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Invariants of 2×2 matrices, irreducible $\mathrm{SL}(2, \mathbb{C})$ characters and the Magnus trace map.
 (English summary)

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This well-written paper concerns the study of representations of a finitely generated group Γ into the algebraic Lie group $G = \mathrm{SL}(2, \mathbb{C})$. Let Γ be generated by $\bar{\varepsilon}_1, \dots, \bar{\varepsilon}_n$, and let $X_n = G^{\times n}$ be the representation variety of a rank n free group into G . In this case, a representation is stable with respect to the conjugation action if its orbit is closed and its isotropy subgroup is of dimension zero. Recall that a representation $\rho: \Gamma \rightarrow G$ is said to be reducible if there is a $\rho(\Gamma)$ -invariant subspace of \mathbb{C}^2 . In these terms the author shows that a representation $\rho: \Gamma \rightarrow G$ is reducible if and only if $(\rho(\bar{\varepsilon}_1), \dots, \rho(\bar{\varepsilon}_n)) \in X_n$ is not stable if and only if every factor $\rho(\bar{\varepsilon}_i)$ is simultaneously conjugate to an upper triangular matrix if and only if all triples $(\rho(\bar{\varepsilon}_i), \rho(\bar{\varepsilon}_j), \rho(\bar{\varepsilon}_k)) \in X_3$ correspond to reducible representations of a rank 3 free group. This gives a finiteness result for reducibility. Next, the author determines a numerical characterization for reducibility. In particular, it is shown that ρ is reducible if and only if for all j, k, l , $\mathrm{tr}([\rho(\bar{\varepsilon}_j), \rho(\bar{\varepsilon}_k)]) = 2$ and $\mathrm{tr}(\rho(\bar{\varepsilon}_j)\rho(\bar{\varepsilon}_k)\rho(\bar{\varepsilon}_l)) = \mathrm{tr}(\rho(\bar{\varepsilon}_l)\rho(\bar{\varepsilon}_k)\rho(\bar{\varepsilon}_j))$, where $[,]$ is the commutator. It is very interesting to note that reducibility of $\mathrm{SL}(2, \mathbb{C})$ representations of finitely generated groups is determined by reducibility of representations of free groups, themselves determined by representations of rank 3 subgroups. As noted by the author, this extends a result of M. Culler and P. B. Shalen [*Ann. of Math. (2)* **117** (1983), no. 1, 109–146; [MR0683804 \(84k:57005\)](#)].

Having characterized reducibility (and consequently irreducibility) the author moves on to describe the orbits of G representations of free groups in terms of trace functions. In particular, extending a result of W. Magnus [*Math. Z.* **170** (1980), no. 1, 91–103; [MR0558891 \(81a:20043\)](#)], the author first shows that for any $B_1, B_2 \in G$ such that $\mathrm{tr}([B_1, B_2]) \neq 2$, there exists $g \in G$ so that $A_i = gB_i g^{-1}$ are transposition invariant. Then, fixing any vector $\mathbf{s} \in \mathbb{C}^{3n-6}$ there exist $A_3, \dots, A_n \in G$ so that

$$\mathbf{s} = (\mathrm{tr}(A_3), \mathrm{tr}(A_1 A_3), \mathrm{tr}(A_2 A_3), \dots, \mathrm{tr}(A_n), \mathrm{tr}(A_1 A_n), \mathrm{tr}(A_2 A_n)).$$

Conversely, all such solutions are characterized, the number of G -orbits being finite. Interestingly, the author shows that the map $X_n \rightarrow \mathbb{C}^{3n-3}$ given by appending $(\mathrm{tr}(A_1), \mathrm{tr}(A_2), \mathrm{tr}(A_1 A_2))$ to the $(3n-6)$ -tuple of trace functions above is only surjective when n equals 2 or 3; an explicit obstruction is obtained in the appendix. However, the map is almost surjective since it omits a set contained in a subvariety of codimension 1.

The methods used to prove these theorems are all constructive, as to provide algorithms for computations. Consequently, the proofs themselves are direct and computational. The invariant theory of C. Procesi [*Advances in Math.* **19** (1976), no. 3, 306–381; [MR0419491 \(54 #7512\)](#)] is used as a starting point. Consequently, many of the results are in terms of the conjugation invariants of n copies of arbitrary 2×2 matrices (not assuming determinant 1). Some suggestion is made

how to generalize some results to $SL(m, \mathbb{C})$. It would be very interesting to have corresponding theorems about 3×3 matrices, in particular, to generalize the obstruction obtained in the appendix.

Reviewed by *Sean Lawton*

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.