Symmetry in $\mathrm{SL}(3,\mathbb{C})$-Character Varieties

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Outline of Presentation

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- $SL(3, \mathbb{C})$-Character Varieties

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- The ith fundamental theorem

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- Outer Automorphisms and Symmetry

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- In other words, these polynomials are defined on orbits. But they do not distinguish orbits whose closures intersect.

- Since $\mathfrak{G}$ is (linearly) reductive, $\mathbb{C}[\mathfrak{R}]^{\mathfrak{G}}$ is a finitely generated domain and so $\mathfrak{X} = \operatorname{Spec}_{max}(\mathbb{C}[\mathfrak{R}]^{\mathfrak{G}})$ is an affine variety.

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- $\mathfrak{X}$ is called the character variety since it is the largest variety which parametrizes conjugacy classes of representations (characters).

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Theorem (1st **Fundamental Theorem of Invariants of** $n \times n$ **Matrices**)**.** *Any polynomial invariant of r matrices $A_1, ..., A_r$ of size $n \times n$ is a polynomial in the invariants* $\text{tr}(A_{i_1} A_{i_2} \cdots A_{i_j})$*; where $A_{i_1} A_{i_2} \cdots A_{i_j}$ run over all possible noncommutative monomials.*

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Remark. *Procesi showed in his work that $j \le 2^n - 1$. In 1974 Razmyslov had shown that $j \le n^2$. For $1 \le j \le 4$, it is known that $j = \frac{n(n+1)}{2}$ (and conjectured to be true in general).*

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Comment. *The difference between invariants of arbitrary $n \times n$ matrices and those with unitary determinant is the invariants* $\mathrm{tr}(\mathbf{X}^3)$*. In other words,*

$$\mathbb{C}[\mathrm{SL}_n(\mathbb{C})^{\times r} /\!\!/ \mathrm{SL}_n(\mathbb{C})] \approx \mathbb{C}[\mathrm{M}_n(\mathbb{C})^{\times r} /\!\!/ \mathrm{GL}_n(\mathbb{C})] / \mathfrak{I},$$

where $\mathfrak{I} = (\mathrm{tr}(\mathbf{X}_1^3) - P(\mathbf{X}_1), ..., \mathrm{tr}(\mathbf{X}_r^3) - P(\mathbf{X}_r))$.

Theorem $\left(2^{\mathbf{nd}}\ \textbf{Fundamental Theorem of Invariants of}\ n \times n\ \textbf{Matrices}\right).$
All relations among the generators of $\mathrm{tr}(A_{i_1} A_{i_2} \cdots A_{i_j})$ *are*
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Quote (from Procesi, 1976)**.** *According to the general theory, we will split the description into two steps. The so called "first fundamental theorem," i.e., a list of generators for* $T_{i,n}$*, and the "second fundamental theorem," i.e., a list of relations among the previously found generators. Of course, it would be very interesting to continue the process by giving the "i^{th} fundamental theorem," i.e., the full theory of syzigies; unfortunately, this seems to be still out of the scope of the theory as presented in this paper.*

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  1. $\operatorname{tr}(\mathbf{X}), \operatorname{tr}(\mathbf{Y}), \operatorname{tr}(\mathbf{XY}), \operatorname{tr}(\mathbf{X}^{-1}), \operatorname{tr}(\mathbf{Y}^{-1}), \operatorname{tr}(\mathbf{YX}^{-1}),$ $\operatorname{tr}(\mathbf{XY}^{-1}), \operatorname{tr}(\mathbf{X}^{-1}\mathbf{Y}^{-1}), \operatorname{tr}(\mathbf{XYX}^{-1}\mathbf{Y}^{-1})$ are minimal.

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We wish to generalize this case.

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- So we may freely replace any polynomial generator $\mathrm{tr}(\mathbf{U}\mathbf{X}^2\mathbf{V})$ with $\mathrm{tr}(\mathbf{U}\mathbf{X}^{-1}\mathbf{V})$ since,

$$\mathrm{tr}(\mathbf{U}\mathbf{X}^2\mathbf{V}) = \mathrm{tr}(\mathbf{U}\mathbf{X}^{-1}\mathbf{V}) + \mathrm{tr}(\mathbf{X})\mathrm{tr}(\mathbf{U}\mathbf{X}\mathbf{V}) - \mathrm{tr}(\mathbf{X}^{-1})\mathrm{tr}(\mathbf{U}\mathbf{V}).$$

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$$\operatorname{tr}(\mathbf{U}\mathbf{X}^2\mathbf{V}) = \operatorname{tr}(\mathbf{U}\mathbf{X}^{-1}\mathbf{V}) + \operatorname{tr}(\mathbf{X})\operatorname{tr}(\mathbf{U}\mathbf{X}\mathbf{V}) - \operatorname{tr}(\mathbf{X}^{-1})\operatorname{tr}(\mathbf{U}\mathbf{V}).$$

- Therefore, the ring of invariants is generated by traces of words whose letters have exponent ± 1.

- By linearizing the Cayley-Hamilton polynomial we get
$\mathbf{Y}\mathbf{X}^2 + \mathbf{X}^2\mathbf{Y} + \mathbf{X}\mathbf{Y}\mathbf{X} = \mathrm{tr}(\mathbf{Y})\mathbf{X}^2 + \mathrm{tr}(\mathbf{X})\mathbf{Y}\mathbf{X} + \mathrm{tr}(\mathbf{X})\mathbf{X}\mathbf{Y} - \mathrm{tr}(\mathbf{X})\mathrm{tr}(\mathbf{Y})\mathbf{X} + \mathrm{tr}(\mathbf{X}\mathbf{Y})\mathbf{X} + \mathrm{tr}(\mathbf{Y}\mathbf{X}^2)\mathbf{I} - \mathrm{tr}(\mathbf{X})\mathrm{tr}(\mathbf{X}\mathbf{Y})\mathbf{I} - \frac{1}{2}\left(\mathrm{tr}(\mathbf{X})^2\mathbf{Y} - \mathrm{tr}(\mathbf{X}^2)\mathbf{Y} - \mathrm{tr}(\mathbf{Y})\mathrm{tr}(\mathbf{X})^2\mathbf{I} + \mathrm{tr}(\mathbf{Y})\mathrm{tr}(\mathbf{X}^2)\mathbf{I}\right).$

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- Define $\mathrm{pol}(\mathbf{X}, \mathbf{Y}) = \mathbf{Y}\mathbf{X}^2 + \mathbf{X}^2\mathbf{Y} + \mathbf{X}\mathbf{Y}\mathbf{X}.$

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- Define $\operatorname{pol}(\mathbf{X}, \mathbf{Y}) = \mathbf{Y}\mathbf{X}^2 + \mathbf{X}^2\mathbf{Y} + \mathbf{X}\mathbf{Y}\mathbf{X}$.

- Then $\operatorname{tr}(\mathbf{W}_1\mathbf{X}^{\pm 1}\mathbf{W}_2\mathbf{X}^{\pm 1}\mathbf{W}_3) = -\operatorname{tr}(\mathbf{W}_1\mathbf{X}^{\pm 2}\mathbf{W}_2\mathbf{W}_3) - \operatorname{tr}(\mathbf{W}_1\mathbf{W}_2\mathbf{X}^{\pm 2}\mathbf{W}_3) + \operatorname{tr}(\mathbf{W}_1\operatorname{pol}(\mathbf{X}^{\pm 1}, \mathbf{W}_2)\mathbf{W}_3)$. However, by subsequently reducing the words having letters with exponent not ± 1, we eliminate expressions $\operatorname{tr}(\mathbf{W}_1\mathbf{X}^{\pm 1}\mathbf{W}_2\mathbf{X}^{\pm 1}\mathbf{W}_3)$.

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- Letting $\mathbf{W}_3 = \mathbf{X}$ we deduce: $\operatorname{tr}(\mathbf{W}_1\mathbf{X}\mathbf{W}_2\mathbf{X}^2) = -\operatorname{tr}(\mathbf{W}_2\mathbf{X}\mathbf{W}_1\mathbf{X}^2) - \operatorname{tr}(\mathbf{W}_1\mathbf{W}_2\mathbf{X}^3) + \operatorname{tr}(\mathbf{W}_1\operatorname{pol}(\mathbf{X}, \mathbf{W}_2)\mathbf{X}).$

Putting this together we deduce that the following traces generate:
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where $1 \le i \ne j \ne k \ne l \ne m \ne n \le r$.

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- It remains to count how many of each type are necessary.

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where $1 \le i \ne j \ne k \ne l \ne m \ne n \le r$.

- It remains to count how many of each type are necessary.

- Using the representation theory of $\operatorname{GL}_r(\mathbb{C})$, Abeasis and Pittaluga (1989) determined a method to count the minimal number of generators with respect to word length and with respect to the invariants of *arbitrary* matrices.

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- If there was a further reduction after passing to unimodular invariants, then there would be a relation of the form $\mathrm{tr}(\mathbf{W}) - \mathrm{tr}(\mathbf{U}) = Q(\mathbf{X}_1, ..., \mathbf{X}_r)$ where $\mathbf{W}$ and $\mathbf{U}$ are words of the same form.

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- Since the unimodular invariants are filtered (NOT graded), the homogeneous left-hand-side has the same degree as the possibly non-homogeneous right-hand-side.

- Therefore, there exists polynomial trace expressions $f_1, ..., f_r$ so $\text{tr}(\mathbf{W}) - \text{tr}(\mathbf{U}) - Q(\mathbf{X}_1, ..., \mathbf{X}_r) = \sum f_i \left(\text{tr}(\mathbf{X}_i^3) - P(\mathbf{X}_i) \right)$ in the ring of arbitrary 3×3 invariants.

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- But since the degree of the left-hand-side and the degree on the right-hand-side must be equal the expressions $\mathrm{tr}(\mathbf{W})$ or $\mathrm{tr}(\mathbf{U})$ cannot be part of any f_i (unless $\mathrm{tr}(\mathbf{U})$ or $\mathrm{tr}(\mathbf{W})$ is of the form $\mathrm{tr}(\mathbf{X}^3)$, but these were removed to begin with).

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- And so we would have a further reduction in the ring of arbitrary invariants, which is a contradiction.

Let $N_r(x, y)$ be the number of generators with respect to the free group of rank r of word length x in y letters. Note that letters with exponent -1 have length 2.

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$N_r(1, 1) = r$	$\mathrm{tr}(\mathbf{X})$
$N_r(2, 1) = r$	$\mathrm{tr}(\mathbf{X}^{-1})$
$N_r(2, 2) = \binom{r}{2}$	$\mathrm{tr}(\mathbf{XY})$
$N_r(3, 2) = 2\binom{r}{2}$	$\mathrm{tr}(\mathbf{XY}^{-1})$
$N_r(3, 3) = 2\binom{r}{3}$	$\mathrm{tr}(\mathbf{XYZ}), \mathrm{tr}(\mathbf{YXZ})$
$N_r(4, 2) = \binom{r}{2}$	$\mathrm{tr}(\mathbf{X}^{-1}\mathbf{Y}^{-1})$
$N_r(4, 3) = 6\binom{r}{3}$	$\mathrm{tr}(\mathbf{XYZ}^{-1}), \mathrm{tr}(\mathbf{YXZ}^{-1})$
$N_r(4, 4) = 5\binom{r}{4}$	$\mathrm{tr}(\mathbf{WXYZ}), \mathrm{tr}(\mathbf{WXZY}), \mathrm{tr}(\mathbf{WYXZ}),$ $\mathrm{tr}(\mathbf{WYZX}), \mathrm{tr}(\mathbf{WZXY})$

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GL_r	Dimension	Basis
$r = 2$	1	$\text{tr}(\mathbf{X}^{-1}\mathbf{Y}^{-1})$
$r = 3$	9	$\text{tr}(\mathbf{X}^{-1}\mathbf{Y}^{-1}), \text{tr}(\mathbf{X}^{-1}\mathbf{Z}^{-1}), \text{tr}(\mathbf{Y}^{-1}\mathbf{Z}^{-1})$ $\text{tr}(\mathbf{X}^{-1}\mathbf{YZ}), \text{tr}(\mathbf{X}^{-1}\mathbf{ZY}), \text{tr}(\mathbf{Y}^{-1}\mathbf{XZ}),$ $\text{tr}(\mathbf{Y}^{-1}\mathbf{ZX}), \text{tr}(\mathbf{Z}^{-1}\mathbf{XY}), \text{tr}(\mathbf{Z}^{-1}\mathbf{YX})$
$r = 4$	35	$\binom{r}{2}$ of $\text{tr}(\mathbf{X}^{-1}\mathbf{Y}^{-1}), 6\binom{r}{3}$ of $\text{tr}(\mathbf{X}^{-1}\mathbf{YZ}),$ $\Longrightarrow 5$ of $\text{tr}(\mathbf{WXYZ})$

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  $\mathbf{XZY} + \mathbf{ZXY} + \mathbf{YXZ} + \mathbf{YZX} + \mathbf{XYZ} + \mathbf{ZYX}$.

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- Thus, $\mathrm{tr}(\mathbf{WXZY}) + \mathrm{tr}(\mathbf{WZXY}) + \mathrm{tr}(\mathbf{WYXZ}) +$
  $\mathrm{tr}(\mathbf{WYZX}) + \mathrm{tr}(\mathbf{WXYZ}) + \mathrm{tr}(\mathbf{WZYX}) =$
  $\mathrm{tr}(\mathbf{W}\,(\mathrm{pol}(\mathbf{X} + \mathbf{Z}, \mathbf{Y}) - \mathrm{pol}(\mathbf{X}, \mathbf{Y}) - \mathrm{pol}(\mathbf{Z}, \mathbf{Y})))$, which allows us to eliminate exactly one of the six generators on the left-hand-side.

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- In all other cases, we have such explicit reductions. However, they are not all as succinct and uniform.

- However, there are 6 such generators. We want to explicitly construct the minimal generating set, so we want the relation.

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  $\mathrm{tr}(\mathbf{W}\,(\mathrm{pol}(\mathbf{X} + \mathbf{Z}, \mathbf{Y}) - \mathrm{pol}(\mathbf{X}, \mathbf{Y}) - \mathrm{pol}(\mathbf{Z}, \mathbf{Y})))$, which allows us to eliminate exactly one of the six generators on the left-hand-side.

- In all other cases, we have such explicit reductions. However, they are not all as succinct and uniform.

- For instance, in the case of words of length 6 in 6 letters there are 120 generators but only 15 are necessary. The relations are of two types: 37 of $\mathrm{tr}(\mathbf{W}\sum\mathbf{XYZ})$ and 68 of $\mathrm{tr}(\mathbf{UVWXYZ}) +$
  $\mathrm{tr}(\mathbf{UVWYXZ}) + \mathrm{tr}(\mathbf{VUWXYZ}) + \mathrm{tr}(\mathbf{VUWYXZ})$.

Here is the rest of the table:

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$N_r(5,3) = 9\binom{r}{3}$	$3\binom{r}{3}$: $\mathrm{tr}(\mathbf{XYZY^{-1}})$, $6\binom{r}{3}$: $\mathrm{tr}(\mathbf{XY^{-1}Z^{-1}})$
$N_r(5,4) = 20\binom{r}{4}$	$\mathrm{tr}(\mathbf{WXYZ^{-1}})$
$N_r(5,5) = 12\binom{r}{5}$	$\mathrm{tr}(\mathbf{UVWXY})$
$N_r(6,2) = \binom{r}{2}$	$\mathrm{tr}(\mathbf{XYX^{-1}Y^{-1}})$
$N_r(6,3) = 7\binom{r}{3}$	$6\binom{r}{3}$: $\mathrm{tr}(\mathbf{XYZ^{-1}Y^{-1}})$, $\binom{r}{3}$: $\mathrm{tr}(\mathbf{X^{-1}Y^{-1}Z^{-1}})$
$N_r(6,4) = 26\binom{r}{4}$	$18\binom{r}{4}$: $\mathrm{tr}(\mathbf{WXY^{-1}Z^{-1}})$, $8\binom{r}{4}$: $\mathrm{tr}(\mathbf{WXYZY^{-1}})$
$N_r(6,5) = 35\binom{r}{5}$	$\mathrm{tr}(\mathbf{VWXYZ^{-1}})$
$N_r(6,6) = 15\binom{r}{6}$	$\mathrm{tr}(\mathbf{UVWXYZ})$

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$\operatorname{tr}(\mathbf{X}_1^{-1}\mathbf{X}_2^{-1}), \operatorname{tr}(\mathbf{X}_i), \operatorname{tr}(\mathbf{X}_i^{-1}), \operatorname{tr}(\mathbf{X}_1\mathbf{X}_i), \operatorname{tr}(\mathbf{X}_2\mathbf{X}_i), \operatorname{tr}(\mathbf{X}_1\mathbf{X}_i^{-1})\}$

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 and any of the following sets of three:
$\{\operatorname{tr}(\mathbf{X}_1^{-1}\mathbf{X}_i^{-1}), \operatorname{tr}(\mathbf{X}_1^{-1}\mathbf{X}_i), \operatorname{tr}(\mathbf{X}_2^{-1}\mathbf{X}_i^{-1})\},$
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The index i ranges from 3 to r. So in all cases we get the Krull dimension of $8r - 8$. Consequently, these sets are maximal.

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- For $r \geq 3$ we calculate the Jacobian matrix of these $8r - 8$ functions in the $8r - 8$ independent variables:
  (from $\mathbf{X}_1$) x_{11}^1, x_{22}^1,
  (from $\mathbf{X}_2$) x_{ij}^2 $(1 \leq i \leq j \leq 3)$,
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- Putting the last 8 functions in the last 8 rows we get a block diagonal matrix and so by induction is remains to show that the last eight traces are independent in the variables from the last matrix $\mathbf{X}_r$.

- Using *Mathematica*, we calculate this subdeterminant and evaluate at random unimodular matrices; finding it non-zero.

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- If there was a relation the determinant would be identically zero and so any non-zero evaluation shows independence.   □

Outer Automorphisms and Symmetry

Given any $\alpha \in \mathrm{Aut}(\mathbf{F}_r)$, we define $a_\alpha \in \mathrm{End}(\mathbb{C}[\mathfrak{X}])$ by extending the following mapping

$$a_\alpha(\mathrm{tr}(\mathbf{W})) = \mathrm{tr}(\alpha(\mathbf{W})).$$

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If $\alpha \in \mathrm{Inn}(\mathbf{F}_r)$, then there exists $\mathtt{u} \in \mathbf{F}_r$ so for all $\mathtt{w} \in \mathbf{F}_r$,

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Therefore,

$$a_\alpha(\mathrm{tr}(\mathbf{W})) = \mathrm{tr}(\mathbf{U}\mathbf{W}\mathbf{U}^{-1}) = \mathrm{tr}(\mathbf{W}).$$

Thus $\mathrm{Out}(\mathbf{F}_r) = \mathrm{Aut}(\mathbf{F}_r)/\mathrm{Inn}(\mathbf{F}_r)$ acts on $\mathbb{C}[\mathfrak{X}]$.

By results of Nielsen $\mathrm{Out}(\mathbf{F}_2)$ is generated by the following mappings

$$\tau = \begin{cases} \mathbf{x}_1 \mapsto \mathbf{x}_2 \\ \mathbf{x}_2 \mapsto \mathbf{x}_1 \end{cases} \tag{1}$$

$$\iota = \begin{cases} \mathbf{x}_1 \mapsto \mathbf{x}_1^{-1} \\ \mathbf{x}_2 \mapsto \mathbf{x}_2 \end{cases} \tag{2}$$

$$\eta = \begin{cases} \mathbf{x}_1 \mapsto \mathbf{x}_1\mathbf{x}_2 \\ \mathbf{x}_2 \mapsto \mathbf{x}_2 \end{cases} \tag{3}$$

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- We wish to generalize this to arbitrary rank r.

For the general case described above we have:

- The order two morphism which transposes x_1 and x_2 preserves only the last two sets of independent variables. So we have order two symmetry there.

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- Again it seems that these morphisms are distinguishing algebraic independence by their action (on other generators they do not act as permutations, but as polynomial maps).

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- Although we discussed fives independent sets, there are many more. It appears that the other morhisms, like in form to the two described above, permute the sets themselves.

- The general theory suggests that there should be a set of independent variables for which the entire ring is integral over. It seems that these sets are not the correct ones, but perhaps linear combinations of the elements, chosen so the sets have the full order 8 symmetry, will work (in the rank 2 case the ring is integral over the independent generators!).

- In the rank 2 case, the symmetry helped to describe the ideal of relations. The next problem, to finish "Procesi's challenge", is to desribe the ideal. We hope the symmetry described, and further symmetry left to uncover, will help make this task possible. (Note: in the rank 3 case the ideal is already very complex, 45 generators and 16 dimensions!)

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Reference available by request.