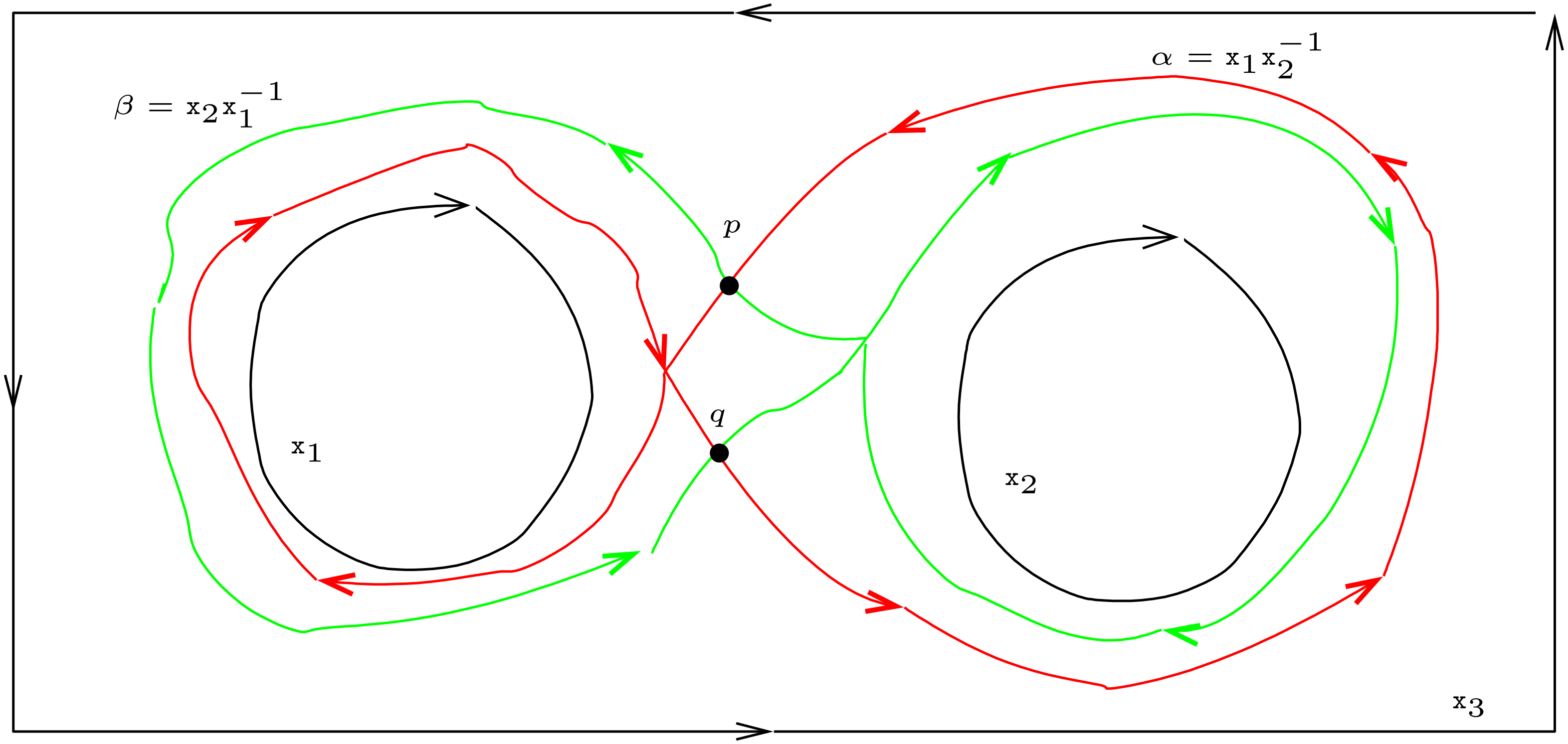

Figure 4: α and β in S

Let α_p and β_p be the curves corresponding to α and β based at the point p in $\pi_1(S, p)$.

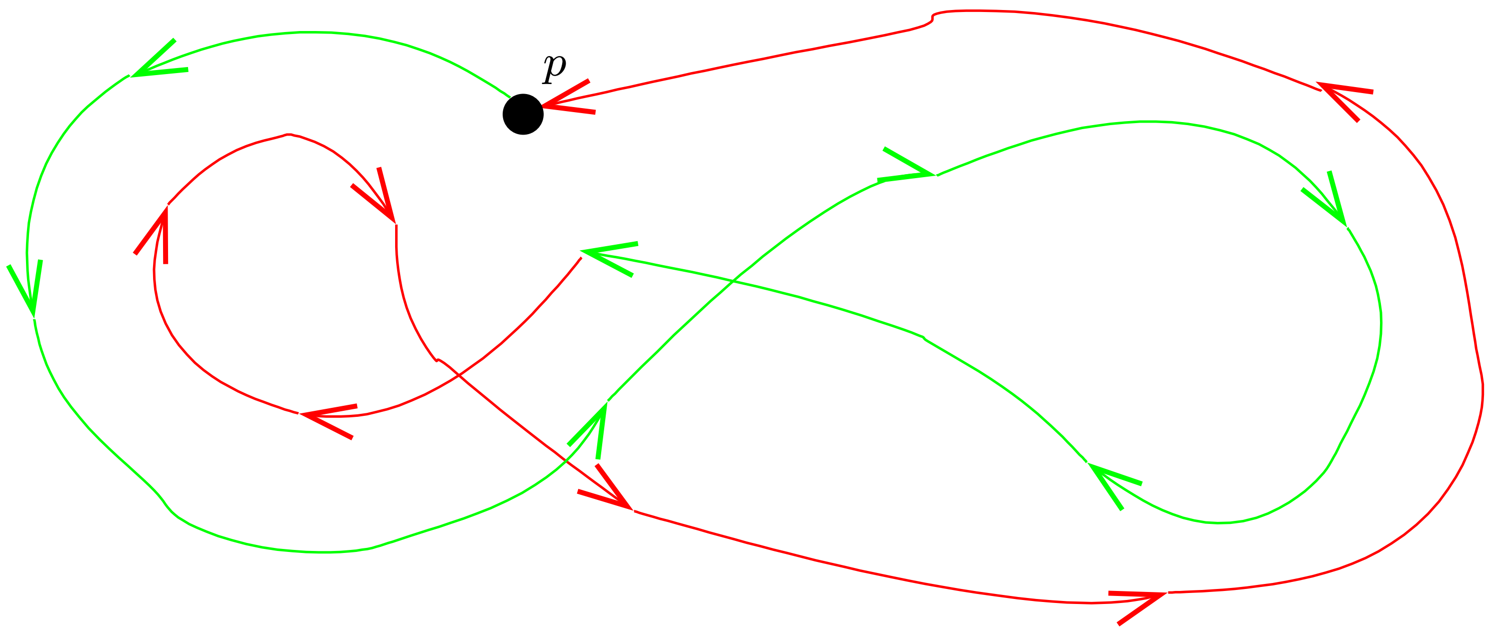

Figure 5: $\alpha_p\beta_p = \mathtt{x}_2^{-1}\mathtt{x}_1\mathtt{x}_2\mathtt{x}_1^{-1}$

Respectively, let α_q and β_q be the corresponding curves in $\pi_1(S, q)$.

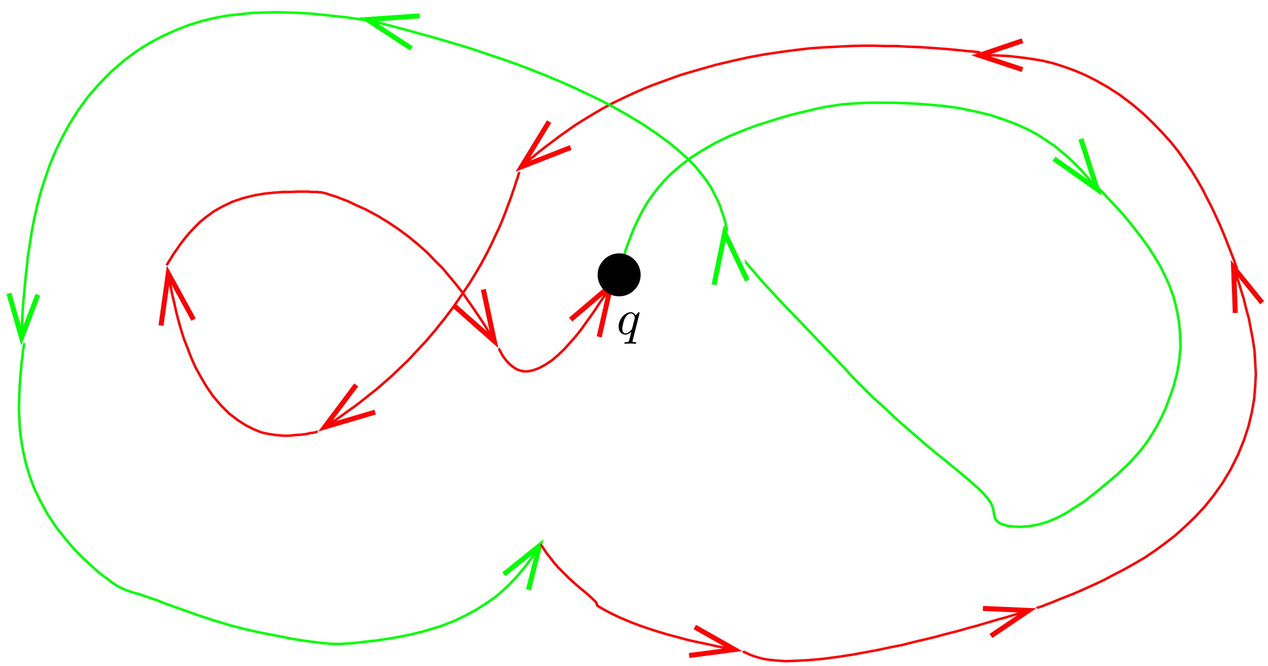

Figure 6: $\alpha_q \beta_q = \mathtt{x}_1 \mathtt{x}_2^{-1} \mathtt{x}_1^{-1} \mathtt{x}_2$

Calculating the **intersection number** at p and q we find $\epsilon(p, \alpha, \beta) = -1$ and $\epsilon(q, \alpha, \beta) = 1$.

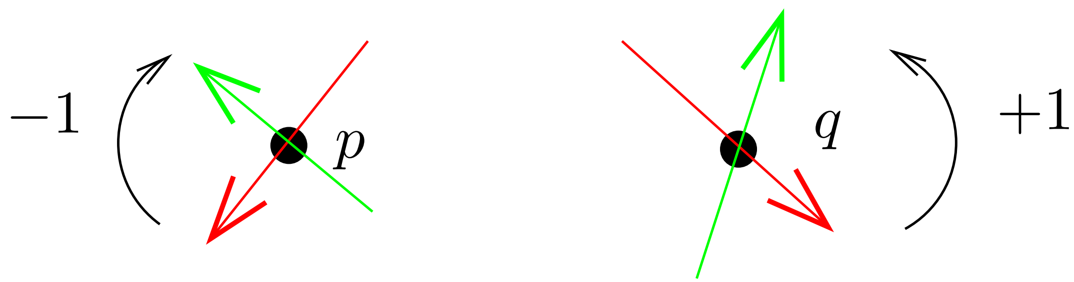

Figure 7: Intersection numbers at p and q

Putting it all together:

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$$\{t_{(4)}, t_{(-4)}\} = \{\mathrm{tr}(\rho(\alpha)), \mathrm{tr}(\rho(\beta))\}$$

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$$= \epsilon(p,\alpha,\beta)\big(\mathrm{tr}(\rho(\alpha_p\beta_p)) - (1/3)\mathrm{tr}(\rho(\alpha))\mathrm{tr}(\rho(\beta))\big) +$$
$$\epsilon(q,\alpha,\beta)\big(\mathrm{tr}(\rho(\alpha_q\beta_q)) - (1/3)\mathrm{tr}(\rho(\alpha))\mathrm{tr}(\rho(\beta))\big)$$

Putting it all together:

$$\{t_{(4)}, t_{(-4)}\} = \{\mathrm{tr}(\rho(\alpha)), \mathrm{tr}(\rho(\beta))\}$$

$$= \epsilon(p, \alpha, \beta)\big(\mathrm{tr}(\rho(\alpha_p\beta_p)) - (1/3)\mathrm{tr}(\rho(\alpha))\mathrm{tr}(\rho(\beta))\big) + \epsilon(q, \alpha, \beta)\big(\mathrm{tr}(\rho(\alpha_q\beta_q)) - (1/3)\mathrm{tr}(\rho(\alpha))\mathrm{tr}(\rho(\beta))\big)$$

$$= -\mathrm{tr}(\rho(\alpha_p\beta_p)) + \mathrm{tr}(\rho(\alpha_q\beta_q))$$

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$$= -\mathrm{tr}(\mathbf{X}_2^{-1}\mathbf{X}_1\mathbf{X}_2\mathbf{X}_1^{-1}) + \mathrm{tr}(\mathbf{X}_1\mathbf{X}_2^{-1}\mathbf{X}_1^{-1}\mathbf{X}_2)$$

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$$= -t_{(5)} + \mathrm{tr}(\mathbf{X}_2\mathbf{X}_1\mathbf{X}_2^{-1}\mathbf{X}_1^{-1})$$

Putting it all together:

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$$= -t_{(5)} + \mathrm{tr}(\mathbf{X}_2\mathbf{X}_1\mathbf{X}_2^{-1}\mathbf{X}_1^{-1})$$

$$= -t_{(5)} + (P - t_{(5)}) = P - 2t_{(5)}.$$

Putting it all together:

$$\{t_{(4)}, t_{(-4)}\} = \{\mathrm{tr}(\rho(\alpha)), \mathrm{tr}(\rho(\beta))\}$$

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$$= -\mathrm{tr}(\rho(\alpha_p\beta_p)) + \mathrm{tr}(\rho(\alpha_q\beta_q))$$

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QED

One-Holed Torus

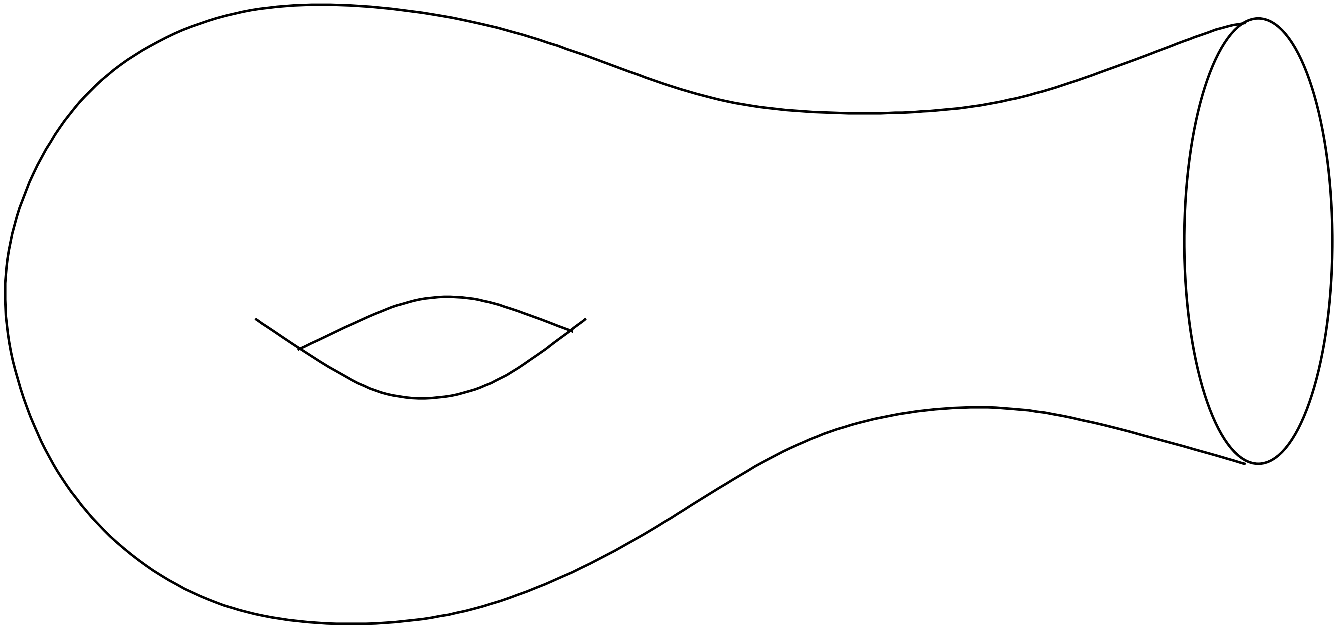

Figure 8: One-Holed Torus

- With respect to the presentation $\{\mathtt{x}_1, \mathtt{x}_2, \mathtt{x}_3 \ : \ [\mathtt{x}_1, \mathtt{x}_2]\mathtt{x}_3 = 1\}$ the boundary $\mathtt{x}_3$ corresponds to the inverse of the word $\mathtt{x}_1\mathtt{x}_2\mathtt{x}_1^{-1}\mathtt{x}_2^{-1}$.

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- Consequently, the only Casimir that is also a generator is $t_{(5)}$ since it corresponds to $\mathrm{tr}(\mathbf{X}_1\mathbf{X}_2\mathbf{X}_1^{-1}\mathbf{X}_2^{-1})$.

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- Consequently, the only Casimir that is also a generator is $t_{(5)}$ since it corresponds to $\mathrm{tr}(\mathbf{X}_1\mathbf{X}_2\mathbf{X}_1^{-1}\mathbf{X}_2^{-1})$.
- Thus the 81 pairings coming from the 9 generators is reduced to 64. Anti-commutativity reduces this number to 28 pairings.

- Observe that $\{t_{(i)}, t_{(-i)}\} = 0$ since the corresponding curves are homotopic in $S_{1,1}$ to parallel curves and so have *no intersection.*

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- We next address the pairings which come from cycles of word length one and so have only *one intersection*.
- Let α be homotopic to $\mathtt{x}_1$ and β be homotopic to $\mathtt{x}_2$, as in the next figure.

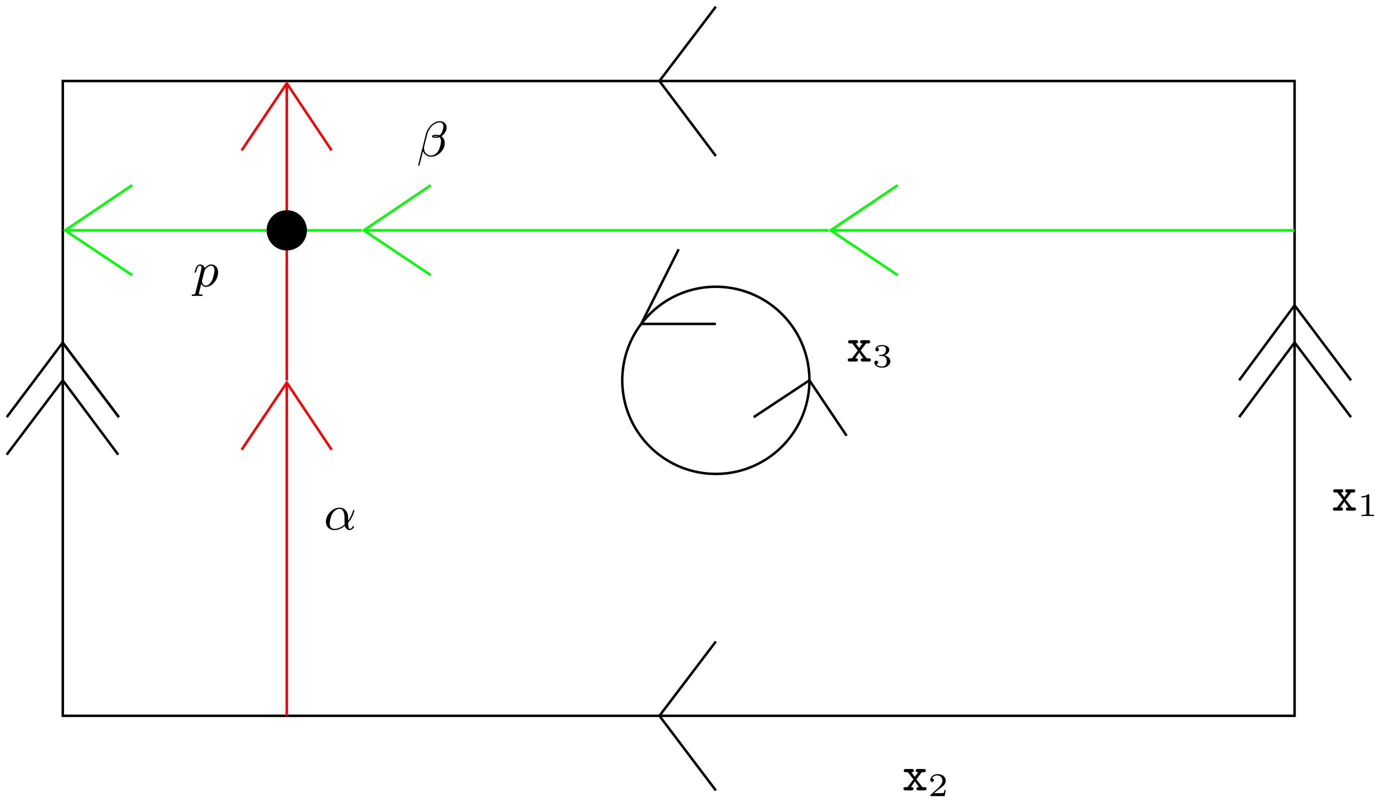

Figure 9: $\alpha = \mathtt{x}_1$ and $\beta = \mathtt{x}_2$ in $S_{1,1}$

Let $\epsilon(p, \alpha, \beta) = \epsilon_p$ to simplify the notation.

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$$\begin{aligned}\{t_{(1)},t_{(2)}\} &= \epsilon_p\big(\mathrm{tr}(\rho(\alpha_p\beta_p)) - (1/3)\mathrm{tr}(\rho(\alpha))\mathrm{tr}(\rho(\beta))\big)\\ &= \epsilon_p\big(\mathrm{tr}(\mathbf{X}_1\mathbf{X}_2) - (1/3)\mathrm{tr}(\mathbf{X}_1)\mathrm{tr}(\mathbf{X}_2)\big)\\ &= \mathrm{tr}(\mathbf{X}_1\mathbf{X}_2) - (1/3)\mathrm{tr}(\mathbf{X}_1)\mathrm{tr}(\mathbf{X}_2)\\ &= t_{(3)} - (1/3)t_{(1)}t_{(2)}.\end{aligned}$$

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Likewise, we can compute

$$\begin{aligned}\{t_{(-1)}, t_{(2)}\} &= -t_{(-4)} + (1/3)t_{(-1)}t_{(2)}\\ \{t_{(1)}, t_{(-2)}\} &= -t_{(4)} + (1/3)t_{(1)}t_{(-2)}\\ \{t_{(-1)}, t_{(-2)}\} &= t_{(-3)} - (1/3)t_{(-1)}t_{(-2)}\end{aligned}$$

We can already see symmetry coming from $D_4 \subset \mathrm{Out}(\mathtt{F}_2)$.
Define $\mathfrak{i}_2 = \mathfrak{t}\mathfrak{i}_1\mathfrak{t}$; the mapping which sends $\mathtt{x}_2 \mapsto \mathtt{x}_2^{-1}$. Then

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$$\{t_{(-1)}, t_{(2)}\} = -\mathfrak{i}_1\{t_{(1)}, t_{(2)}\} = \{\mathfrak{i}_1 t_{(1)}, \mathfrak{i}_1 t_{(2)}\}$$

$$\{t_{(1)}, t_{(-2)}\} = -\mathfrak{i}_2\{t_{(1)}, t_{(2)}\} = \{\mathfrak{i}_2 t_{(1)}, \mathfrak{i}_2 t_{(2)}\}$$

$$\{t_{(-1)}, t_{(-2)}\} = \mathfrak{i}\{t_{(1)}, t_{(2)}\} = \{\mathfrak{i}t_{(1)}, \mathfrak{i}t_{(2)}\}$$

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$$\{t_{(-1)}, t_{(2)}\} = -\mathfrak{i}_1\{t_{(1)}, t_{(2)}\} = \{\mathfrak{i}_1 t_{(1)}, \mathfrak{i}_1 t_{(2)}\}$$

$$\{t_{(1)}, t_{(-2)}\} = -\mathfrak{i}_2\{t_{(1)}, t_{(2)}\} = \{\mathfrak{i}_2 t_{(1)}, \mathfrak{i}_2 t_{(2)}\}$$

$$\{t_{(-1)}, t_{(-2)}\} = \mathfrak{i}\{t_{(1)}, t_{(2)}\} = \{\mathfrak{i} t_{(1)}, \mathfrak{i} t_{(2)}\}$$

We are left with 20 computations.

Next we consider curves which again intersect *only once*, but at least one of them has word length two.

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There will be two cases: either you have a repeated letter in the pair of words OR you get a cancelation after cyclic reduction.

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<u>Case 1</u>

For instance, let $\alpha = \mathtt{x}_1$ and $\beta = \mathtt{x}_1\mathtt{x}_2$ (both have the letter $\mathtt{x}_1$).

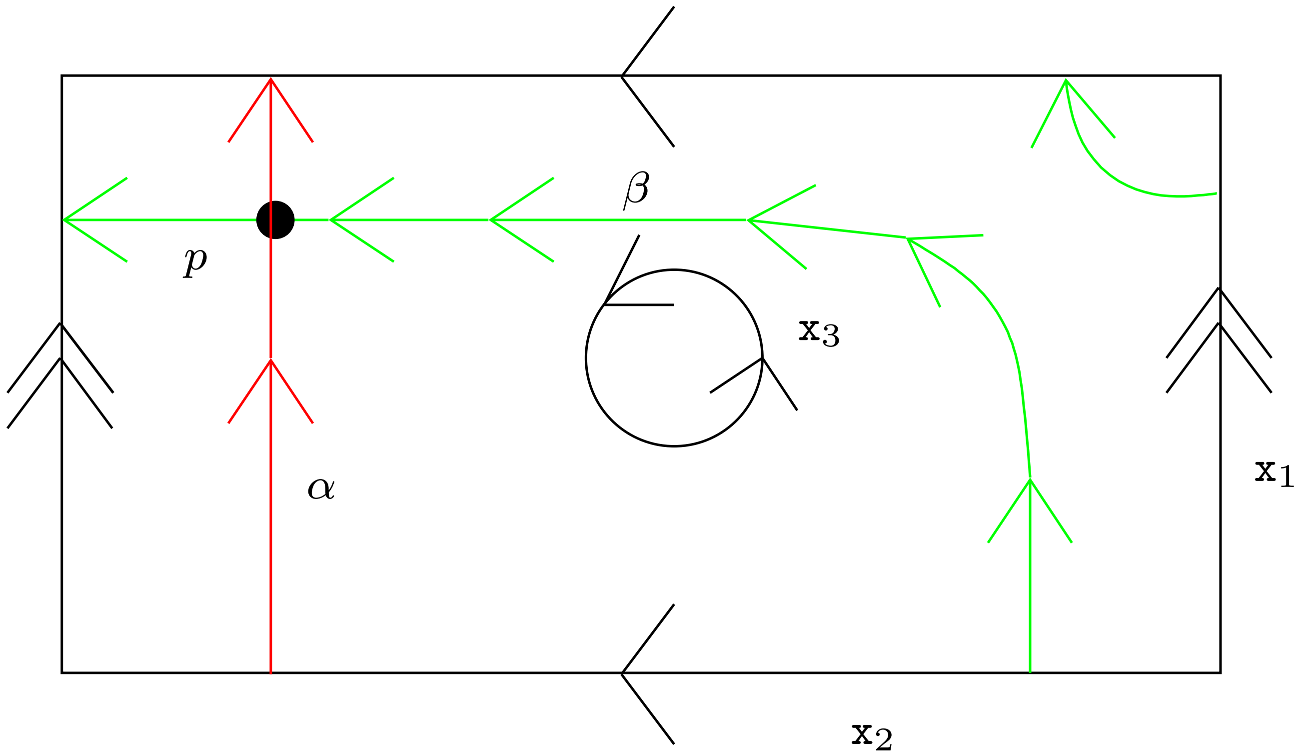

Figure 10: $\alpha = \mathtt{x}_1$ and $\beta = \mathtt{x}_1\mathtt{x}_2$ in $S_{1,1}$

$$
\begin{aligned}
\{t_{(1)}, t_{(3)}\} &= \epsilon_p\big(\mathrm{tr}(\rho(\alpha_p\beta_p)) - (1/3)\mathrm{tr}(\rho(\alpha))\mathrm{tr}(\rho(\beta))\big) \\
&= \mathrm{tr}(\mathbf{X}_1^2\mathbf{X}_2) - (1/3)\mathrm{tr}(\mathbf{X}_1)\mathrm{tr}(\mathbf{X}_1\mathbf{X}_2) \\
&= -t_{(-1)}t_{(2)} + t_{(-4)} + (2/3)t_{(1)}t_{(3)}.
\end{aligned}
$$

$$
\begin{aligned}
\{t_{(1)}, t_{(3)}\} &= \epsilon_p \big(\mathrm{tr}(\rho(\alpha_p \beta_p)) - (1/3)\mathrm{tr}(\rho(\alpha))\mathrm{tr}(\rho(\beta))\big) \\
&= \mathrm{tr}(\mathbf{X}_1^2 \mathbf{X}_2) - (1/3)\mathrm{tr}(\mathbf{X}_1)\mathrm{tr}(\mathbf{X}_1 \mathbf{X}_2) \\
&= -t_{(-1)} t_{(2)} + t_{(-4)} + (2/3) t_{(1)} t_{(3)}.
\end{aligned}
$$

There are seven more pairings like the one above; again there is symmetry.

$$\begin{aligned}\{t_{(1)}, t_{(3)}\} &= \epsilon_p\big(\mathrm{tr}(\rho(\alpha_p\beta_p)) - (1/3)\mathrm{tr}(\rho(\alpha))\mathrm{tr}(\rho(\beta))\big)\\ &= \mathrm{tr}(\mathbf{X}_1^2\mathbf{X}_2) - (1/3)\mathrm{tr}(\mathbf{X}_1)\mathrm{tr}(\mathbf{X}_1\mathbf{X}_2)\\ &= -t_{(-1)}t_{(2)} + t_{(-4)} + (2/3)t_{(1)}t_{(3)}.\end{aligned}$$

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$$\begin{aligned}
\{t_{(-1)}, t_{(-3)}\} &= -t_{(1)}t_{(-2)} + t_{(4)} + (2/3)t_{(-1)}t_{(-3)} = \mathfrak{i}\{t_{(1)}, t_{(3)}\} = \{\mathfrak{i}t_{(1)}, \mathfrak{i}t_{(3)}\}\\
\{t_{(1)}, t_{(4)}\} &= t_{(-1)}t_{(-2)} - t_{(-3)} - (2/3)t_{(1)}t_{(4)} = -\mathfrak{i}_2\{t_{(1)}, t_{(3)}\} = \{\mathfrak{i}_2 t_{(1)}, \mathfrak{i}_2 t_{(3)}\}\\
\{t_{(-1)}, t_{(-4)}\} &= t_{(1)}t_{(2)} - t_{(3)} - (2/3)t_{(-1)}t_{(-4)} = -\mathfrak{i}_1\{t_{(1)}, t_{(3)}\} = \{\mathfrak{i}_1 t_{(1)}, \mathfrak{i}_1 t_{(3)}\}\\
\{t_{(2)}, t_{(3)}\} &= t_{(-2)}t_{(1)} - t_{(4)} - (2/3)t_{(2)}t_{(3)} = -\mathfrak{t}\{t_{(1)}, t_{(3)}\} = \{\mathfrak{t}t_{(1)}, \mathfrak{t}t_{(3)}\}\\
\{t_{(-2)}, t_{(-3)}\} &= t_{(2)}t_{(-1)} - t_{(-4)} - (2/3)t_{(-2)}t_{(-3)} = -\mathfrak{i}\mathfrak{t}\{t_{(1)}, t_{(3)}\} = \{\mathfrak{i}\mathfrak{t}t_{(1)}, \mathfrak{i}\mathfrak{t}t_{(3)}\}\\
\{t_{(2)}, t_{(-4)}\} &= -t_{(-2)}t_{(-1)} + t_{(-3)} + (2/3)t_{(2)}t_{(-4)} = \mathfrak{i}_1\mathfrak{t}\{t_{(1)}, t_{(3)}\} = \{\mathfrak{i}_1\mathfrak{t}t_{(1)}, \mathfrak{i}_1\mathfrak{t}t_{(3)}\}\\
\{t_{(-2)}, t_{(4)}\} &= -t_{(2)}t_{(1)} + t_{(3)} + (2/3)t_{(-2)}t_{(4)} = \mathfrak{i}_2\mathfrak{t}\{t_{(1)}, t_{(3)}\} = \{\mathfrak{i}_2\mathfrak{t}t_{(1)}, \mathfrak{i}_2\mathfrak{t}t_{(3)}\}
\end{aligned}$$

$$
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\{t_{(1)}, t_{(3)}\} &= \epsilon_p\big(\mathrm{tr}(\rho(\alpha_p\beta_p)) - (1/3)\mathrm{tr}(\rho(\alpha))\mathrm{tr}(\rho(\beta))\big)\\
&= \mathrm{tr}(\mathbf{X}_1^2\mathbf{X}_2) - (1/3)\mathrm{tr}(\mathbf{X}_1)\mathrm{tr}(\mathbf{X}_1\mathbf{X}_2)\\
&= -t_{(-1)}t_{(2)} + t_{(-4)} + (2/3)t_{(1)}t_{(3)}.
\end{aligned}
$$

There are seven more pairings like the one above; again there is symmetry.

$$
\begin{aligned}
\{t_{(-1)}, t_{(-3)}\} &= -t_{(1)}t_{(-2)} + t_{(4)} + (2/3)t_{(-1)}t_{(-3)} = \mathfrak{i}\{t_{(1)}, t_{(3)}\} = \{\mathfrak{i}t_{(1)}, \mathfrak{i}t_{(3)}\}\\
\{t_{(1)}, t_{(4)}\} &= t_{(-1)}t_{(-2)} - t_{(-3)} - (2/3)t_{(1)}t_{(4)} = -\mathfrak{i}_2\{t_{(1)}, t_{(3)}\} = \{\mathfrak{i}_2t_{(1)}, \mathfrak{i}_2t_{(3)}\}\\
\{t_{(-1)}, t_{(-4)}\} &= t_{(1)}t_{(2)} - t_{(3)} - (2/3)t_{(-1)}t_{(-4)} = -\mathfrak{i}_1\{t_{(1)}, t_{(3)}\} = \{\mathfrak{i}_1t_{(1)}, \mathfrak{i}_1t_{(3)}\}\\
\{t_{(2)}, t_{(3)}\} &= t_{(-2)}t_{(1)} - t_{(4)} - (2/3)t_{(2)}t_{(3)} = -\mathfrak{t}\{t_{(1)}, t_{(3)}\} = \{\mathfrak{t}t_{(1)}, \mathfrak{t}t_{(3)}\}\\
\{t_{(-2)}, t_{(-3)}\} &= t_{(2)}t_{(-1)} - t_{(-4)} - (2/3)t_{(-2)}t_{(-3)} = -\mathfrak{i}\mathfrak{t}\{t_{(1)}, t_{(3)}\} = \{\mathfrak{i}\mathfrak{t}t_{(1)}, \mathfrak{i}\mathfrak{t}t_{(3)}\}\\
\{t_{(2)}, t_{(-4)}\} &= -t_{(-2)}t_{(-1)} + t_{(-3)} + (2/3)t_{(2)}t_{(-4)} = \mathfrak{i}_1\mathfrak{t}\{t_{(1)}, t_{(3)}\} = \{\mathfrak{i}_1\mathfrak{t}t_{(1)}, \mathfrak{i}_1\mathfrak{t}t_{(3)}\}\\
\{t_{(-2)}, t_{(4)}\} &= -t_{(2)}t_{(1)} + t_{(3)} + (2/3)t_{(-2)}t_{(4)} = \mathfrak{i}_2\mathfrak{t}\{t_{(1)}, t_{(3)}\} = \{\mathfrak{i}_2\mathfrak{t}t_{(1)}, \mathfrak{i}_2\mathfrak{t}t_{(3)}\}
\end{aligned}
$$

We are now left with 12 computations.

Case 2 (letters cancel) Let $\alpha = \mathtt{x}_1$ and $\beta = \mathtt{x}_1^{-1}\mathtt{x}_2^{-1}$.

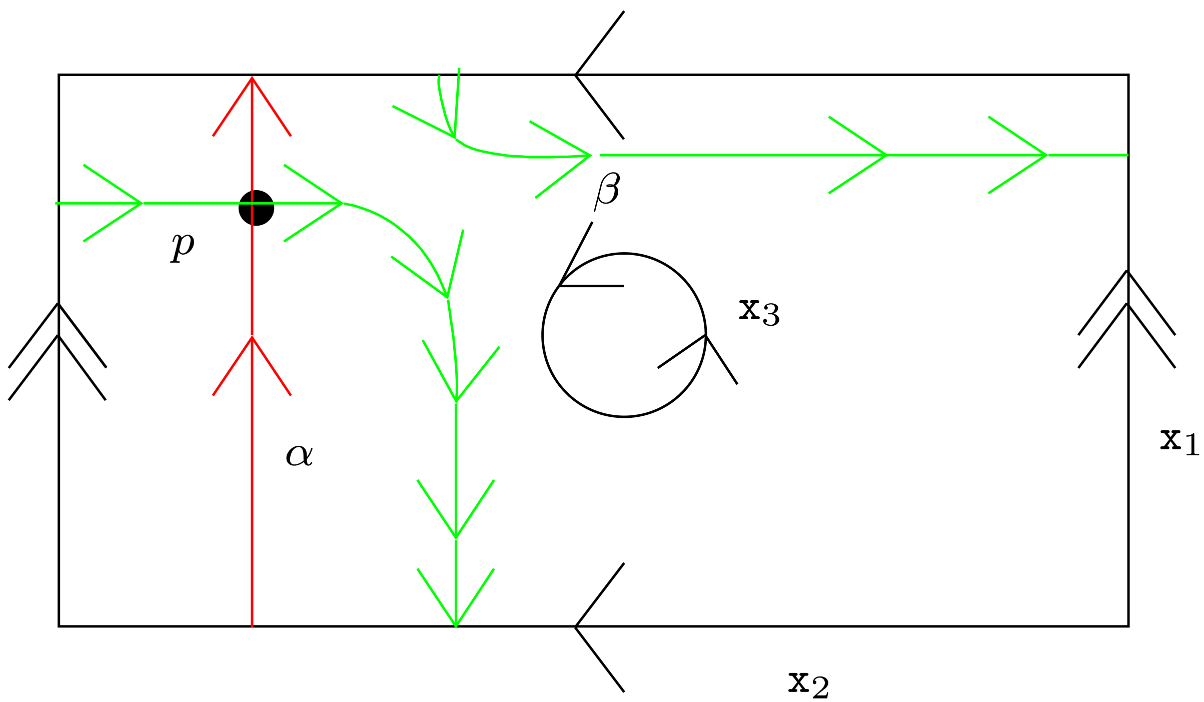

Figure 11: $\alpha = \mathtt{x}_1$ and $\beta = \mathtt{x}_1^{-1}\mathtt{x}_2^{-1}$ in $S_{1,1}$

$$
\begin{aligned}
\{t_{(1)}, t_{(-3)}\} &= \epsilon_p\big(\mathrm{tr}(\rho(\alpha_p\beta_p)) - (1/3)\mathrm{tr}(\rho(\alpha))\mathrm{tr}(\rho(\beta))\big) \\
&= (-1)\big(\mathrm{tr}(\mathbf{X}_1\mathbf{X}_2^{-1}\mathbf{X}_1^{-1}) - (1/3)\mathrm{tr}(\mathbf{X}_1)\mathrm{tr}(\mathbf{X}_1^{-1}\mathbf{X}_2^{-1})\big) \\
&= -t_{(-2)} + (1/3)t_{(1)}t_{(-3)}.
\end{aligned}
$$

$$\begin{aligned}\{t_{(1)}, t_{(-3)}\} &= \epsilon_p\big(\mathrm{tr}(\rho(\alpha_p\beta_p)) - (1/3)\mathrm{tr}(\rho(\alpha))\mathrm{tr}(\rho(\beta))\big)\\ &= (-1)\big(\mathrm{tr}(\mathbf{X}_1\mathbf{X}_2^{-1}\mathbf{X}_1^{-1}) - (1/3)\mathrm{tr}(\mathbf{X}_1)\mathrm{tr}(\mathbf{X}_1^{-1}\mathbf{X}_2^{-1})\big)\\ &= -t_{(-2)} + (1/3)t_{(1)}t_{(-3)}.\end{aligned}$$

There are seven more pairings computed like the one above and again there is symmetry.

$$\begin{aligned}\{t_{(1)}, t_{(-3)}\} &= \epsilon_p\big(\mathrm{tr}(\rho(\alpha_p\beta_p)) - (1/3)\mathrm{tr}(\rho(\alpha))\mathrm{tr}(\rho(\beta))\big)\\ &= (-1)\big(\mathrm{tr}(\mathbf{X}_1\mathbf{X}_2^{-1}\mathbf{X}_1^{-1}) - (1/3)\mathrm{tr}(\mathbf{X}_1)\mathrm{tr}(\mathbf{X}_1^{-1}\mathbf{X}_2^{-1})\big)\\ &= -t_{(-2)} + (1/3)t_{(1)}t_{(-3)}.\end{aligned}$$

There are seven more pairings computed like the one above and again there is symmetry.

$$\begin{aligned}\{t_{(-1)}, t_{(3)}\} &= -t_{(2)} + (1/3)t_{(-1)}t_{(3)} = \mathfrak{i}\{t_{(1)}, t_{(-3)}\} = \{\mathfrak{i}t_{(1)}, \mathfrak{i}t_{(-3)}\},\\ \{t_{(1)}, t_{(-4)}\} &= t_{(2)} - (1/3)t_{(1)}t_{(-4)} = -\mathfrak{i}_2\{t_{(1)}, t_{(-3)}\} = \{\mathfrak{i}_2t_{(1)}, \mathfrak{i}_2t_{(-3)}\},\\ \{t_{(-1)}, t_{(4)}\} &= t_{(-2)} - (1/3)t_{(-1)}t_{(4)} = -\mathfrak{i}_1\{t_{(1)}, t_{(-3)}\} = \{\mathfrak{i}_1t_{(1)}, \mathfrak{i}_1t_{(-3)}\},\\ \{t_{(2)}, t_{(-3)}\} &= t_{(-1)} - (1/3)t_{(2)}t_{(-3)} = -\mathfrak{t}\{t_{(1)}, t_{(-3)}\} = \{\mathfrak{t}t_{(1)}, \mathfrak{t}t_{(-3)}\},\\ \{t_{(-2)}, t_{(3)}\} &= t_{(1)} - (1/3)t_{(-2)}t_{(3)} = -\mathfrak{i}\mathfrak{t}\{t_{(1)}, t_{(-3)}\} = \{\mathfrak{i}\mathfrak{t}t_{(1)}, \mathfrak{i}\mathfrak{t}t_{(-3)}\},\\ \{t_{(2)}, t_{(4)}\} &= -t_{(1)} + (1/3)t_{(2)}t_{(4)} = \mathfrak{i}_1\mathfrak{t}\{t_{(1)}, t_{(-3)}\} = \{\mathfrak{i}_1\mathfrak{t}t_{(1)}, \mathfrak{i}_1\mathfrak{t}t_{(-3)}\},\\ \{t_{(-2)}, t_{(-4)}\} &= -t_{(-1)} + (1/3)t_{(-2)}t_{(-4)} = \mathfrak{i}_2\mathfrak{t}\{t_{(1)}, t_{(-3)}\} = \{\mathfrak{i}_2\mathfrak{t}t_{(1)}, \mathfrak{i}_2\mathfrak{t}t_{(-3)}\}.\end{aligned}$$

$$\begin{aligned}\{t_{(1)}, t_{(-3)}\} &= \epsilon_p\big(\mathrm{tr}(\rho(\alpha_p\beta_p)) - (1/3)\mathrm{tr}(\rho(\alpha))\mathrm{tr}(\rho(\beta))\big)\\ &= (-1)\big(\mathrm{tr}(\mathbf{X}_1\mathbf{X}_2^{-1}\mathbf{X}_1^{-1}) - (1/3)\mathrm{tr}(\mathbf{X}_1)\mathrm{tr}(\mathbf{X}_1^{-1}\mathbf{X}_2^{-1})\big)\\ &= -t_{(-2)} + (1/3)t_{(1)}t_{(-3)}.\end{aligned}$$

There are seven more pairings computed like the one above and again there is symmetry.

$$\begin{aligned}\{t_{(-1)}, t_{(3)}\} &= -t_{(2)} + (1/3)t_{(-1)}t_{(3)} = \mathfrak{i}\{t_{(1)}, t_{(-3)}\} = \{\mathfrak{i}t_{(1)}, \mathfrak{i}t_{(-3)}\},\\ \{t_{(1)}, t_{(-4)}\} &= t_{(2)} - (1/3)t_{(1)}t_{(-4)} = -\mathfrak{i}_2\{t_{(1)}, t_{(-3)}\} = \{\mathfrak{i}_2 t_{(1)}, \mathfrak{i}_2 t_{(-3)}\},\\ \{t_{(-1)}, t_{(4)}\} &= t_{(-2)} - (1/3)t_{(-1)}t_{(4)} = -\mathfrak{i}_1\{t_{(1)}, t_{(-3)}\} = \{\mathfrak{i}_1 t_{(1)}, \mathfrak{i}_1 t_{(-3)}\},\\ \{t_{(2)}, t_{(-3)}\} &= t_{(-1)} - (1/3)t_{(2)}t_{(-3)} = -\mathfrak{t}\{t_{(1)}, t_{(-3)}\} = \{\mathfrak{t}t_{(1)}, \mathfrak{t}t_{(-3)}\},\\ \{t_{(-2)}, t_{(3)}\} &= t_{(1)} - (1/3)t_{(-2)}t_{(3)} = -\mathfrak{i}\mathfrak{t}\{t_{(1)}, t_{(-3)}\} = \{\mathfrak{i}\mathfrak{t}t_{(1)}, \mathfrak{i}\mathfrak{t}t_{(-3)}\},\\ \{t_{(2)}, t_{(4)}\} &= -t_{(1)} + (1/3)t_{(2)}t_{(4)} = \mathfrak{i}_1\mathfrak{t}\{t_{(1)}, t_{(-3)}\} = \{\mathfrak{i}_1\mathfrak{t}t_{(1)}, \mathfrak{i}_1\mathfrak{t}t_{(-3)}\},\\ \{t_{(-2)}, t_{(-4)}\} &= -t_{(-1)} + (1/3)t_{(-2)}t_{(-4)} = \mathfrak{i}_2\mathfrak{t}\{t_{(1)}, t_{(-3)}\} = \{\mathfrak{i}_2\mathfrak{t}t_{(1)}, \mathfrak{i}_2\mathfrak{t}t_{(-3)}\}.\end{aligned}$$

We are now left with 4 computations.

The last case to consider is when there are two intersections. For instance, let $\alpha = \mathtt{x}_1\mathtt{x}_2$ and $\beta = \mathtt{x}_1\mathtt{x}_2^{-1}$.

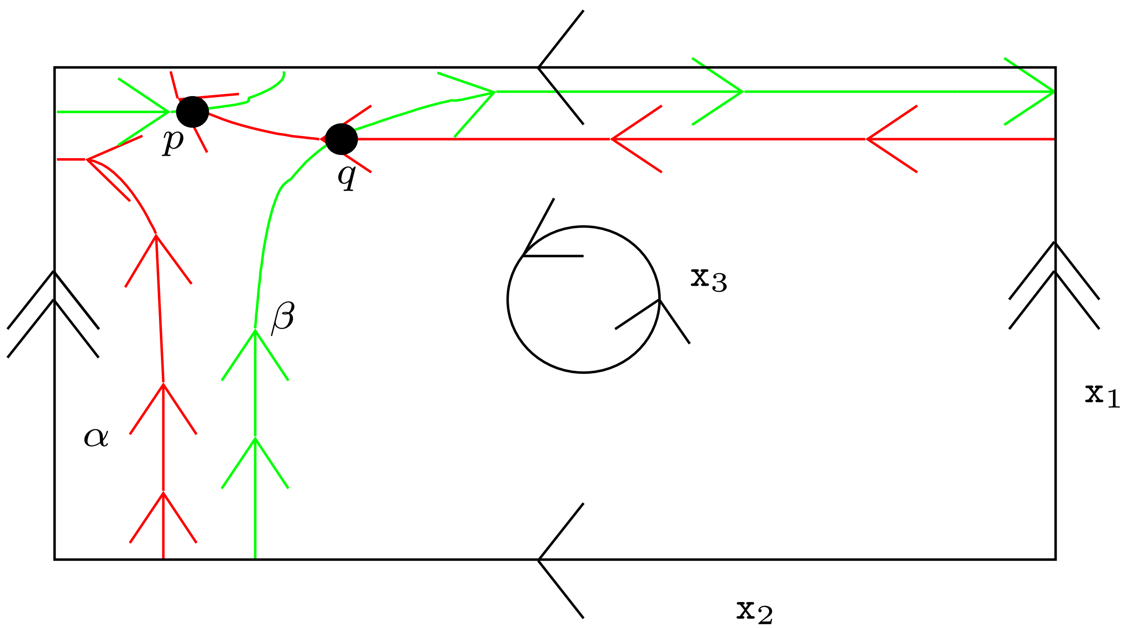

Figure 12: $\alpha = \mathtt{x}_1\mathtt{x}_2$ and $\beta = \mathtt{x}_1\mathtt{x}_2^{-1}$ in $S_{1,1}$

$$
\begin{aligned}
\{t_{(3)}, t_{(4)}\} &= \epsilon_p\big(\mathrm{tr}(\rho(\alpha_p\beta_p)) - (1/3)\mathrm{tr}(\rho(\alpha))\mathrm{tr}(\rho(\beta))\big) \\
&\quad + \epsilon_q\big(\mathrm{tr}(\rho(\alpha_q\beta_q)) - (1/3)\mathrm{tr}(\rho(\alpha))\mathrm{tr}(\rho(\beta))\big) \\
&= (-1)\big(\mathrm{tr}(\mathbf{X}_2\mathbf{X}_1\mathbf{X}_2^{-1}\mathbf{X}_1) - (1/3)\mathrm{tr}(\mathbf{X}_1\mathbf{X}_2)\mathrm{tr}(\mathbf{X}_1\mathbf{X}_2^{-1})\big) \\
&\quad + (-1)\big(\mathrm{tr}(\mathbf{X}_2\mathbf{X}_1\mathbf{X}_1\mathbf{X}_2^{-1}) - (1/3)\mathrm{tr}(\mathbf{X}_1\mathbf{X}_2)\mathrm{tr}(\mathbf{X}_1\mathbf{X}_2^{-1})\big) \\
&= -\mathrm{tr}(\mathbf{X}_1^2) - \mathrm{tr}(\mathbf{X}_1\mathbf{X}_2\mathbf{X}_1\mathbf{X}_2^{-1}) + (2/3)\mathrm{tr}(\mathbf{X}_1\mathbf{X}_2)\mathrm{tr}(\mathbf{X}_1\mathbf{X}_2^{-1}) \\
&= -t_{(1)}^2 + t_{(-1)} - t_{(-4)}t_{(-2)} - t_{(2)}t_{(-3)} + t_{(-1)}t_{(2)}t_{(-2)} - \frac{1}{3}t_{(3)}t_{(4)}
\end{aligned}
$$

$$\begin{aligned}
\{t_{(3)}, t_{(4)}\} &= \epsilon_p\big(\mathrm{tr}(\rho(\alpha_p\beta_p)) - (1/3)\mathrm{tr}(\rho(\alpha))\mathrm{tr}(\rho(\beta))\big) \\
&\quad + \epsilon_q\big(\mathrm{tr}(\rho(\alpha_q\beta_q)) - (1/3)\mathrm{tr}(\rho(\alpha))\mathrm{tr}(\rho(\beta))\big) \\
&= (-1)\big(\mathrm{tr}(\mathbf{X}_2\mathbf{X}_1\mathbf{X}_2^{-1}\mathbf{X}_1) - (1/3)\mathrm{tr}(\mathbf{X}_1\mathbf{X}_2)\mathrm{tr}(\mathbf{X}_1\mathbf{X}_2^{-1})\big) \\
&\quad + (-1)\big(\mathrm{tr}(\mathbf{X}_2\mathbf{X}_1\mathbf{X}_1\mathbf{X}_2^{-1}) - (1/3)\mathrm{tr}(\mathbf{X}_1\mathbf{X}_2)\mathrm{tr}(\mathbf{X}_1\mathbf{X}_2^{-1})\big) \\
&= -\mathrm{tr}(\mathbf{X}_1^2) - \mathrm{tr}(\mathbf{X}_1\mathbf{X}_2\mathbf{X}_1\mathbf{X}_2^{-1}) + (2/3)\mathrm{tr}(\mathbf{X}_1\mathbf{X}_2)\mathrm{tr}(\mathbf{X}_1\mathbf{X}_2^{-1}) \\
&= -t_{(1)}^2 + t_{(-1)} - t_{(-4)}t_{(-2)} - t_{(2)}t_{(-3)} + t_{(-1)}t_{(2)}t_{(-2)} - \frac{1}{3}t_{(3)}t_{(4)}
\end{aligned}$$

The last identity is a consequence of trace reduction formulas.

In a like manner one can compute the remaining pairings of this type.

In a like manner one can compute the remaining pairings of this type.

$$\{t_{(-3)}, t_{(-4)}\} = -t^2_{(-1)} + t_{(1)} - t_{(4)}t_{(2)} - t_{(-2)}t_{(3)} + t_{(1)}t_{(-2)}t_{(2)} - \frac{1}{3}t_{(-3)}t_{(-4)}$$

$$\{t_{(3)}, t_{(-4)}\} = t^2_{(2)} - t_{(-2)} + t_{(4)}t_{(-1)} + t_{(1)}t_{(-3)} - t_{(-2)}t_{(1)}t_{(-1)} + \frac{1}{3}t_{(3)}t_{(-4)}$$

$$\{t_{(-3)}, t_{(4)}\} = t^2_{(-2)} - t_{(2)} + t_{(-4)}t_{(1)} + t_{(-1)}t_{(3)} - t_{(2)}t_{(-1)}t_{(1)} + \frac{1}{3}t_{(-3)}t_{(4)}$$

In a like manner one can compute the remaining pairings of this type.

$$\{t_{(-3)}, t_{(-4)}\} = -t^2_{(-1)} + t_{(1)} - t_{(4)}t_{(2)} - t_{(-2)}t_{(3)} + t_{(1)}t_{(-2)}t_{(2)} - \frac{1}{3}t_{(-3)}t_{(-4)}$$

$$\{t_{(3)}, t_{(-4)}\} = t^2_{(2)} - t_{(-2)} + t_{(4)}t_{(-1)} + t_{(1)}t_{(-3)} - t_{(-2)}t_{(1)}t_{(-1)} + \frac{1}{3}t_{(3)}t_{(-4)}$$

$$\{t_{(-3)}, t_{(4)}\} = t^2_{(-2)} - t_{(2)} + t_{(-4)}t_{(1)} + t_{(-1)}t_{(3)} - t_{(2)}t_{(-1)}t_{(1)} + \frac{1}{3}t_{(-3)}t_{(4)}$$

Again there is symmetry:

$$\{t_{(-3)}, t_{(-4)}\} = \mathfrak{i}\{t_{(3)}, t_{(4)}\} = \{\mathfrak{i}t_{(3)}, \mathfrak{i}t_{(4)}\}$$

$$\{t_{(3)}, t_{(-4)}\} = -\mathfrak{t}\{t_{(3)}, t_{(4)}\} = \{\mathfrak{t}t_{(3)}, \mathfrak{t}t_{(4)}\}$$

$$\{t_{(-3)}, t_{(4)}\} = -\mathfrak{i}\mathfrak{t}\{t_{(3)}, t_{(4)}\} = \{\mathfrak{i}\mathfrak{t}t_{(3)}, \mathfrak{i}\mathfrak{t}t_{(4)}\}$$

In a like manner one can compute the remaining pairings of this type.

$$\{t_{(-3)}, t_{(-4)}\} = -t_{(-1)}^2 + t_{(1)} - t_{(4)}t_{(2)} - t_{(-2)}t_{(3)} + t_{(1)}t_{(-2)}t_{(2)} - \frac{1}{3}t_{(-3)}t_{(-4)}$$

$$\{t_{(3)}, t_{(-4)}\} = t_{(2)}^2 - t_{(-2)} + t_{(4)}t_{(-1)} + t_{(1)}t_{(-3)} - t_{(-2)}t_{(1)}t_{(-1)} + \frac{1}{3}t_{(3)}t_{(-4)}$$

$$\{t_{(-3)}, t_{(4)}\} = t_{(-2)}^2 - t_{(2)} + t_{(-4)}t_{(1)} + t_{(-1)}t_{(3)} - t_{(2)}t_{(-1)}t_{(1)} + \frac{1}{3}t_{(-3)}t_{(4)}$$

Again there is symmetry:

$$\{t_{(-3)}, t_{(-4)}\} = \mathfrak{i}\{t_{(3)}, t_{(4)}\} = \{\mathfrak{i}t_{(3)}, \mathfrak{i}t_{(4)}\}$$

$$\{t_{(3)}, t_{(-4)}\} = -\mathfrak{t}\{t_{(3)}, t_{(4)}\} = \{\mathfrak{t}t_{(3)}, \mathfrak{t}t_{(4)}\}$$

$$\{t_{(-3)}, t_{(4)}\} = -\mathfrak{i}\mathfrak{t}\{t_{(3)}, t_{(4)}\} = \{\mathfrak{i}\mathfrak{t}t_{(3)}, \mathfrak{i}\mathfrak{t}t_{(4)}\}$$

This finishes the bracket computations for the one-holed torus.

Define the following elements of the group ring of D_4:

- $\Sigma_1 = 1 + \mathfrak{i} - \mathfrak{i}_1 - \mathfrak{i}_2$
- $\Sigma_2 = 1 + \mathfrak{i} - \mathfrak{t} - \mathfrak{i}\mathfrak{t}$

and let $\mathfrak{a}_{i,j} = \{t_{(i)}, t_{(j)}\}$.

Define the following elements of the group ring of D_4:

- $\Sigma_1 = 1 + \mathfrak{i} - \mathfrak{i}_1 - \mathfrak{i}_2$
- $\Sigma_2 = 1 + \mathfrak{i} - \mathfrak{t} - \mathfrak{i}\mathfrak{t}$

and let $\mathfrak{a}_{i,j} = \{t_{(i)}, t_{(j)}\}$.

Note

$$\frac{1}{2}\Sigma_1\Sigma_2 = 1 + \mathfrak{i} - \mathfrak{i}_1 - \mathfrak{i}_2 - \mathfrak{t} - \mathfrak{i}\mathfrak{t} + \mathfrak{i}_1\mathfrak{t} + \mathfrak{i}_2\mathfrak{t}.$$

Then putting our work together proves:

Then putting our work together proves:

Theorem 0.3. *The Poisson bivector field for the* $\mathrm{SL}(3,\mathbb{C})$*-relative character variety of the one-holed torus is*

$$\Sigma_1\left(\mathfrak{a}_{1,2}\frac{\partial}{\partial t_{(1)}}\wedge\frac{\partial}{\partial t_{(2)}}\right)+\Sigma_2\left(\mathfrak{a}_{3,4}\frac{\partial}{\partial t_{(3)}}\wedge\frac{\partial}{\partial t_{(4)}}\right)$$
$$+\frac{1}{2}\Sigma_1\Sigma_2\left(\mathfrak{a}_{1,3}\frac{\partial}{\partial t_{(1)}}\wedge\frac{\partial}{\partial t_{(3)}}+\mathfrak{a}_{1,-3}\frac{\partial}{\partial t_{(1)}}\wedge\frac{\partial}{\partial t_{(-3)}}\right),$$

Then putting our work together proves:

Theorem 0.3. *The Poisson bivector field for the* SL$(3,\mathbb{C})$*-relative character variety of the one-holed torus is*

$$\Sigma_1\left(\mathfrak{a}_{1,2}\frac{\partial}{\partial t_{(1)}}\wedge\frac{\partial}{\partial t_{(2)}}\right)+\Sigma_2\left(\mathfrak{a}_{3,4}\frac{\partial}{\partial t_{(3)}}\wedge\frac{\partial}{\partial t_{(4)}}\right)$$
$$+\frac{1}{2}\Sigma_1\Sigma_2\left(\mathfrak{a}_{1,3}\frac{\partial}{\partial t_{(1)}}\wedge\frac{\partial}{\partial t_{(3)}}+\mathfrak{a}_{1,-3}\frac{\partial}{\partial t_{(1)}}\wedge\frac{\partial}{\partial t_{(-3)}}\right),$$

where:

- $\mathfrak{a}_{1,2}=t_{(3)}-\frac{1}{3}t_{(1)}t_{(2)}$
- $\mathfrak{a}_{1,3}=\frac{2}{3}t_{(1)}t_{(3)}-t_{(-1)}t_{(2)}+t_{(-4)}$
- $\mathfrak{a}_{1,-3}=-t_{(-2)}+\frac{1}{3}t_{(1)}t_{(-3)}$
- $\mathfrak{a}_{3,4}=-t_{(1)}^2+t_{(-1)}-t_{(-4)}t_{(-2)}-t_{(2)}t_{(-3)}+t_{(-1)}t_{(2)}t_{(-2)}-\frac{1}{3}t_{(3)}t_{(4)}.$

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