MR2319079 (2008c:32019) 32G15 (37F30 37F45)
Jiang, Yunping (1-CUNYQ); **Mitra, Sudeb** (1-CUNYQ)
Some applications of universal holomorphic motions. (English summary)
Kodai Math. J. **30** (2007), *no. 1,* 85–96.

This paper mainly concerns the extension problem for holomorphic motions of subsets of the Riemann sphere. In particular, the first result concerns limitations of such extensions and the second result explains a case where extensions always exist. Throughout the paper, the concept of a universal holomorphic motion is utilized. Lastly, this notion and the second result are together used to study a holomorphic family of hyperbolic dynamical systems.

Let V be a connected complex manifold with basepoint and let E be a subset of the Riemann sphere $\widehat{\mathbb{C}}$. A holomorphic motion of E over V is a map $V \times E \to \widehat{\mathbb{C}}$ that is the identity at the basepoint, an injection at all points in V, and a holomorphic mapping for all points in E. In other words, V holomorphically parametrizes injections $E \to \widehat{\mathbb{C}}$, and always includes the identity. In his 1989 doctoral thesis ["Holomorphic motions and Teichmüller space", Cornell Univ., Ithaca, NY] G. S. Lieb showed that when E is closed and contains $\{0, 1, \infty\}$, there always exists a contractible complex Banach manifold, referred to in this paper as the Teichmüller space of E and denoted by $T(E)$, that is the parameter space for a holomorphic motion $\Psi_E\colon T(E) \times E \to \widehat{\mathbb{C}}$. It was subsequently established that if V is simply connected, then any holomorphic motion $\varphi\colon V \times E \to \widehat{\mathbb{C}}$ uniquely factors through Ψ_E; it is universal [S. Mitra, J. Anal. Math. **81** (2000), 1–33; MR1785276 (2001g:32037)].

A holomorphic motion $\varphi\colon V \times E \to \widehat{\mathbb{C}}$ is said to be extendable if there exists a set $\widehat{E}$ that properly contains E, and another holomorphic motion $\widehat{\varphi}\colon V \times \widehat{E} \to \widehat{\mathbb{C}}$ that equals φ on $V \times E$. In 1991, Z. Slodkowski showed that any motion over the open unit disk can be extended to the entire sphere [Proc. Amer. Math. Soc. **111** (1991), no. 2, 347–355; MR1037218 (91f:58078)]. Later, J. H. Hubbard showed that this theorem does not generalize to higher-dimensional parameter spaces [Mem. Amer. Math. Soc. **4** (1976), no. 166, ix+137 pp.; MR0430321 (55 #3326)].

Theorem 1 in this paper offers a new proof that Slodkowski's theorem cannot be so generalized. In particular it is shown that if E is finite, contains $\{0, 1, \infty\}$, and at least two other distinct points, then the holomorphic motion of E, Ψ_E, cannot be extended to the entire Riemann sphere.

The second theorem of this paper proves that over any complex Banach manifold V a holomorphic motion $\varphi\colon V \times E \to \widehat{\mathbb{C}}$ can always be extended to the closure of E. It was known that this was true if the parameter space was the open disk [R. Mañé, P. Sad and D. P. Sullivan, Ann. Sci. École Norm. Sup. (4) **16** (1983), no. 2, 193–217; MR0732343 (85j:58089)], and this theorem generalizes this fact to any complex Banach manifold.

In addition to the universality of Ψ_E, to prove these theorems, the authors use the fact that $T(E)$ is equivalent to the classical Teichmüller space of $\widehat{\mathbb{C}} \setminus E$ when E is finite. Hence, Teichmüller theory is employed.

The third and last theorem concerns holomorphic families of hyperbolic dynamical systems.

Given such a system, it is shown that the Julia sets associated to each member of the family are quasiconformally equivalent and deform holomorphically across the parameter space. Such a family is a holomorphic mapping $V \times E \to \widehat{\mathbb{C}}$ where V is a based simply connected complex Banach manifold and E is a simply connected domain. Additionally, it is required that fixing any $x \in V$ yields a proper holomorphic mapping. Iterating such mappings gives dynamics. If all periodic points are either attracting or repelling then the family is called hyperbolic. The Julia set is the closure of the repelling periodic points. So most of the work in proving this theorem is in constructing a holomorphic motion of the periodic repelling points over V from the given holomorphic family of hyperbolic dynamical systems. Once this is done the paper's Theorem 2 and a consequence of the universal property of Ψ_E are used to finish the proof.

Reviewed by *Sean Lawton*

References

1. L. V. Ahlfors and L. Bers, Riemann's mapping theorem for variable metrics, Ann. of Math. **72** (1960), 385–404. MR0115006 (22 #5813)
2. L. Bers and H. L. Royden, Holomorphic families of injections, Acta Math. **157** (1986), 259–286. MR0857675 (88i:30034)
3. L. Carleson and T. Gamelin, Complex dynamics, Universitext: tracts in mathematics, Springer-Verlag, New York, 1993. MR1230383 (94h:30033)
4. A. Douady, Prolongement de mouvements holomorphes [d'après Slodkowski et autres], Séminaire N. Bourbaki, 1993/94, Exp. 775, Astérisque **227** (1995), 7–20. MR1321641 (95m:58104)
5. A. Douady and J. H. Hubbard, On the dynamics of polynomial-like maps, Ann. Sci. École Norm. Sup. (4) **18** (1985), 287–344. MR0816367 (87f:58083)
6. C. J. Earle, On holomorphic cross-sections in Teichmüller spaces, Duke Math. J. **33** (1969), 409–416. MR0254233 (40 #7442)
7. C. J. Earle, Some maximal holomorphic motions, Contemp. Math. **211** (1997), 183–192. MR1476987 (99e:58142)
8. C. J. Earle, F. P. Gardiner and N. Lakic, Isomorphisms between generalized Teichmüller spaces, Contemporary Mathematics **240** (1999), 97–110. MR1703554 (2001h:32018)
9. C. J. Earle and I. Kra, On holomorphic mappings between Teichmüller spaces, Contributions to analysis, Academic Press, New York, 1974, 107–124. MR0430319 (55 #3324)
10. C. J. Earle and S. Mitra, Variation of moduli under holomorphic motions, Contemporary Mathematics **256** (2000), 39–67. MR1759669 (2001f:30031)
11. F. P. Gardiner and N. Lakic, Quasiconformal Teichmüller theory, Mathematical surveys and monographs **76**, AMS, 2000. MR1730906 (2001d:32016)
12. J. H. Hubbard, Sur les sections analytiques de la courbe universelle de Teichmüller, Mem. Amer. Math. Soc. **166** (1976), 1–137. MR0430321 (55 #3326)
13. J. H. Hubbard, Teichmüller theory and applications to geometry, topology, and dynamics **I**: Teichmüller theory, Matrix Editions, Ithaca, NY, 2006. MR2245223
14. Y. Jiang, α-Asymptotically conformal fixed points and holomorphic motions, Complex analysis and its applications, the proceedings of the 13th ICFIDCAA, 2006, World Scientific Publishing

Co., to appear.
15. G. Lieb, Holomorphic motions and Teichmüller space, Ph.D. dissertation, Cornell University, 1990.
16. R. Ma ne, P. Sad and D. Sullivan, On the dynamics of rational maps, Ann. Sci. École Norm. Sup. **16** (1983), 193–217. MR0732343 (85j:58089)
17. J. Milnor, Dynamics in one complex variable: introductory lectures, 2nd ed., Vieweg, 2000. MR1721240 (2002i:37057)
18. S. Mitra, Teichmüller spaces and holomorphic motions, J. d'Analyse Math. **81** (2000), 1–33. MR1785276 (2001g:32037)
19. S. Mitra, Extensions of holomorphic motions, to appear in Israel Journal of Mathematics. cf. MR2342482
20. S. Mitra, Extensions of holomorphic motions to quasiconformal motions, preprint. cf. MR2342817
21. S. Nag, The complex analytic theory of Teichmüller spaces, Canadian Math. Soc. monographs and advanced texts, Wiley-Interscience, 1988. MR0927291 (89f:32040)
22. Z. Slodkowski, Holomorphic motions and polynomial hulls, Proc. Amer. Math. Soc. **111** (1991), 347–355. MR1037218 (91f:58078)
23. D. Sullivan and W. P. Thurston, Extending holomorphic motions, Acta Math. **157** (1986), 243–257. MR0857674 (88i:30033)

Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.