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On symmetric, smooth and Calabi-Yau algebras. (English summary)
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This paper mainly concerns a "symmetry" condition in smooth PI algebras, and examples satisfying it. Quoting from the abstract: "One possible definition for a Calabi-Yau algebra is a symmetric smooth PI algebra." A PI algebra is an algebra that satisfies a nontrivial polynomial identity. A ring R is called smooth if it has finite global dimension and the projective dimension of all simple R-modules is constant along each clique. R is symmetric over a central subdomain C if R is a finite C-module and $\mathrm{Hom}_C(R,C) \approx R$. The author determines some necessary and sufficient properties for a smooth PI ring to be (locally) Calabi-Yau. For instance, in one direction, it is shown that if R is prime, smooth, locally Calabi-Yau, and has invertible PI degree, then its center $Z(R)$ is Gorenstein. A more general version is likewise established, and also a converse when additional conditions are assumed. Using these results, the author shows that many examples are locally Calabi-Yau. For instance, all quantum enveloping algebras of complex semi-simple Lie algebras (in the root of unity case) are examples. Also, a more limited theorem (when the PI degree is not assumed to be invertible) shows that the enveloping algebra of $\mathfrak{sl}(n)$ in characteristic p (for $(n,p)=1$) is a Calabi-Yau algebra. Other examples are also considered. A very interesting corollary is that the mixed trace ring of m generic $n \times n$ matrices over a field of characteristic 0 has finite injective dimension. This is contrasted with the fact that the global dimension is only finite in the cases $(m,n)=(1,n),(m,1),(2,2),(2,3)$, and $(3,2)$ [see M. Van den Bergh, J. Amer. Math. Soc. **2** (1989), no. 4, 775–799; MR1001850 (90j:14065)].

Reviewed by *Sean Lawton*

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.