MR2330155 (2008d:14065) 14L24 (14F43 14L30)
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Towards non-reductive geometric invariant theory. (English summary)
Pure Appl. Math. Q. **3** (2007), *no. 1,* 61–105.

Given a scheme X and an algebraic group G acting on X, when is X/G a scheme (or even defined)? Mumford's Geometric Invariant Theory (GIT) is remarkably successful at answering this question for affine, projective, and even general quasi-projective schemes when G is reductive [see D. B. Mumford, J. Fogarty and F. C. Kirwan, *Geometric invariant theory*, Third edition, Springer, Berlin, 1994; MR1304906 (95m:14012)]. However, there are occasions for which one would like a quotient scheme when G is not reductive. In this case, G may be assumed to be unipotent. In the paper under review, both well written and containing many illustrative examples, the authors make significant advances towards generalizing GIT to non-reductive linear group actions on projective varieties.

When G is reductive and X is affine, the ring of invariants $k[X]^G$ is finitely generated (k is an algebraically closed field of characteristic 0). Moreover, these invariants separate disjoint closed invariant sets. Consequently, there is an affine quotient $X//G = \mathrm{Spec}(k[X]^G)$ and when X is not affine (and there is a linearization of the action of G) a quotient exists by patching together suitable open affine pieces. One of the important points of this paper is to illustrate how these properties fail for non-reductive groups and to discuss ways to work around this problem by approximating the good behavior from the reductive theory.

Let X be a projective variety with an ample line bundle, on which an affine algebraic group H acts linearly; then the inclusion $k[X]^H \hookrightarrow k[X]$ always gives a rational "quotient" mapping of schemes $q\colon X \dashrightarrow \mathrm{Spec}(k[X]^H)$. The mapping q is undefined exactly where all invariants vanish (the unstable points). In the reductive theory, the points where q is defined are the semistable points and if the orbits are full-dimensional and closed the points are called stable. To obtain a quotient when H is not reductive, the basic idea is either to take large enough finitely generated approximations to $k[X]^H$ or to extend the action of H to a reductive action and use the established theory. Both approaches are successful to some extent and are related to one another as is discussed as the paper develops.

With this in mind, the notions of stability and semistability for non-reductive groups are defined: $x \in X$ is naively semistable if $q(x)$ is well defined and is almost stable if further the dimension of a fiber in the image of q is equal to the dimension of H and has trivial stabilizer. On the other hand, in reductive GIT the stable set is the union of principal affine opens X_f where the action is closed and orbits have maximal dimension. In this case, the geometric quotient (a quotient that is an honest orbit space) is given by piecing together the affine varieties $\mathrm{Spec}(k[X_f]^G)$. The notion of naively stable for unipotent actions is then defined to be the set of points (with respect to a linearization) that have invariant sections f so $k[X_f]^H$ is finitely generated and $X_f \to \mathrm{Spec}(k[X_f]^H)$ is a geometric quotient; assuming only $k[X_f]^H$ is finitely generated gives the set of finitely generated semistable points.

Using this notion the authors define the enveloping quotient $X//H$ to be the union of $\mathrm{Spec}(k[X_f]^H)$ with respect to the sections defining the finitely generated semistable points. When the entire coordinate ring is finitely generated then the enveloping quotient is $\mathrm{Proj}(k[X]^H)$ and so is a projective variety.

Other notions of semistable and stable points are then defined as well: locally trivial stable, Mumford stable, Mumford semistable, completely stable, and completely semistable.

The next character to introduce in this story is the induced reductive action from a given unipotent one. Given a unipotent group H choose a reductive group G with H as a closed subgroup. Let $G \times_H X$ denote the quotient of $G \times X$ by the following H action: $h \cdot (g, x) \mapsto (gh^{-1}, hx)$. If the action of H extends to G then $G \times_H X \approx (G/H) \times X$ where G/H is quasi-affine. Under general conditions, $k[G \times_H X]^G \approx k[X]^H$. Therefore, one would like to choose a G-equivariant projective completion of $G \times_H X$ and use the reductive theory here, but this alone will not work well in general because of issues with extending invariants to the boundary of such a completion. Consequently, the authors define the notion of a reductive envelope which loosely is a projective completion $\overline{G \times_H X}$ together with a G-linearization that restricts to the H-linearization on X that additionally has the property that sufficiently many separating H-invariant sections f extend over the boundary to G-invariant sections F of $\overline{G \times_H X}$ with $(\overline{G \times_H X})_F$ affine. Further refinements of the notion of a reductive envelope are also introduced: ample reductive envelope, fine reductive envelope, and strong reductive envelope.

In these terms we come to our first theorem: If X is a normal projective variety with a linear action of a connected unipotent group H, then with respect to any fine reductive envelope, the completely stable points are contained in the locally trivial stable points which are equal to the Mumford stable points which equal the Mumford semistable points which are contained in the naively stable points which are all almost stable points and are in turn contained in the finitely generated semistable points which are all completely semistable and naively semistable; the latter two sets being equal. All stable points admit quasi-projective geometric quotients, the finitely generated semistable points give an enveloped quotient and its image, the enveloping quotient, is an open subvariety of $\overline{G \times_H X}//G$.

This theorem relates all previously defined notions of stability and tells when there are quotients and how they relate.

The authors define X^{ss} to be the set of finitely generated semistable points and X^s to be the Mumford stable points. The general picture is that there exists an enveloped quotient $q\colon X^{ss} \to X//H \subset \mathrm{Proj}(k[X]^H)$ that admits a geometric quotient by restricting to X^s, the enveloping quotient $X//H$ is in general quasi-projective but equals $\mathrm{Proj}(k[X]^H)$ when the invariant ring is finitely generated, and the image of q is in general dense and constructible but not categorical.

The second theorem of this paper shows that when the projective completion of the associated reductive G-variety $\overline{G \times_H X}$ is normal and has a fine strong reductive envelope then the completely stable points are the Mumford stable points and the finitely generated semistable points are the naively semistable points. In particular, in this case all notions of semistability are equivalent. The authors then go on to discuss when strong envelopes exist, introducing the notion of a gentle completion.

The third and final main result gives a geometric criterion for a nonsingular projective variety to have its ring of invariants finitely generated. Loosely stated, $k[X]^H$ is finitely generated if and only if there exists a linearization so the codimension 1 components of the boundary of $G \times_H X \subset \overline{G \times_H X}$ are unstable.

To conclude this seminal work the authors compute the intersection cohomology of an example of an enveloping quotient arising from a non-reductive action; namely, the moduli of n unordered points on $\mathbb{P}^1$ for $n = 3$ and $n = 4$.

REVISED (February, 2008)

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