Estimated transversality and rational maps

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Abstract

In his work on symplectic Lefschetz pencils, Donaldson introduced the notion of estimated transversality for a sequence of sections of a bundle. Together with asymptotic holomorphicity, it is the key ingredient allowing the construction of symplectic submanifolds. Despite its importance in the area, estimated transversality has remained a mysterious property. One of the aims of this thesis is to shed some light into this notion by studying it in the simplest possible case namely that of S^2. We state some new results about high degree rational maps on the 2-sphere that can be seen as consequences of Donaldson's existence theorem for pencils, and explain how one might go about answering a question of Donaldson: what is the best estimate for transversality that can be obtained? We also show how the methods applied to S^2 can be further generalized.

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Chapter 1

Introduction

The notion of linear system is one of importance in complex geometry. For example, it is often using a linear system that one is able to realize a complex manifold (when it satisfies certain constrains) as a submanifold of $\mathbb{CP}^N$. This is the content of Kodaira's embedding theorem. More trivially, a generic linear system of dimension 0 (whose existence is easy to establish) gives rise to a divisor i.e. a complex submanifold. Moving one step up, a generic 1 dimensional linear system gives rise to a Lefschetz pencil i.e., a map defined on X minus a codimension 1 subvariety to $\mathbb{CP}^1$ with the simplest possible singularities in the fiber. It is well know that, every symplectic manifold has an almost-complex structure, which is the same as saying that the symplectic category generalizes the Kähler category. The natural question is then: is it possible to generalize the notion of linear system to this new setting and use it to study symplectic manifolds, as it was used to study complex manifolds? Even in the simplest case of linear systems of dimension 1 this poses problems. The most important difficulty with which one is faced is the non-existence of holomorphic sections of complex bundles over symplectic manifolds (except in the integrable case, where the manifold is actually complex). In his paper [Do1], Donaldson, resolved this issue by substituting the holomorphic condition by what is called asymptotic holomorphicty. One looks for sections of an increasing power of a line bundle, which have an increasingly small $\bar{\partial}$. In the holomorphic setting, a non-singular complex submanifold is obtained as the zero set of a transverse holomorphic section. The holomorphicity condition becomes asymptotic holomorphicty, how about the transverse condition? Even though this condition could easily be translated into the symplectic picture, it is no longer enough. It needs to become "estimated transversality", that is, transverse with a good estimate independent of the (increasing) degree of the bundle. Using this notion in a very key manner, Donaldson establishes the existence of sections of bundles whose zero set is symplectic, therefore proving an important existence theorem for symplectic submanifolds. In [Do3], he goes one step further and proves the existence of the analogue of pencils. These symplectic Lefschetz pencils completely characterize symplectic manifolds. Very roughly, a manifold is symplectic exactly when it can be seen (after blow up) as a bundle over $\mathbb{CP}^1$ with symplectic fibers, some of which have simple singularities. Surprisingly, the techniques used to prove this, and in particular the notion of estimated transversality, can be used to prove new theorems in the complex setting. Even for S^2, one can prove new and unexpected results as the one that follows:

Proposition 1 *There is $0 < \eta \leq 1$, such that, for each k large enough, there exists a pair of homogeneous polynomials, (p_k, q_k), of degree k, in two complex variables, defining a function from $\mathbb{C}^2$ to $\mathbb{C}^2$, that takes $S^3 \subset \mathbb{C}^2$ into an annulus of outer radius 1 and inner radius $\sqrt{\eta}$.*

This proposition is a consequence of a more general result for complex Kähler manifolds that comes from applying Donaldson's techniques to the complex setting (using the techniques appearing in [Do1] for the Kähler setting; this is an exercise which we carry out for the sake of completeness). The special case of S^2 is treated in more detail in section 4. To get a feeling for how strong the above result is, let us try to take $p_k(\mathbf{z}, \mathbf{w}) = \mathbf{z}^k$ and $q_k(\mathbf{z}, \mathbf{w}) = \mathbf{w}^k$. Then, the image of S^3 by each of the maps (p_k, q_k) is contained in an annulus of outer radius 1 but whose inner radius is $1/\sqrt{2^{k-1}}$.

In fact, a sequence of pairs of homogeneous polynomials satisfying Proposition 1 is very special. We prove:

Theorem 1 *Let (p_k, q_k) be a sequence of pairs of homogeneous polynomials as above. The map p_k/q_k, thought of as a degree k map of $\mathbb{CP}^1$ to itself, has asymptotically uniformly distributed fibers, in the sense that, if x_i^k denote the points in one fiber for $i = 1, \cdots k$, counted with multiplicity, and f is a $\mathcal{C}^2$ function on S^2, then*

$$\left| \frac{1}{k} \sum_{i=1}^{k} f(x_i^k) - \frac{1}{|S^2|} \int_{S^2} f \right| \leq \frac{C||\triangle f||_\infty}{k}, \tag{1.1}$$

and, in particular, tends to zero.

A similar result holds for the branch points of p_k/q_k. The problem of distributing points on S^2 is an old and important problem with many applications. It has been addressed by several branches of mathematics, for example in potential theory (see [RSZ]) and in arithmetic number theory (see [BSS]). This result is sharper than results found through potential theory or arithmetic number theory methods (although in this last case the bound for the expression in inequality (1.1) is for functions f in L^2 and not simply in $\mathcal{C}^2$). Indeed the bound (1.1) is optimal in the sense that we cannot expect to get a better asymptotic bound in k for $\mathcal{C}^2$ functions using second derivatives. The statement of the above Proposition 1, has nothing to do with symplectic geometry, it is simply a statement about rational maps. We try to look at it without using the techniques of [Do3] and give an explicit construction of polynomials which are experimentally seen to satisfy the required property. This involves the choice of two sets of k points, the zeroes of p_k and q_k on S^2. We will choose two sets of asymptotically uniformly distributed points which are a slight modification of the so called generalized spiral points. Generalized spiral points come from trying to solve a problem in potential theory, that of distributing a big number of charges on the 2 sphere subject to a logarithmic potential. They are described in [RSZ] as a good approximation of the actual solution to this problem. Even though the problem itself remains unsolved, some things are known about the optimal distribution. We will discuss the relations between this problem and our own. One of the upshots of this explicit construction is that it allows to experimentally determine a lower bound for the constant η appearing above thus answering part of a question asked by Donaldson in [Do1]: What is the best estimate one can give on the transversality of an asymptotically holomorphic sequence of sections of bundles? As for an upper bound, we have also established a method to calculate one, for each symplectic manifold. We prove the following in chapter 7:

Theorem 2 *Let X be a symplectic manifold of dimension $2n$ with symplectic form ω such that $[\omega/2\pi]$ lies in $H^2(X,\mathbb{Z})$, and a compatible almost complex structure. Let $L \to X$ be a Hermitian line bundle whose Chern class is $[\omega/2\pi]$. There exits $\eta_0 < 1$ such that, if we have n asymptotically holomorphic sequences of sections $s_0, \cdots, s_n$ of L^k satisfying $\eta \leq ||s_0||^2 + \cdots + ||s_n||^2 \leq 1$, then $\eta < \eta_0$.*

Although the method can be made totally explicit, thus yielding a numerical value for η_0, this value is probably not at all optimal, as one can see in the case of S^2, (for which it was calculated in section 4.2) by comparing with the experimental lower bound obtained in 6.2 .

A brief outline of this thesis is the following: In chapter 2, we give a review of the results in [Do1] and [Do3] as well as a proof that, in the complex setting, symplectic Lefschetz pencils can be taken to be holomorphic. In chapter 3, we explain how these results imply Proposition 1 and we also calculate an explicit upper bound for η appearing in the proposition, this bound turns out to be of the order $e^{-10^{36}}$! Chapter 4 deals with some properties of rational maps arising as quotients of polynomials as in Proposition 1. Namely, the property of having uniformly distributed fibers and uniformly distributed branch points (which is proved) and the property of minimizing a certain functional related to the dilation factor of the map (which is conjectured). Chapter 5 gives a construction of a sequence of polynomials which are experimentally know to satisfy the condition in Proposition 1, as well as some steps towards the proof of the fact that they, indeed, satisfy this property. We also describe some relations between estimated transversality and a famous problem in potential theory. Chapter 6 generalizes the method described in chapter 3 to find an upper bound for η in the general case, thus proving Theorem 2. We finish with some conjectures in Chapter 7.

Chapter 2

Existence results for Lefschetz Pencils

2.1 Complex Lefschetz pencils

Let X be a complex manifold of dimension n. Let $L \to X$ be a holomorphic line bundle.

Definition 1 *A divisor on X is the zero set of a section in $H^0(X, L)$.*

Next, we define linear system:

Definition 2 *A linear system on X is a set of divisors corresponding to a subspace of $H^0(X, L)$ for some complex line bundle $L \to X$.*

The base locus of a linear system is the intersection of all the divisors in it. The dimension of a linear system is the dimension of the subspace it corresponds to minus 1. A linear system of dimension N defines a map a map to $\mathbb{CP}^N$ away from its base locus. In fact, if one locally chooses a basis for the subspace of $H^0(X, L)$ corresponding to the linear system by trivializing the bundle L, denoting the basis $s_0, \cdots s_N$, this map is given by $[s_1 : \cdots : s_N]$.

Definition 3 *A pencil is a linear system corresponding to a subspace of dimension 2 of $H^0(X, L)$ for some complex line bundle $L \to X$.*

A Lefschetz pencil is the simplest type of pencil there can be. Namely, its fibers have only double points and no other type of singularities away from the base locus. A Lefschetz pencil is also the most generic type of pencil. Any pencil gives rise to a map from X minus a codimension 1 subvariety (its base locus) to $\mathbb{CP}^1$, that is, a meromorphic function. We often confuse the pencil with the map that it gives rise to since they contain the same information. A possible characterization of a Lefschetz pencil is the following: A holomorphic map $F : X \dashrightarrow \mathbb{CP}^1$ is a Lefschetz pencil if, at each point p, either

- F is a submersion at p,
- F is not defined at p and there are complex coordinates $z_1, \cdots z_n$, centered at p, with $F = z_1/z_2$,
- F is defined at p but it is not a submersion there and there are complex coordinates $z_1, \cdots z_n$, centered at p, with $F = z_1^2 + \cdots + z_n^2$.

There is an important existence theorem for nice linear systems:

Theorem 3 (Kodaira's Embedding) *Let X be a complex manifold and assume that it has a line bundle L and a connection on L whose curvature is a positive 2-form ω (i.e. $\omega(v, Jv) > 0$ for all non zero v). Then, for k big* *enough, the map defined by the linear system $H^0(X, L^k)$ is an embedding of X into $\mathbb{CP}^N$, for some N.*

Complex manifolds with 2-forms satisfying the above condition have a special name and play an important role in complex geometry

Definition 4 *A complex manifold, X, is said to be Kähler if it carries a 2-form ω such that:*

- *ω is compatible with the complex structure on X i.e. $\omega(., J.)$ defines a metric on X,*
- *ω is closed.*

From Kodaira's embedding theorem, it follows that every Kähler manifold with integer cohomology class has complex submanifolds which are also Kähler and even Lefschetz pencils. This is simply because, if X is Kähler and ω is its symplectic form satisfying $[\omega/2\pi] \in H^2(X, \mathbb{Z})$, then, there is a line bundle L whose first Chern class is $[\omega/2\pi]$. This line bundle carries a connection whose curvature is $i\omega$. We can apply Kodaira's embedding theorem above to conclude that, for k big enough, the map defined by the linear system $H^0(X, L^k)$ is an embedding. A general element of that linear system will give rise to a submanifold, Poincaré dual to $\omega/2\pi$. A generic 2 dimensional subspace of $H^0(X, L^k)$ will give rise to a Lefschetz pencil.

2.2 Symplectic Lefschetz pencils

The notion of Kähler manifold has been generalized to the following notion:

Definition 5 *A manifold, X, is said to be symplectic if it carries a 2-form, ω, such that:*

- *ω is non-degenerate, i.e., $\omega(\mathbf{v}, \mathbf{w}) = 0$ for all $\mathbf{w}$ implies $\mathbf{v} = 0$,*
- *ω is closed.*

A Kähler manifold is an example of a symplectic manifold.

A almost complex structure on X is a map $J : TX \to TX$ whose square is minus the identity. An almost complex structure, J, is said to be compatible with the symplectic structure, if $\omega(., J.)$ defines a metric on X. The almost complex structure, J, is said to be integrable, if there is a structure of complex manifold on X such that the complexification of J corresponds to multiplication by i in complex coordinates. Said in another way, a if X admits a complex chart such that, in coordinates $z_1, \cdots z_n$, the complexification of J is given by

$$J \frac{\partial}{\partial z_l} = i \frac{\partial}{\partial z_l}.$$

Every symplectic manifold admits almost-complex structures compatible with the symplectic form, but, not all symplectic manifolds admit integrable almost complex structures. Not all symplectic manifolds are Kähler.

Next, we give the definition of submanifold in the symplectic context:

Definition 6 *Given a symplectic manifold, X, with symplectic form ω, a symplectic submanifold of X is a submanifold Y, such that $\omega_{|Y}$ is non-degenerate.*

Clearly $\omega_{|Y}$ is closed therefore a symplectic submanifold is itself symplectic.

A first natural question to ask about symplectic manifolds would be: does every symplectic manifold have symplectic submanifolds? There have, so far, been two approaches to this question

1. The oldest one, by Gromov, is, given an almost-complex structure J on X, to look for 2-dimensional submanifolds which are J invariant. More precisely, Y a dimension 2 submanifold of X is a J holomorphic curve if, for all $y \in Y$, $J(T_yY) = T_yY$. Every holomorphic curve is symplectic.

2. The more recent one, the one we are going to be concerned with here, by Donaldson, is, assuming that $[\omega/2\pi]$ is an integral class and that $L \to X$ is a complex line bundle with $c_1(L) = [\omega/2\pi]$, to look for sections of the bundle $L^k \to X$ with $|\bar{\partial}s| < |\partial s|$ (here ∂ and $\bar{\partial}$ are with respect to some almost-complex structure). Their zero set will be a codimension 2 symplectic submanifold of X.

When a line bundle over a manifold with an almost-complex structure has a connection ∇, one can split the connection into a complex linear part and a complex anti-linear part. That is $\nabla = \partial + \bar{\partial}$ with:

$$\partial = \frac{\nabla \circ J + i\nabla}{2}, \quad \bar{\partial} = \frac{\nabla \circ J - i\nabla}{2}.$$

In his paper [Do1], Donaldson proves the existence of such sections. Let us state the main result in that paper:

Theorem 4 (Donaldson, [Do1]) *Let X be a manifold with a symplectic form ω, such that $[\omega/2\pi] \in H^2(X, \mathbb{Z})$ and let $L \to X$ be a complex Hermitian line bundle with a connection form, whose curvature is $i\omega$. For sufficiently large k, there is a sequence, s_k of sections of L^k such that:*

1. *$|s_k|$ is bounded by 1, $|\nabla s_k| \leq C\sqrt{k}$ and $|\nabla\nabla s_k| \leq Ck$ where C is independent of k,*
2. *$|\bar{\partial} s_k|$ is bounded by some constant C, independent of k,*
3. *there is a constant η, independent of k such that $|s_k| \leq \eta \implies |\partial s_k| \geq \eta\sqrt{k}$.*

When k is large enough, $C < \eta\sqrt{k}$ along the zero set of s_k, therefore $|\bar{\partial} s_k| < |\partial s_k|$ which implies that $s_k^{-1}(0)$ is a submanifold of X and that it is symplectic. Condition 3 in Theorem 4 plays an extremely important role in the story. A sequence satisfying it is said to be η transverse to zero (or simply η transverse). In fact, we can generalize this notion further:

Definition 7 *Let X be a manifold with a metric, $L \to X$ a Hermitian complex line bundle, $E \to X$ a Hermitian complex vector bundle and $\{\tau_k\}$ a sequence of sections of $E \otimes L^k$. Let η be a positive number. Then, we say that $\{\tau_k\}$ is*

η transverse to zero if

$$|\tau_k| \leq \eta \implies \langle [\nabla \tau_k]^* \mathbf{v}, [\nabla \tau_k]^* \mathbf{v} \rangle \geq \eta^2 k |\mathbf{v}|^2,$$

for all $\mathbf{v}$ section of $E \otimes L^k$.

Note that, if $\partial \tau_k$ is not surjective, this will not be possible, since for some $\mathbf{v} \neq 0$, $[\partial \tau_k]^* \mathbf{v} = 0$. The above definition is the same as asking that $|\tau_k| \leq \eta$, needs to imply that $\partial \tau_k$ is surjective and has a pointwise right inverse, whose norm is smaller than $\eta^{-1} k^{-1/2}$. Condition 2 in Theorem 4 is referred to as asymptotic holomorphicity. We will not reproduce the proof of Theorem 4 because in the next section we will give a proof of a similar result.

Generalizing this result further, Auroux in [Au1], proves Theorem 4 with L replaced by $L \otimes E$, where E is a complex vector bundle of any rank. Theorem 4 is an existence theorem for the symplectic analogs of linear systems of dimension 0. What about pencils, i.e., linear systems of dimension 1? The first thing to do is to generalize the notion to this new setting.

Definition 8 *Let X be a manifold with a symplectic form. Then, a map F, defined on X minus a codimension 4 manifold, is a symplectic Lefschetz pencil if, for every point $p \in X$, one of the following conditions is satisfied:*

- *F is a submersion at p,*
- *F is not defined at p, in which case, there are compatible complex coordinates $z_1, \cdots z_n$, centered at p, with $F = z_1/z_2$,*

- *F is defined at p, but it is not a submersion at p, there are compatible complex coordinates $z_1, \cdots z_n$, centered at p, with $F = z_1^2 + \cdots z_n^2$.*

Here, "compatible complex coordinates" simply means a map, defined locally around p to $\mathbb{C}^n$, such that, the pullback of the standard symplectic form on $\mathbb{C}^n$ is ω at the origin. Donaldson in [Do3], proves the following:

Theorem 5 (Donaldson, [Do3]) *Let X be a manifold with a symplectic form ω, such that $[\omega/2\pi] \in H^2(X, \mathbb{Z})$ and let $L \to X$ be a complex Hermitian line bundle with a connection form, whose curvature is $i\omega$. For sufficiently large k, there is a sequence of pairs of sections (s_0, s_1) of L^k such that:*

1. *$|s_0|^2 + |s_1|^2 \leq 1$, $|\nabla s_i| \leq C\sqrt{k}$ and $|\nabla\nabla s_i| \leq Ck$, $i = 0, 1$,*
2. *$|\bar{\partial} s_0|$ and $|\bar{\partial} s_1|$ are bounded by a constant independent of k,*
3. *s_0 is η transverse to zero, for some η independent of k,*
4. *(s_0, s_1) is η transverse to zero,*
5. *$\partial(s_1/s_0)$ is η transverse to zero, away from the zero locus of (s_0, s_1).*

As explained in [Do3], for large k, after perturbing the sections s_1 slightly, the map s_1/s_0 will give rise to a Lefschetz pencil. Again, we will prove a theorem similar to this one in the next subsection.

2.3 Estimated transversality in the complex setting

Since any Kähler manifold is also symplectic, one could apply Theorems 4 and 5 to the Kähler case. At first sight, these theorems seem not to generalize the existence theorems for complex submanifolds and holomorphic Lefschetz pencils, since they do not produce holomorphic sections of bundles. But Donaldson, in [Do1], proves that in the Kähler case, the asymptotically holomorphic condition in Theorem 4 can be strengthened to holomorphic. The result then becomes a new theorem for Kähler manifolds:

Theorem 6 (Donaldson, [Do1]) *Let X be a Kähler manifold with integral cohomology. Let ω be the symplectic form on X and let $L \to X$ be a complex Hermitian line bundle with a connection form, whose curvature is $i\omega$. For sufficiently large k, there is a sequence of holomorphic sections of L^k, $\{s_k\}$, such that:*

- *$|s_k|$ is bounded by 1,*
- *there is a constant η, independent of k, such that $|s_k| \leq \eta \implies |\partial s_k| \geq \eta\sqrt{k}$.*

It is then natural to ask: what does Theorem 5 say for a Kähler manifold?

Theorem 7 *Let X be a Kähler manifold with symplectic form ω, such that $[\omega/2\pi]$ is in $H^2(X, \mathbb{Z})$ and a complex Hermitian line bundle $L \to X$ with a* *connection form, whose curvature is* $i\omega$*. For sufficiently large* k*, there is a sequence of holomorphic sections of* L^k*,* (s_0, s_1)*, such that:*

1. $|s_0|^2 + |s_1|^2 \leq 1$*,* $|\nabla s_i| \leq C\sqrt{k}$ *and* $|\nabla\nabla s_i| \leq Ck$*,* $i = 0, 1$*,*

2. s_0 *is* η *transverse to zero, for some* η *independent of* k*,*

3. (s_0, s_1) *is* η *transverse to zero,*

4. $\partial(s_1/s_0)$ *is* η *transverse to zero, away from the zero locus of* (s_0, s_1)*.*

The existence of a sequence of holomorphic η transverse sections is proven in [Do1]. Our proof of Theorem 7 is very similar to the proof of the corresponding statement for sections appearing in [Do1]. For the sake of completeness, we will still proceed to give the details. We make use of the following lemma appearing in [Do3]:

Lemma 1 *Given a holomorphic map* $f : B^m(11/10) \to B^n(1)$ *and any small positive number* δ *there is an* $w \in \mathbb{C}^n$ *with norm smaller than* δ *such that* $f + w$ *is* $\eta = \delta/\log^p \delta$ *transverse to zero.*

This is the fundamental lemma in the proof of the main theorem in [Do3]. Its simpler version when $n = 1$ is the fundamental ingredient in the proof of the main result in [Do1].

We will also make use of a simpler lemma appearing in [Do1]:

Lemma 2 *There are constants a, b and c such that, given any $p \in X$ and any k large, there is a holomorphic section σ_p of L^k satisfying the following estimates:*

- $e^{-bkd^2(p,q)} \leq |\sigma_p(q)| \leq e^{-akd^2(p,q)}$, *if* $d(p,q) \leq ck^{-1/3}$,
- $|\sigma_p(q)| \leq e^{-ak^{1/3}}$, *if* $d(p,q) \geq ck^{-1/3}$.

The section σ_p also satisfies:

- $|\partial\sigma_p(q)| \leq \sqrt{k}e^{-akd^2(p,q)}$, *if* $d(p,q) \leq ck^{-1/3}$,
- $|\partial\sigma_p(q)| \leq \sqrt{k}e^{-ak^{1/3}}$, *if* $d(p,q) \geq ck^{-1/3}$.

proof: We will first construct a section, defined in a neighborhood of radius $k^{-1/3}$ of q, which is holomorphic and satisfies the first inequality in the lemma. If X were locally flat around p, then, this would be easy. In fact, we would be able to choose complex coordinates centered at p, $z = \{z_\alpha\}$, such that $\omega = \sum dz_\alpha \wedge d\bar{z_\alpha} = \omega_0$. We could then choose $A_0 = \sum z_\alpha d\bar{z_\alpha} - \bar{z_\alpha} dz_\alpha$ to be the connection form on L. Then, $e^{-|z|^2} \otimes \mathbf{1}$, would be a holomorphic section because:

$$\nabla^{(0,1)} e^{-|z|^2} = \sum \bar{\partial} e^{-|z|^2} + z_\alpha d\bar{z_\alpha} e^{-|z|^2} = \sum (-z_\alpha d\bar{z_\alpha} + z_\alpha d\bar{z_\alpha}) e^{-|z|^2} = 0.$$

It would clearly satisfy the required inequality. The general case follows from this one by noting that there are always complex coordinates, $z = \{z_\alpha\}$, centered at p, such that $\omega = \omega_0 + \alpha$, where $||\alpha|| \leq C|z|^2$. Locally, one can write

$\alpha = \bar{\partial}\partial f$ for some f. Set $B = \partial f$ and $A = A_0 + B$. Then A is a primitive of ω and its $(0,1)$ part is the same as that of A_0, therefore (using A for the connection on L), the section $e^{-|z|^2} \otimes \mathbf{1}$ is still holomorphic. Its norm is not $e^{-|z|^2}$ though. Instead, it is $e^{-|z|^2}\sqrt{u}$, where u satisfies $d\ln(u) = B + \bar{B}$. Now, $dB = \alpha$ and B can be chosen to be 0 at p so, $||B|| \leq C|z|^3$. Choosing u to be 1 at p, we then have, $|\ln(u)| \leq C|z|^4$ and in particular

$$-\epsilon|z|^2 \leq \ln(u) \leq \epsilon|z|^2,$$

for small ϵ. The norm of $e^{-|z|^2} \otimes \mathbf{1}$ is therefore smaller than $e^{-(1-\epsilon/2)|z|^2}$ and bigger than $e^{-(1+\epsilon/2)|z|^2}$. Hence, the sections of L^k, $e^{-k|z|^2} \otimes \mathbf{1}^{\otimes k}$ satisfy the first inequality in the lemma. The next step is to extend these sections to X. Let β be a $\mathcal{C}^\infty$ bump function that is 1 in $[0, 1/2[$ and is 0 in $[1, \infty[$. Set

$$s = \beta(k^{2/3}|z|^2)e^{-k|z|^2} \otimes \mathbf{1}^{\otimes k}.$$

Then, there exits ξ, a section of L^k, such that $s + \xi = \sigma$ is holomorphic and $||\xi||_{L^2(X)} \leq ||\bar{\partial}s||_{L^2(X)}$. Simply take $\xi = -\bar{\partial}^* G\bar{\partial}s$, where G is the Green's operator. Over the ball of radius $k^{-1/3}/2$, because s is holomorphic, σ coincides with it and satisfies the required inequalities. To see that it also does so outside this ball, we start by noting that, in the ball of radius $k^{-1/2}$, the L^∞ norm of a holomorphic section is bounded by a constant (independent of k) times its L^2 norm in a ball of radius $Ck^{-1/2}$ for some constant C greater than 1 and independent of k. The bundle $L^k \to B(k^{-1/2})$ is isomorphic to $L \to B(1)$ trough dilation, so, we need only check the corresponding statement for sections of L. Choose a non vanishing, holomorphic section of L, τ (we may have to shrink the size of the neighborhood of p to be able to do this but only by a fixed amount independent of k). Any holomorphic section of L can be written as $f\tau$, for some holomorphic function f. The statement will follow from the corresponding statement for f. Now take any point with coordinates z, where $|z| < 1$. For any $r \leq 1$, the Cauchy formula for f says:

$$f(z) = \int_{\triangle_r} \frac{f(w_1, \cdots, w_n)}{(z_1 - w_1) \cdots (z_n - w_n)} dw_1 \cdots dw_n,$$

where $\triangle_r$ is a polydisc of radius r centered at z. So,

$$f(z) = 2 \int_{1/2}^{1} \int_{\triangle_r} \frac{f(w_1, \cdots, w_n)}{(z_1 - w_1) \cdots (z_n - w_n)} dw_1 \cdots dw_n dr.$$

Now,

$$|z_1 - w_1| \geq 1/2, \cdots, |z_n - w_n| \geq 1/2,$$

so we obtain

$$|f(z)| \leq ||f||_{L^1(\triangle_1)} \leq C||f||_{L^1(B(C))} \leq C||f||_{L^2(B(C))},$$

where C is some constant for which $\triangle_1$ is contained in $B(C)$. Next, we show that the L^2 norms of both s and ξ, on the complement of the ball of radius $k^{-1/3}/4$ and center p, are bounded by $e^{-ak^{1/3}}$, for some positive constant a independent of k. This will be enough, because, given a point q on the complement of the ball of radius $k^{-1/3}/2$ centered at p,

$$|\sigma|(q) \leq ||\sigma||_{L^\infty(B(q,k^{-1/2}))} \leq ||\sigma||_{L^2(B(q,Ck^{-1/2}))},$$

since σ is holomorphic. In turn,

$$||\sigma||_{L^2(B(q,Ck^{-1/2}))} \leq ||s||_{L^2(B(q,Ck^{-1/2}))} + ||\xi||_{L^2(B(q,Ck^{-1/2}))}.$$

The ball of radius $Ck^{-1/2}$ centered at q will be contained in the complement of $B(k^{-1/3}/4)$ (as long as k is big enough) so

$$||s||_{L^2(B(q,Ck^{-1/2}))} \leq ||s||_{L^2(B^c(k^{-1/3}/4)} \leq e^{-ak^{1/3}}$$

and

$$||\xi||_{L^2(B(q,Ck^{-1/2})} \leq ||\xi||_{L^2(B^c(k^{-1/3}/4)} \leq e^{-ak^{1/3}},$$

therefore $|\sigma|(q) \leq e^{-ak^{1/3}}$ (for a different a, but one which is still independent of k). The bound on the L^2 norm of s comes from the pointwise bound that $|s|$ satisfies in the ball of radius $k^{-1/3}$

$$\int_{B^c(k^{-1/3}/4)} |s| = \int_{k^{-1/3}/4 \leq |z| \leq k^{-1/3}} e^{-k|z|^2} \leq e^{-ak(k^{-1/3})^2} = e^{-ak^{1/3}}.$$

As for the bound on the L^2 norm of ξ, it comes from the inequality

$$||\xi||_{L^2(X)} \leq ||\bar{\partial}s||_{L^2(X)}.$$

Since $e^{-k|z|^2} \otimes \mathbf{1}^{\otimes k}$ is holomorphic then

$$\bar{\partial}s = k^{2/3}\beta'\bar{\partial}|z|^2 e^{-k|z|^2} \otimes \mathbf{1}^{\otimes k}$$

and $|\bar{\partial}s| \leq k^{1/3}e^{-ak|z|^2}$. So, for a different a,

$$||\xi||_{L^2(X)} \leq \int_{k^{-1/3}/4 \leq |z| \leq k^{-1/3}} k^{1/3}e^{-ak|z|^2} \leq e^{-ak^{1/3}}.$$

As for the estimates on the derivatives, they come from the same type of reasoning.

We will also need a refinement of this lemma, namely:

Lemma 3 *There are constants a, b and c such that, given any $p \in X$ and any k, there are holomorphic section σ_p^α of L^k, satisfying the following estimates:*

- $e^{-bkd^2(p,q)} \leq |\sigma_p(q)| \leq e^{-akd^2(p,q)}$, *if* $d(p,q) \leq ck^{-1/2}$,
- $|\sigma_p(q)| \leq e^{-ak^{1/3}}$, *if* $d(p,q) \geq ck^{-1/2}$,

and such that:

- $\sigma_p^\alpha(p) = 0$,
- $\bar{\partial}\sigma_p^\alpha(p) = d\bar{z}_\alpha \otimes \mathbf{1}$.

The section σ_p^α also satisfies:

- $|\partial\sigma_p^\alpha(q)| \leq \sqrt{k}e^{-akd^2(p,q)}$, *if* $d(p,q) \leq ck^{-1/3}$,
- $|\partial\sigma_p^\alpha(q)| \leq \sqrt{k}e^{-ak^{1/3}}$, *if* $d(p,q) \geq ck^{-1/3}$.

proof: Consider the section σ_p from the previous lemma. Let $z = \{z_\alpha\}$ be centered coordinates at p and β the same bump function as before. Set $s^\alpha = \beta(k^{2/3}|z|^2)z_\alpha\sigma_p$ on $B(k^{-1/3})$. Close to p, this section coincides with $z_\alpha\sigma_p$ and since σ_p satisfies the inequalities in Lemma 2 in that region, s^α also does. We have:

$$\bar{\partial}s^\alpha(p) = (\beta d\bar{z}_\alpha + z_\alpha k^{2/3}\beta'\bar{\partial}|z|^2) \otimes \sigma_p,$$

because σ_p is holomorphic. At the origin, $\bar{\partial}s^\alpha(p) = d\bar{z_\alpha} \otimes \mathbf{1}$. Again, we need to perturb s^α to make it holomorphic in all of X. If we apply the process from the previous lemma, we note that the essential fact there, was, that the section to be perturbed had small $\bar{\partial}$ (and satisfied the inequalities 2 close to p). To be more precise, the relevant inequality was

$$||\bar{\partial}s^\alpha||_{L^2(B^c(k^{-1/3}/2)} \leq e^{-ak^{1/3}}.$$

Now,

$$\bar{\partial}s^\alpha(p) = (\beta d\bar{z}_\alpha + z_\alpha k^{2/3}\beta'\bar{\partial}|z|^2) \otimes \sigma_p,$$

and the form $\beta d\bar{z}_\alpha + z_\alpha k^{2/3}\beta'\bar{\partial}|z|^2$ is bounded independently of k. Since

$$|z| \geq k^{-1/3}/2 \implies |\sigma_p| \leq e^{-ak^{1/3}},$$

for some constant a, the result follows.

The next result which is needed is a covering lemma from [Do1]:

Lemma 4 *There is a constant C such that, for any k, there is a collection of points of X, $\{p_i\}$, satisfying*

$$\sum_{p_i} e^{-kd^2(p,p_i)} \leq C,$$

such that, the balls of radius $k^{-1/2}$ centered at those points cover X. Also, given any $D \geq 0$ the points can be chosen in such a way that, it is possible to partition them into $N \sim D^{2n}$ sets (we label these with "colors") with the following property: if any two points p_i and p_j belong to the same partition, *then,*

$$d(p_i, p_j) \geq k^{-1/2} D.$$

Also,

$$d(p, p_j) \leq ck^{1/2} \implies \sum_{p_i, c(p_i)=j, i\neq j} e^{-kd^2(p,p_i)} \leq Ce^{-aD^2}.$$

Here we have used the notation $c(p_i)$ to mean the color of p_i and we assume that we have numbered the colors. The point is that, if you choose such a covering $\{p_i\}$, then there is a constant C independent of k such that

$$|\sum_{p_i} \sigma_{p_i}| \leq C.$$

This is because, given a point $p \in X$,

$$|\sum_{p_i} \sigma_{p_i}(p)| = \sum_{p_i, d(p,p_i)\leq ck^{-1/3}} |\sigma_{p_i}(p)| + \sum_{p_i, d(p,p_i)\geq ck^{-1/3}} |\sigma_{p_i}(p)|.$$

The first term in this sum is bounded by

$$\sum_{p_i} e^{-kd^2(p,p_i)},$$

which is bounded by C. As for the second term it is bounded by

$$\sharp\{p_i\} e^{-ak^{1/3}}.$$

Since

$$\sharp\{p_i\} \sim k^{n/2},$$

this is bounded by C as well, as long as k is big enough. Also, given p_j with color l,

$$p \in B(p_j, ck^{1/2}) \implies |\sum_{p_i, c(p_i)=l} \sigma_{p_i}| \leq Ce^{-aD^2}.$$

This inequality is again a consequence of the fact that,

$$\sum_{p_i, d(p,p_i)\geq ck^{-1/3}} |\sigma_{p_i}| \leq Ck^{n/2}e^{-ak^{1/3}} \leq e^{-aD^2}.$$

We have to change a a bit but we still denote it by the same letter. The same type of inequality holds true for the derivatives of σ_p, namely

$$\sum_{p_i, c(p_i)=l} |\partial\sigma_{p_i}| \leq Ce^{-aD^2}.$$

We can also substitute σ_{p_i} by $\sigma^{\alpha}_{p_i}$ in these inequalities.

We are now in a position to get started with the proof of Theorem 7. The first step is to get s_0, a section of L^k, for big k, which is holomorphic and η transverse to zero, for some η, independent of k. The strategy is as follows: apply Lemma 4 with some D, (to be determined later) to get a collection of points $\{p_i\}$. We will look for a section, s_0, of the form

$$\sum_{p_i} w_i\sigma_{p_i}, \quad |w_i| \leq 1.$$

This will be done in $N \sim D^{2n}$ steps, one for each color (where N is the number coming from Lemma 4). We first make the two following two observations,

1. Consider a sequence of section s_k of L^k which are η transverse to zero for some η independent of k. Let ξ_k denote another sequence of sections of L^k which satisfy $|\xi_k| \leq C\sqrt{k}$ and $|\nabla\xi_k| \leq C$, for some constant C, independent of k. Then, $s_k + \xi_k$ is $\eta - C$ transverse to zero. In fact, if $|s_k + \xi_k| \leq \eta - C$, then $|s_k| \leq \eta$ and, by transversality, $|\nabla s_k| \geq \eta\sqrt{k}$ which in turn implies $|\nabla(s_k + \xi_k)| \geq (\eta - C)\sqrt{k}$.

2. Let f be a function defined on a neighborhood of radius of order $k^{-1/2}$ of a point p in X. Using charts, interpret f as a function on a ball of center 0 and radius $11/10k^{-1/2}$. Also, let w be a complex number and η a positive number satisfying:

$$|f - w| \leq \eta e^b \implies |\partial f| \geq \eta(1 + C)e^b k^{1/2},$$

where b and C come from Lemma 4. Then, the section $f\sigma_p - w\sigma_p$ is η transversal to zero. This is, again, very simple to check. If $|f\sigma_p - w\sigma_p| \leq \eta$ then $|f - w| \leq \eta e^b$. Now,

$$\partial(f\sigma_p - w\sigma_p) = (\partial f)\sigma_p + (f - w)\partial\sigma_p.$$

Since $|f - w| \leq \eta e^b$, then

$$|(\partial f)\sigma_p| \geq \eta(1 + C)e^b\sqrt{k}e^{-b},$$

and

$$|(f - w)\partial\sigma_p| \leq C\eta e^b\sqrt{k}e^{-b},$$

therefore

$$|\partial(f\sigma_p - w\sigma_p)| \geq \eta(1 + C)e^b\sqrt{k}e^{-b} - \eta e^b\sqrt{k}e^{-b} = \eta\sqrt{k}.$$

Start with any holomorphic section, s, of L^k

- In the first step consider, for each of the balls $B(p_i, k^{-1/2})$ in the first color, the restriction of s to that ball. In the trivialization given by

σ_{p_i}, s is given by a function, which we can interpret as a function f_i on $B(11/10)$, (by using a chart and dilation). Apply Lemma 1 (in fact a slight modification of it) to this function and $\delta_1 = 1/2$. One obtains w_i, such that $|w_i| \leq \delta_1$ and

$$|f - w| \leq \eta e^b \implies |\partial f| \geq \eta(1 + C)e^b k^{1/2},$$

where $\eta_1 = \delta_1 / \log^p \delta_1$. Now consider the section $s - w_i \sigma_{p_i}$. By the second observation above, this is η_1 transverse to zero over $B(p_i, k^{-1/2})$. Set

$$s_0^1 = s - \sum_{c(p_i)=1} w_i \sigma_{p_i}.$$

We will check that s_0^1 is μ_1 transverse to zero over the balls of radius $k^{-1/2}$ centered at points whose color is, 1 for some number μ_1. Given p_j with $c(p_j) = 1$, over $B(p_j, ck^{-1/2})$,

$$s_0^1 = s - w_j \sigma_{p_j} + \sum_{p_i, c(p_i)=1, i \neq j} w_i \sigma_{p_i},$$

and

$$\sum_{c(p_i)=1, i \neq j} w_i \sigma_{p_i}$$

is bounded in norm by $\delta_1 e^{-aD^2} C$, for some constant C, independent of k. Also, the derivative of this sum is bounded by $\sqrt{k} \delta_1 e^{-aD^2} C$. By the first observation above, this allows us to conclude that s_0^1 is $\mu_1 = \eta_1 - \delta_1 e^{-aD^2} C$ transverse to zero over $B(p_j, ck^{-1/2})$.

- Suppose we are given sections of L^k, s_0^l which are μ_l transverse to zero, for some $\mu_l > 0$, over all balls of radius $ck^{-1/2}$, centered at points of the covering that have color smaller than l. We will construct s_0^{l+1}, μ_{l+1} transverse to zero over all balls of radius $ck^{-1/2}$ centered at points of the covering that have color smaller than $l+1$. Over each of the balls $B(p_i, k^{-1/2})$ in the color $l+1$, with respect to the trivialization given by σ_{p_i}, s_l is given by a function, which we can interpret as a function f_i on $B(11/10)$ (by using a chart and dilation). We apply Lemma 1 (in fact a slight modification of it) to this function and $\delta_{l+1} = \mu_{l+1}/2C$. We get w_i, such that $|w_i| \leq \delta_l$ and

$$|f - w| \leq \eta e^b \implies |\partial f| \geq \eta(1+C)e^b\sqrt{k},$$

where

$$\eta_{l+1} = \frac{\delta_{l+1}}{\log^p \delta_{l+1}}.$$

Now consider the section $s_0^l - w_i\sigma_{p_i}$. By a previous observation, this is η_{l+1} transverse to zero over the ball of radius $k^{-1/2}$, centered at p_i. Set

$$s_0^{l+1} = s_0^l - \sum_{p_i, c(p_i)=l+1} w_i\sigma_{p_i}.$$

The sum

$$\sum_{p_i, c(p_i)=l+1} w_i\sigma_{p_i}$$

is bounded from above by $C\delta_{l+1}$ (because all the $|w_i|$ are) and, for the same reason, the derivative of this sum is bounded from above by $C\delta_{l+1}\sqrt{k}$. This implies that s_0^{l+1} is $\mu_l - C\delta_{l+1} = \mu_l/2$ transverse to zero, over the balls centered at points with color $\leq l$. If p_j is a point with color $l+1$, over the ball of center p_j and radius $ck^{-1/2}$,

$$s_0^{l+1} = s_0^l - w_j \sigma_{p_j} + \sum_{p_i, c(p_i)=l+1, i \neq j} w_i \sigma_{p_i}.$$

By construction, $s_0^l - w_j \sigma_{p_j}$ is η_{l+1} transverse to zero and

$$\sum_{c(p_i)=l+1, i \neq j} w_i \sigma_{p_i}$$

is bounded in norm by $\delta_1 e^{-aD^2} C$ for some constant C, independent of k. Also, the derivative of this sum is bounded by $\sqrt{k}\delta_1 e^{-aD^2} C$. This allows us to conclude that s_0^{l+1} is $\eta_{l+1} - \delta_{l+1} e^{-aD^2} C$ transverse to zero, over the ball centered at p_j and radius $ck^{-1/2}$, as long as this quantity is positive. So, over the balls centered at points whose color is smaller than $l+1$, s_0^{l+1} is

$$\mu_{l+1} = \min\left(\frac{\mu_l}{2}, \eta_{l+1} - \delta_{l+1} e^{-aD^2} C\right)$$

(which is $\eta_{l+1} - \delta_{l+1} e^{-aD^2} C$) transverse to zero.

This would allow us to conclude by induction, as long as we knew that, at all steps

$$\eta_{l+1} - \delta_{l+1} e^{-aD^2} C > 0.$$

We need to prove that we can choose D make

$$\eta_{l+1} - \delta_{l+1} e^{-aD^2} C > 0, \quad \forall l \leq N \sim D^{2n}.$$

Lemma 5 *Let $\{\delta_l\}$ be a sequence defined by induction by $\delta_0 = 1/2$ and*

$$\delta_{l+1} = \frac{\delta_l}{2C \log^p(1/\delta_l)} - \frac{\delta_l e^{-aD^2}}{2}.$$

Then, there is a D such that $\delta_N > 0$ where $N \sim D^{2n}$.

First, note that $\{\delta_l\}$ is decreasing so this lemma is enough for our purposes. What we want is D, such that,

$$\log^p(1/\delta_{N)} < \frac{e^{aD^2}}{C}, \;\; N \sim D^{2n}$$

(C has changed). Consider the sequence given by induction trough $\lambda_1 = 1/2$ and

$$\lambda_{l+1} = \frac{\lambda_l}{4C\log^p(1/\lambda_l)}. \tag{2.1}$$

Suppose we find D such that

$$\frac{e^{-aD^2}}{C} < \frac{1}{2\log^p(1/\lambda_{N)}}, \;\; N \sim D^{2n}.$$

Then, this will be enough. In fact, then,

$$\lambda_{l+1} < \frac{\lambda_l}{2C\log^p(1/\lambda_l)} - \frac{\lambda_l e^{-aD^2}}{2}$$

and so by induction $\lambda_l < \delta_l$, in particular $\lambda_N < \delta_N$ and $\delta_N > 0$. If we set $u_l = \log(1/\lambda_l)$ then, equation (2.1) becomes:

$$u_{l+1} = u_l + \log 4C + p\log u_l.$$

We want to find D such that $2u_N^p < e^{aD^2}$. It will be enough to find D for which $u_N < e^{aD^2/p}$. Let f be a function such that, for all $x \geq 1$,

$$f(x+1) - f(x) \geq p\log f(x) + \log 4C$$

and $f(1) \geq 1/2$. Then, $f(l) \geq u_l$, for all integer l and we are done if we find $D > 0$, for which $f(N) < e^{aD^2/p}$. Just take $f(x) = (x+c)^2$. We have,

$$f(x+1) - f(x) - p \log f(x) = 2x + 2c + 1 - 2p \log (x + c),$$

which is $3 + 2c - 2p \log (1 + c)$ at 1 (in particular greater than $\log 4C$, if c is big enough). On the other hand equation (2.3) defines an increasing function, as long as $c > p$, for $x \geq 1$. So, for c big enough, this function does the job.

We move to the next step, namely, building a sequence of pairs of sections of L^k, with small norm, that are η transverse to zero. This can be done by using a modification of the method used to build s_0. The main difference is in what corresponds to the second observation above, in this setting. Consider a section of $L^k \oplus L^k$ over $B(p, rk^{-1/2})$, which we represent by a pair of functions (f, g), with respect to the trivialization σ_p. Assume we knew that $F = (f, g)$ was η transverse to some vector in $\mathbb{C}^2$, w i.e.

$$|F - w| \leq \eta \implies \langle [dF]^*(\mathbf{v}), [dF]^*(\mathbf{v}) \rangle \geq \eta^2 k |\mathbf{v}|^2, \forall \mathbf{v}.$$

Without loss of generality, let us assume that $w = 0$. Suppose we are at a point where $|\sigma_p F| \leq e^{-b}\eta$, then, if we represent a section of $L^k \oplus L^k$ over $B(p, rk^{-1/2})$ by a pair of functions (u, v), $\langle [\nabla \sigma_p (f, g)]^*(u, v), [\nabla \sigma_p (f, g)]^*(u, v) \rangle$ is the sum of the following terms:

$$|\sigma_p|^4 [|u|^2 |\nabla f|^2 + |v|^2 |\nabla g|^2 + 2\Re(u\bar{v} \langle \nabla f, \nabla g \rangle)],$$

which is greater than a constant times $\eta^2 k|(u,v)|^2$, and

$$|[\nabla\sigma_p]^*\sigma_p|^2[|f|^2|u|^2+|g|^2|v|^2+2\Re(\bar{f}gu\bar{v})],$$

which is smaller than a constant times $\eta^2 k|(u,v)|^2$, and

$$|\sigma_p|^2[f\langle|u|^2\nabla f+u\bar{v}\nabla g,[\nabla\sigma_p]^*\sigma_p\rangle+g\langle|v|^2\nabla g+u\bar{v}\nabla f,[\nabla\sigma_p]^*\sigma_p\rangle.$$

Now this term is smaller than a constant times $\eta k|(u,v)|^2$, so, it is not good enough to have the first term greater than $\eta^2 k|(u,v)|^2$. Instead, we need

$$|F-w|\leq\eta \implies \langle[dF]^*(\mathbf{v}),[dF]^*(\mathbf{v})\rangle\geq\sqrt{\eta}k^{1/2}|\mathbf{v}|^2.$$

Alternatively, with the first condition we can ensure that $F\sigma_p$ is $\sqrt{\eta}$ transverse, given that F is η transverse. This is enough, since, then, the argument for the construction of a single transverse section follows trough, with the only difference that, we need to consider a sequence defined by

$$\delta_{l+1}=\frac{\sqrt{\delta_l}}{2C\log^p(1/\delta_l)}-\frac{\delta_l e^{-aD^2}}{2}.$$

In fact p should be $p/2$, but the argument holds for every p. This is again decreasing and we can use the same reasoning to show that $\delta_N>0$. Now that we have constructed s_0 and s_1 that satisfy the two first conditions in the main theorem, we will modify s_1 so as to have $\partial(s_0/s_1)$ η transverse to zero. This is done exactly as in [Do3] with the exception of lemma 3 which has its asymptotically holomorphic counter-part there. We will reproduce the argument here, for the sake of completeness. It makes use of the following lemma in [Do3]:

Lemma 6 *Let (s_0, s_1) be a sequence of pairs of holomorphic sections of L^k, such that, for k big enough,*

1. $|s_0|^2 + |s_1|^2 \leq 1$,

2. *s_0 is η transverse to zero,*

3. *(s_0, s_1) is η transverse to zero.*

Then, there is $\chi = \chi(\eta)$ such that

$$|\partial(\frac{s_1}{s_0})| \leq 2\eta \implies |s_0| \geq \chi.$$

This is proven in [Do3] in a more general setting, namely, that of asymptotically holomorphic s_0 and s_1. There is yet another lemma in [Do3] which comes into play:

Lemma 7 *Let δ be a positive number smaller than $1/2$. For sequences s_0 and s_1 as above and small r, there is a $\pi \in \mathbb{C}^n$, with norm smaller than δ such that*

$$\partial \left(\frac{s_1 + \sigma_{p,\pi}}{s_0} \right)$$

is $\delta / \log^p(1/\delta)$ transverse to zero, over the ball of center p and radius $rk^{-1/2}$, where

$$\sigma_{p,\pi} = \sum_{\alpha} \pi^\alpha \sigma_p^\alpha,$$

and σ_p^α comes from Lemma 3.

Now use Lemma 4 to cover X with balls of radius $rk^{-1/2}$. Discard those balls that have points where $|s_0|$ is smaller than χ. Inside these balls,

$$\left|\partial\left(\frac{s_1}{s_0}\right)\right| \geq 2\eta,$$

so $\partial(s_1/s_0)$ is η transverse. We construct a modification of s_1, s_1^N in N steps

- In the first step, we modify s_1 over the balls centered at points colored with the first color. Let p_i be one such point. Apply Lemma 7 to $\delta_1 = \eta/2C$ to obtain π_i such that

$$\partial\left(\frac{s_1 + \sigma_{p_i,\pi_i}}{s_0}\right)$$

is $\eta_1 = \delta_1/\log^p(1/\delta_1)$ transverse to zero over the ball of radius $rk^{-1/2}$ and center p_i. Set

$$s_1^1 = s_1 + \sum_{p_i, c(p_i)=1} \sigma_{p,\pi}.$$

Over a ball centered at p_j, with $c(p_j) = 1$,

$$\partial\left(\frac{s_1^1}{s_0}\right) = \partial\left(\frac{s_1 + \sigma_{p_j,\pi_j}}{s_0}\right) + \partial\left(\frac{\sum_{c(p_i)=1, i\neq j} \sigma_{p_i,\pi_i}}{s_0}\right).$$

Now

$$\partial\left(\frac{\sum_{c(p_i)=1, i\neq j} \sigma_{p_i,\pi_i}}{s_0}\right) = \sum_{c(p_i)=1, i\neq j} \frac{\partial\sigma_{p_i,\pi_i}}{s_0} - \sum_{c(p_i)=1, i\neq j} \frac{\sigma_{p_i,\pi_i}}{s_0}\frac{\partial s_0}{s_0},$$

therefore, the norm

$$\left|\partial\left(\frac{\sum_{c(p_i)=1, i\neq j} \sigma_{p_i,\pi_i}}{s_0}\right)\right| \leq \frac{C}{\chi}\delta_1 k^{1/2} e^{-aD^2}$$

and the norm of its derivative is bounded by $(C/\chi)k\delta_1 e^{-aD^2}$, so that, $\partial(s_1^1/s_0)$ is $\eta_1 - (C/\chi)\delta_1 e^{-aD^2}$ transverse to zero, over each of these balls.

- Assume we have s_1^l, such that, $\partial(s_1^l/s_0)$ is η_l-transverse to zero, over the balls centered at points whose color is smaller than l. Set $\delta_{l+1} = \chi\eta_l/2C$. Let p_i be a point with $c(p_i) = l+1$. Then, there is π_i such that $|\pi_i| \leq \delta_l$ and such that
$$\partial\left(\frac{s_1^l + \sigma_{p_i,\pi_i}}{s_0}\right)$$
is $\eta_{l+1} = \delta_{l+1}/\log^p(1/\delta_{l+1})$ transverse to zero, over the ball of radius $rk^{-1/2}$ and center p_i. Choose
$$s_1^{l+1} = s_1^l + \sum_{p_i, c(p_i)=l+1} \sigma_{p_i,\pi_i}.$$
As before, if p_j has color $l+1$, then $\partial(s_1^{l+1}/s_0)$ is $\eta_{l+1} - (C/\chi)\delta_1 e^{-aD^2}$ transverse to zero, over the ball of radius $rk^{-1/2}$ and center p_i. If p_j has color $\leq l$, on the other hand, $\partial(s_1^{l+1}/s_0)$ is $\eta_l - (C/\chi)\delta_l = \eta_l/2$ transverse to zero.

We are in the same situation as we were when we were constructing s_0. If we apply Lemma 5 we are done. The only thing that remains to be checked is that we have not destroyed the two previously achieved transversality properties. But this is clear because the new s_1 differs from the old one by a section, whose norm is smaller that $C\delta_1 = \eta/2$, so that, the new pair (s_0, s_1) is $\eta/2$ transverse to zero. Note that this construction could be used to prove the following:

Proposition 2 *Let X be a Kähler manifold of dimension n with symplectic form ω such that $[\omega/2\pi]$ lies in $H^2(X,\mathbb{Z})$. Let $L \to X$ be a line bundle whose*

Chern class is $[\omega/2\pi]$. *There exits* η *and* n *sequences of holomorphic sections* $s_0, \cdots, s_n$ *of* L^k *satisfying*

$$\eta \leq ||s_0||^2 + \cdots + ||s_n||^2 \leq 1.$$

Chapter 3

Transversality and uniform distribution

Donaldson's construction gives rise to sections, which themselves give rise to several submanifolds (the fibers of Lefschetz pencils, away from their singularities for example). An important feature of those, is that they are asymptotically uniformly distributed.

Theorem 8 (Donaldson) *Let X be a symplectic manifold with an almost-complex structure and $L_k \to X$ be complex line bundles with curvature $\omega_k/2\pi$, such that we can cover X by balls $B_i = B(p_i, rk^{-1/2})$, where L_k have local trivializations, σ_i, such that:*

- $0 < \alpha \leq |\sigma_i| \leq 1$, *for some α independent of k,*
- $|\nabla \sigma_i| \leq \beta\sqrt{k}$.

Suppose further that s_k is a sequence of sections of L_k for which there is a sequence a_k of numbers, bounded away from zero such that:

- $|s_k| \leq aa_k$, $|\nabla s_k| \leq b\sqrt{k}a_k$,
- $|\bar{\partial} s_k| = O(a_k)$,
- $|s_k| < a_k \implies |\nabla s_k| \geq 2\frac{\beta}{\alpha} a_k \sqrt{k}$.

Then, $s_k^{-1}(0)$ become uniformly distributed, with the following estimate:

$$| \int_{s_k^{-1}(0)} \psi - \frac{1}{2\pi} \int_X \omega_k \wedge \psi | \leq \sqrt{k} |d\psi|_\infty,$$

for all ψ, a $2n-2$ form on X.

The proof can be found in [Do1]. We will sketch it here for the sake of completeness. **proof:** Let $A = \frac{\nabla s}{s}$. The first thing to note is that, for any $\psi \in \Omega^{2n-2}(X)$,

$$\int_X A \wedge d\psi - \frac{1}{2\pi} \int_X \omega_k \wedge \psi = \int_{s_k^{-1}(0)} \psi.$$

This is because, if you take U to be a shrinking tubular neighborhood of $s_k^{-1}(0)$, then from Stokes' theorem

$$\int_{X \setminus U} A \wedge d\psi - \int_{X \setminus U} dA \wedge \psi = \int_{\partial(X \setminus U)} A \wedge \psi.$$

Now

- dA is the curvature of L_k divided by 2π,
- A has an integrable singularity at any point of U,
- as U shrinks to $s_k^{-1}(0)$, $\int_{\partial(X \setminus U)} A \wedge \psi$ tends to $\int_{s_k^{-1}(0)} \psi$.

The result follows. We see that to prove the theorem, we need only show that $\int_X |A| \leq C\sqrt{k}$. Since we know $|\nabla s_k| \leq b\sqrt{k}a_k$, it is enough to see that

$$\int_X \frac{1}{|s|} \leq \frac{C}{a_k}.$$

Now $s = f_i\sigma_i$ over B_i. Rescaling by a $k^{1/2}$ factor, we get a function $\tilde{f}_i$ defined on a ball of radius r, $\tilde{B}_i$ which satisfies

$$|\tilde{f}_i| \leq a_k \implies |d\tilde{f}_i| \geq Ca_k.$$

This is because

$$|\tilde{f}_i| \leq a_k \implies |s| = |f_i||\sigma_i| \leq a_k,$$

which we know implies $|\nabla s_k| \geq 2\beta/\alpha a_k\sqrt{k}$. Since

$$df_i = \frac{\nabla s}{\sigma_i} - f_i\frac{\nabla \sigma_i}{\sigma_i},$$

then

$$|df_i| \geq 2\frac{\beta}{\alpha}a_k\sqrt{k} - \frac{\beta}{\alpha}a_k\sqrt{k} = \frac{\beta}{\alpha}a_k\sqrt{k}.$$

Also,

$$\int_X \frac{1}{|s|} \leq \sum_i Ck^{-n/2}\int_{\tilde{B}_i} \frac{1}{|\tilde{f}_i|}.$$

Next, we use the estimated transversality property of $\tilde{f}_i$. In fact,

$$\int_{\tilde{B}_i} \frac{1}{|\tilde{f}_i|} = \int_{|\tilde{f}_i|\leq a_k} \frac{1}{|\tilde{f}_i|} + \int_{|\tilde{f}_i|\geq a_k} \frac{1}{|\tilde{f}_i|} \leq \int_{|\tilde{f}_i|\leq a_k} \frac{1}{|\tilde{f}_i|} + \frac{C}{a_k},$$

so we need only to find an upper bound for

$$\int_{|\tilde{f}_i|\leq a_k} \frac{1}{|\tilde{f}_i|}.$$

We use the co-area formula

$$\int_{|\tilde{f}_i|\le a_k} \frac{1}{|\tilde{f}_i|} = \int_{|\xi|\le a_k} \frac{\int_{Z_\xi}[J\tilde{f}_i]^{-1}}{|\xi|} d\xi.$$

Since the Jacobian of $\tilde{f}_i$, $J\tilde{f}_i$, is bounded from below by $|\partial f|^2 - |\bar{\partial} f|^2$, it is greater than Ca_k^2, when $|\tilde{f}_i| \le a_k$. As can easily be checked, Z_ξ, the set where $\tilde{f}_i$ is equal to ξ, has a bound on the volume independent of k. Using these two facts, we conclude

$$\int_{|\xi|\le a_k} \frac{\int_{Z_\xi}[J\tilde{f}_i]^{-1}}{|\xi|} d\xi \le \frac{C}{a_k^2} \int_{|\xi|\le a_k} \frac{1}{|\xi|} d\xi,$$

which in turn is smaller than a constant over a_k.

The equation

$$| \int_{s_0^{-1}(0)} \psi - \frac{k}{2\pi} \int_X \omega \wedge \phi \; | \le \sqrt{k} |d\psi|_\infty,$$

has a simple consequence. If we divide it by k, and take the limit as k tends to ∞, namely,

$$\frac{1}{|s_0^{-1}(0)|} \int_{s_0^{-1}(0)} \psi \to \frac{1}{2\pi|X|} \int_X \omega \wedge \phi.$$

This means that the sets $s_0^{-1}(0)$ become uniformly distributed as k tends to ∞.

We can apply this theorem to the bundles L^k over X, and the sequence of sections s_0, coming from Theorem 5. The sequence a_k can be chosen to be constant equal to η. We can also apply the theorem to the sections $s_0 - bs_1$ when $|b| \le 1/2$. In fact,

$$|s_0 - bs_1| \le (1 - \frac{|b|}{2})\eta \implies |s_0| \le \eta$$

(because $|s_1| \leq \eta/2$), so that,

$$|\partial s_0| \geq \eta\sqrt{k}$$

and

$$|\partial(s_0 - bs_1)| \geq (1 - \frac{|b|}{2})\eta\sqrt{k}.$$

The sequence $s_0 - bs_1$ is actually $(1-|b|/2)\eta$ transverse. The fibers over an open set of S^2 of Donaldson's symplectic Lefschetz pencil are uniformly distributed, with an estimate on how uniformly distributed they are.

Chapter 4

The case of S^2

4.1 The problem on S^2

In [Do3], Donaldson asks the question: Given a symplectic manifold, what is the best η for which we can find $(s_0, s_1) \in \Gamma(L^k)$ satisfying the conditions of Theorem 5? (We need to normalize the pairs to have L^∞ norm 1 for this question to make sense.) We will address this question for S^2 with the Fubini-Study metric. Now, S^2 with the Fubini-Study form is Kähler, so we can apply to it Theorem 7. In fact, the existence of a pair of holomorphic sections of $\mathcal{O}(k)$, satisfying only conditions 1 and 3 in the theorem, seems to already give an interesting result. This last condition becomes somewhat simpler in the context of 2-dimensional manifolds, for there can be no surjective maps from $\mathbb{C}^2$ to $\mathbb{C}$. Namely, it becomes $|s_0|^2 + |s_1|^2 \geq \eta$ so that, when $|s_0|^2 \leq \eta/2$, $|s_1|^2 \geq \eta/2$. Also, holomorphic sections of $\mathcal{O}(k)$ are easy to characterize, they are simply homogeneous polynomials of degree k in two complex variables. In this way, we prove Proposition 1. A way to find a lower bound for the best η appearing in that proposition is then to explicitly determine a sequence of pairs of homogeneous polynomials of degree k, (p_k, q_k), such that the number

$$\frac{\max(||p_k||^2 + ||q_k||^2)}{\min(||p_k||^2 + ||q_k||^2)}$$

is bounded independently of k. Where $||p_k||$ stands for the norm of p_k as a section of $\mathcal{O}(k)$, i.e., letting $[\mathbf{z} : \mathbf{w}]$ be the homogeneous coordinates in $S^2 = \mathbb{CP}^1$,

$$||p_k||[\mathbf{z} : \mathbf{w}] = \frac{|p_k|(\mathbf{z}, \mathbf{w})}{(|\mathbf{z}|^2 + |\mathbf{w}|^2)^{k/2}}.$$

The inverse of this bound will provide the lower bound we are looking for. There is a chart on S^2 minus the south pole, obtained by stereographic projection trough the south pole. It is centered at the north pole and identifies S^2 minus the south pole with $\mathbb{C}$. In homogenous coordinates on S^2, it is simply given by $z = \mathbf{z}/\mathbf{w}$. Over S^2 minus the south pole, $\mathcal{O}(k)$ admits a trivialization, i.e., a global section which we denote by $\mathbf{w}^k$. In fact, when we identify sections of $\mathcal{O}(k)$ with homogeneous polynomials of degree k, this section corresponds precisely to the polynomial $\mathbf{w}^k$. It is actually defined over all of S^2, but vanishes at the south pole. Its norm in the z coordinate is simply,

$$\frac{1}{(1 + |z|^2)^{k/2}}.$$

Often, we do not distinguish between a homogeneous polynomial and its representation in this trivialization, which is simply a polynomial of degree k, in one complex variable z.

As a warm up we make the following simple remark

Proposition 3 *If p_k and q_k are two η transverse homogeneous polynomials of degree k in two complex variables, then (p_k, q_k) cannot map S^3 to S^3.*

proof: We need only show that $||p_k||^2 + ||q_k||^2$ cannot be identically one. If one thinks of the polynomials p_k and q_k as polynomials of one real complex variable z, then

$$||p_k||^2 + ||q_k||^2 = \frac{|p_k(z)|^2 + |q_k(z)|^2}{(1+|z|^2)^k}.$$

Letting F be the rational map on S^2 given by the quotient of p_k and q_k, we have,

$$\frac{F^*\omega}{\omega} = \frac{|p'q - pq'|^2(1+|z|^2)^2}{(|p|^2+|q|^2)^2},$$

(we have dropped the k index) which can be written as

$$\frac{F^*\omega}{\omega} = \frac{|p'q - pq'|^2}{(1+|z|^2)^{2k-2}} \frac{(1+|z|^2)^{2k}}{(|p|^2+|q|^2)^2} = \frac{||s||^2}{(||p||^2+||q||^2)^2}.$$

Here, s denotes the section of $\mathcal{O}(2k-2)$ defined by $p'q - qp'$ (with respect to the trivialization $\mathbf{w}^{2k-2}$) and we use the usual norm in that bundle. In the case where $||p_k||^2 + ||q_k||^2 = 1$, this expression becomes,

$$\frac{F^*\omega}{\omega} = ||s||^2.$$

Using the map F, we can also pullback the metric on S^2 to obtain a conformally equivalent metric, with singularities at the branch points of F. Letting

$$u = \frac{1}{2}\log\left(\frac{F^*\omega}{\omega}\right),$$

e^{2u} is the conformal factor of this metric. Note that u has log type singularities at the branch points of F, so that e^{2u} actually has zeroes. Expressing the curvature of the pullback metric in terms of this factor and noting that this curvature is one away from the branch points of F, one concludes that u satisfies the following PDE, away from the branch points of F

$$\triangle u + e^{2u} = 1.$$

But $\log ||s||^2$ has constant Laplacian, because $\mathcal{O}(2k-1)$ has curvature a multiple of ω. These two facts then imply that, e^{2u} is constant and so is $(F^*\omega/\omega) = ||s||^2$, this in turn implies that s must be zero when $k > 1$ (because the degree of $\mathcal{O}(2k-2)$ is greater than 1 then), so that F' is zero as well, and p is a multiple of q. But then, $||p||^2 + ||q||^2 = 1$, implies that the polynomials have no zeroes, which cannot be true. In the $k = 1$ case, we would get that F is area preserving and so must be $e^{i\theta}/z$ or $e^{i\theta}z$ and one of the polynomials must be constant.

4.2 An upper bound for η on S^2

In this section, we show how to find an upper bound for the best η for which there is a sequence of pairs of polynomials, (p_k, q_k), mapping S^3 into the $(\eta, 1)$ annulus in $\mathbb{C}^2$.

Proposition 4 *If p_k and q_k are two homogeneous polynomials of degree k in two complex variables, and if, for k large, (p_k, q_k) maps S^3 into the annulus of outer radius 1 and inner radius η (for some eta independent of k) then, $\eta \leq \eta_0 < 1$, where η_0 is going to be made explicit ahead.*

proof: Choose complex coordinates on S^2, by using stereographic projection trough the north pole for example. Consider p_k and q_k as polynomials in the chosen complex coordinate. We set

$$F_k = \frac{p_k}{q_k},$$

$$u_k = \frac{1}{2}\log\left(\frac{F_k^*\omega_{FS}}{\omega_{FS}}\right),$$

(note that u has log type singularities) and

$$f_k = \log(||p_k||^2 + ||q_k||^2).$$

We have

$$\frac{F_k{}^*\omega_{FS}}{\omega_{FS}} = \frac{|p_k' q_k - p_k q_k'|^2(1+|z|^2)^2}{(|p_k|^2 + |q_k|^2)^2},$$

which can be written as

$$\frac{F_k^*\omega_{FS}}{\omega_{FS}} = \frac{|p_k' q_k - p_k q_k'|^2}{(1+|z|^2)^{2k-2}} \frac{(1+|z|^2)^{2k}}{(|p_k|^2 + |q_k|^2)^2}$$

and therefore

$$\frac{F_k^*\omega_{FS}}{\omega_{FS}} = \frac{||s_k||^2}{(||p_k||^2 + ||q_k||^2)^2},$$

where s_k is a section of $\mathcal{O}(2k-2)$, so that

$$u_k = \frac{1}{2}\log(||s_k||^2) - f_k.$$

Now, we can use the fact that $\triangle \log ||s_k||^2 = -(2k-2)$, away from the zeroes of s_k, to conclude that

$$\triangle u = -(k-1) - \triangle f_k,$$

(we dropped the k dependence on u). Now, e^{2u} is the conformal factor of the metric defined by $F_k^*\omega_{FS}$. Note that e^{2u} actually has zeroes. Expressing the curvature of the pullback metric in terms of this factor and noting that this curvature is one away from the branch points of F_k, one concludes that u satisfies the following PDE, away from the branch points of F_k,

$$\triangle u + e^{2u} = 1,$$

that is,

$$\triangle u + \frac{F_k^*\omega_{FS}}{\omega_{FS}} = 1.$$

Putting this together with $\triangle u = -(k-1) - \triangle f_k$, we conclude

$$\frac{1}{k}\frac{F_k^*\omega_{FS}}{\omega_{FS}} = 1 + \frac{1}{k}\triangle f_k, \tag{4.1}$$

which holds at all points on S^2. Next, we show that the polynomials p_k, q_k, after rescaling, satisfy bounds independent of k, as do all of their derivatives. We start by recalling that

$$\eta \leq \frac{|p_k|^2(z) + |q_k|^2(z)}{(1+|z|^2)^k} \leq 1. \tag{4.2}$$

Now consider the rescaled polynomials

$$\tilde{p_k}(z) = p_k(\frac{z}{\sqrt{z}}), \quad \tilde{q_k}(z) = q_k(\frac{z}{\sqrt{z}}),$$

as a consequence of inequality 4.2, these satisfy

$$\eta\left(1 + \frac{|z|^2}{k}\right)^k \leq |\tilde{p}_k|^2 + |\tilde{q}_k|^2 \leq \left(1 + \frac{|z|^2}{k}\right)^k,$$

so that, $|\tilde{p}_k|$ and $|\tilde{q}_k|$, when restricted to the disk of center 0 and radius $2c$, D_{2c}, admit an upper bound e^{2c^2} (also valid on the closure of the disc). Now both polynomials are holomorphic functions on D_{2c}. The Cauchy formula gives an upper bound for the norm of their derivatives on D_c, namely $2e^{2c^2}$. In fact, we even get an upper bound for the norm of higher order derivatives. If l denotes the order of the derivative,

$$|\tilde{p}_k^l|, |\tilde{q}_k^l| \leq \frac{2l!e^{2c^2}}{c^{l-1}},$$

and

$$|\tilde{p}_k|, |\tilde{p}_k| \leq e^{\frac{c^2}{2}}.$$

This then proves that $\tilde{p}_k$ and $\tilde{q}_k$ have convergent subsequences. Let us call the limits p and q respectively. These satisfy

$$\eta \leq |p|^2 + |q|^2 \leq e^{c^2}.$$

Also, the derivatives of p and q satisfy the same bounds as those of $\tilde{p}_k$ and $\tilde{q}_k$. From the inequality $\eta \leq |p|^2 + |q|^2$, not both p and q can vanish at zero. In fact, either $|p(0)|$ or $|q(0)|$ is greater than $\sqrt{\eta}/\sqrt{2}$. Let us say that q satisfies this bound. Then, if c is such that

$$2e^{2c^2}c \leq \frac{\sqrt{\eta}}{2\sqrt{2}},$$

in D_c,

$$|q| \geq \frac{\sqrt{\eta}}{2\sqrt{2}}.$$

This comes from the lower bound on $|q(0)|$ and the upper bound on $|q'|$. From now on, we assume that c is such that it satisfies this. Let G_k be the quotient of $\tilde{p}_k$ and $\tilde{q}_k$ (that is F_k rescaled). On D_c, for k big enough,

$$|G_k| \leq \frac{4\sqrt{2}e^{c^2}}{\sqrt{\eta}},$$

because for k sufficiently big, since $\tilde{q}_k$ converges uniformly to p (after passing to a subsequence),

$$|\tilde{q}_k| \geq \frac{\sqrt{\eta}}{4\sqrt{2}}.$$

Furthermore, we get a bound on the derivative of G_k

$$|G_k'| \leq \frac{64\sqrt{2}e^{3c^2}}{\eta},$$

so that, we may conclude that G_k has a convergent subsequence (uniformly on compact subsets of D_c). By bounding the second derivative of G_k, we can even assume that G_k' is convergent in that same subsequence (uniformly on compact subsets of D_c). Consider now $\tilde{f}_k$, which is simply f_k rescaled. These functions define a sequence which is uniformly bounded (in fact f_k varies between $\log(\eta)$ and 0), as is the sequence of derivatives of all orders. In fact

$$\tilde{f}_k = \log(|\tilde{p}_k|^2 + |\tilde{q}_k|^2) - \frac{k}{2}\log\left(1 + \frac{z\bar{z}}{k}\right),$$

and, for example,

$$\frac{\partial \tilde{f}_k}{\partial z} = \frac{\tilde{p}_k'\bar{\tilde{p}}_k + \tilde{q}_k'\bar{\tilde{q}}_k}{(|\tilde{p}_k|^2 + |\tilde{q}_k|^2)} - \frac{\bar{z}}{2}\left(1 + \frac{z\bar{z}}{k}\right)$$

which is clearly bounded in D_c. This proves that $\tilde{f}_k$ has a subsequence which converges, together with its derivatives of higher order, on compact subsets of D_c. Let us denote by f the limit of this subsequence. The function f is smooth and the derivatives of $\tilde{f}_k$ converge to the derivatives of f. Next, we show that G and f satisfy a differential relation, that comes from taking the limit of the rescaled version of the differential relation (4.1). For this purpose, consider the metric

$$\tilde{\omega} = \frac{dzd\bar{z}}{\left(1 + \frac{|z|^2}{k}\right)^2}$$

on D_c, which is k times the pull back of the Fubini-Study metric on S^2 by the rescaling map δ_k (this map is defined in the z coordinate by $\delta_k(z) = z/\sqrt{k}$). This metric has curvature $1/k$ and tends to a flat metric. Now $G_k = F_k \circ \delta_k$ so that

$$\frac{G_k^* \omega_{FS}}{\tilde{\omega}} = \frac{\delta_k^* F_k^* \omega_{FS}}{k \delta_k^* \omega_{FS}} = \frac{1}{k} \frac{F_k^* \omega_{FS}}{\omega_{FS}} \circ \delta_k.$$

As for the Laplacian of f_k with respect to the Fubini-Study metric, it is given by

$$\triangle f_k = (1 + |z|^2)^2 (f_k)_{z\bar{z}},$$

whereas the Laplacian of $\tilde{f}_k$ in the metric $\tilde{\omega}$ is given by

$$\triangle \tilde{f}_k = (1 + \frac{|z|^2}{k})^2 (\tilde{f}_k)_{z\bar{z}}.$$

We see that

$$\frac{1}{k}(\triangle f_k) \circ \delta_k = \frac{1}{k}(1 + \frac{|z|^2}{k})^2 (f_k)_{z\bar{z}} \circ \delta_k = \triangle \tilde{f}_k.$$

Then the differential relation (4.1) becomes

$$\frac{G_k^* \omega_{FS}}{\tilde{\omega}} = 1 + \triangle \tilde{f}_k.$$

Taking the limit of this equation as k tends to infinity (we actually need to consider a subsequence) we get

$$G^*\omega_{FS} = (1 + \triangle f)\omega_{\text{flat}}, \tag{4.3}$$

where ω_{flat} is the flat metric in the disc and the Laplacian is the flat Laplacian. Intuitively, this equation says that, at least when η is very close to 1, the rescaled F_k is trying to be an isometry between flat $\mathbb{C}$ and the sphere. Such isometries do not exist, even locally, so it cannot be that. We now prove the following lemma:

Lemma 8 *Let c, δ and M be positive constants and suppose we have a fixed metric on D_c. Then, there is a constant $\lambda = \lambda(\delta, M, c)$, such that, for every smooth function f from the disc D_c to $\mathbb{C}$ satisfying*

- $|f| \leq \delta$,
- $|d\triangle f| \leq M$,

we have $|\triangle f| \leq \lambda$ in $D_{c/4}$.

proof: Let z_0 be a point in $D_{c/4}$ where $\triangle f$ is greater than λ. We want to see that λ big leads to a contradiction. (The same reasoning would lead to a contradiction if we assumed that $\triangle f \leq -\lambda$). Given ξ in $D_{c/4}$, for some χ in the segment from z_0 to ξ, we have

$$\triangle f(z_0 + \xi) - \triangle f(z_0) = d\triangle f(\chi)\xi.$$

Choosing ξ with norm smaller than $\lambda/2M$, we conclude that $\triangle f(z_0+\xi)$ is greater than $\lambda/2$. Set μ to be the minimum of $\lambda/2M$ and $c/4$, we have seen

$$\triangle f(z_0) \geq \lambda \implies \triangle f(z) \geq \lambda/2, \quad z \in B(z_0, \mu).$$

Let r denote the distance function to z_0 and β a smooth function on $\mathbb{R}^+$ that is zero in $]\mu, +\infty[$ and 1 in $[0, \mu/2]$. Set $g(z) = \beta(r)$, g has compact support contained in D_c and, for any positive number ϵ, β can be chosen so as to have g satisfy $\triangle g \leq (8+\epsilon)/\mu^2$. Since g vanishes in a neighborhood of the boundary of D_c,

$$\int_{D_c} g\triangle f = \int_{D_c} f\triangle g.$$

Now

- the right hand side is equal in norm to

$$\left|\int_{B(z_0,\mu)} f\triangle g\right| \leq \frac{8+\epsilon}{\mu^2}\delta\pi\mu^2 = (8+\epsilon)\pi\delta,$$

- whereas the left hand side is equal to

$$\int_{B(z_0,\mu)} g\triangle f \geq \int_{B(z_0,\mu/2)} g\triangle f \geq \frac{\pi\lambda\mu^2}{8}.$$

Therefore we get a contradiction if $\lambda\mu^2 > (64+\epsilon)\delta$ (a different ϵ...). Taking

$$\lambda = \max(7M^{2/3}\delta^{1/3}, \frac{1025\delta}{c^2})$$

we get $|\triangle f(z_0)| \leq \lambda$ for every z_0 in $D_{c/2}$.

Let us again consider the differential relation (4.3). The left hand side represents a metric conformal to the flat metric on the disc, with conformal factor $1+\triangle f$, its curvature is therefore

$$-\frac{1}{2}\frac{\triangle \log(1+\triangle f)}{1+\triangle f}.$$

As for the right hand side, it represents a metric (with singularities maybe), whose curvature is 1, because it is the pullback of the Fubini-Study metric. We have therefore established the following equality

$$1=-\frac{1}{2}\frac{\triangle \log(1+\triangle f)}{1+\triangle f}. \tag{4.4}$$

We know that $|f_k|\leq \log(1/\eta)$, so that f satisfies the same inequality. From Lemma 8, this implies that $\triangle f$ is small (because $d\triangle f$ is bounded from above by some constant depending only on η as we will see ahead) and, so, $\log(1+\triangle f)$ is small as well. We will show that the derivative of the Laplacian of this quantity is bounded (by a quantity depending only on η), so, we can apply Lemma 8 again, to conclude that $\triangle \log(1+\triangle f)$ is small. Since the denominator of the right hand side of equation (4.4) is bounded from below ($\triangle f$ is small), then, the right handside itself is small. But it needs to equal 1, so it can't be small and we arrive at a contradiction. We still need to get an upper bound on the derivatives of $\triangle f$ and $\log(1+\triangle f)$. From equation (4.3) we write

$$1+\triangle f=\frac{|p'q-pq'|^2}{(|p|^2+|q|^2)^2}.$$

To estimate $d\triangle f$, we estimate

$$d\frac{|p'q-pq'|^2}{(|p|^2+|q|^2)^2},$$

on D_c, when c is sufficiently small by using the inequalities

$$|(p,q)| \leq e^{c^2/2},$$

$$|(p',q')| \leq 2\sqrt{2}e^{2c^2}$$

and

$$|(p'',q'')| \leq \frac{4\sqrt{2}e^{2c^2}}{c}.$$

We also use the fact that $|p|^2+|q|^2 \geq e^{-\delta}$. This gives the bound

$$M = \frac{128\sqrt{2}e^{3\delta}e^{7.5c^2}}{c},$$

for $|d\triangle f|$. Note that, we can only apply Lemma 8 to a c such that

$$e^{2c^2}c \leq \frac{\sqrt{\eta}}{4\sqrt{2}}$$

(since it is only on balls with such radii that there is convergence and therefore f is defined). Choosing

$$c = \frac{e^{\frac{-\delta}{2}}}{8},$$

is enough to ensure the above inequality. Applying Lemma 8 to such c, we conclude that on $D_{c/4}$, $|\triangle f| \leq \lambda$ where

$$\lambda = \max(896e^{2+\frac{1}{3}+\frac{5}{64}}\delta^{1/3}, 65600e\delta).$$

So for example, to ensure $|\triangle f| \leq 1/2$, it is enough to ensure that $\delta \leq 0.5*10^{13}$. As for $\log(1-\triangle f)$, its norm is less than $\log(1/(1-\lambda))$, a small number, as long as δ is small. This, in turn, is smaller than 2λ, if λ is smaller than $1/2$. Now, we apply Lemma 8 again, but this time to $\log(1-\triangle f)$. We need an upper bound for $|d\triangle \log(1-\triangle f)|$. Using the fact that

$$\triangle \log(1-\triangle f) = -2\triangle \log(|p|^2+|q|^2),$$

away from the zeroes of $p'q - qp'$, as well as the inequalities above for $|(p,q)|$, $|(p',q')|$ and $|(p'',q'')|$, we calculate this upper bound on D_c. It can be taken to be

$$\frac{512\sqrt{2}e^{3\delta}e^{7.5c^2}}{c}.$$

We are again in a position to apply Lemma 8 with

$$c = \frac{e^{\frac{-\delta}{2}}}{32}.$$

From it, we get an upper bound for $|\triangle \log(1-\triangle f)|$, namely

$$\max(7859e^{2+\frac{1}{3}+\frac{5}{64}}\lambda^{1/3}, 2099200e\lambda),$$

so if we substitute for λ when $\delta \leq 0.5*10^{13}$, this gives an upper bound for the curvature of $(1-\triangle f)\omega_{\text{flat}}$. If that upper bound happens to smaller than 1, we get a contradiction. Following trough with these calculations we conclude that

$$\delta \geq \frac{1}{(74708)^9 e^{28+\frac{15}{16}}},$$

which is of the order 10^{-36}, giving an upper bound for η of the order $e^{-10^{-36}} <$

1.

Chapter 5

Rational maps and estimated transversality

To find polynomials in one complex variable corresponding to homogeneous polynomials in two variables satisfying the conditions of Theorem 7 for S^2, it is helpful to realize that the zero sets of these polynomial, as well as the rational map determined by them, satisfy restrictive properties.

5.1 Uniform distribution revisited

As we saw before in the general case, some fibers of a Lefschetz Pencil coming from Donaldson's construction, are uniformly distributed. In the case of S^2, we will show that we can go a step further. Then, the fibers of the pencil are simply sets of k points p_i and they satisfy:

$$| \frac{1}{k} \sum_{i=1}^{k} f(p_i) - \frac{1}{|S^2|} | \leq C \frac{|df|_\infty}{\sqrt{k}}, \tag{5.1}$$

for all $\mathcal{C}^\infty$ functions f. On the sphere, another interesting result of the same type can be obtained by applying Theorem 8 to the bundles $\mathcal{O}(k) \otimes \mathcal{O}(k) \otimes$

T^*S^2, that is $\mathcal{O}(2k-2)$, and their sections $s_1\partial s_0 - s_0\partial s_1$. Here, we take $a_k = \epsilon\sqrt{k}$ for some ϵ to be made explicit shortly. Let us first check that the sections satisfy the hypotheses in the theorem. We know

$$\nabla\left(\frac{s_1}{s_0}\right) = \frac{s_1\nabla s_0 - s_0\nabla s_1}{s_0^2}$$

and

$$\nabla(s_1\partial s_0 - s_0\partial s_1) = s_0^2\nabla\nabla\left(\frac{s_1}{s_0}\right) - (s_1\nabla s_0 - s_0\nabla s_1)\frac{2\nabla s_0}{s_0}.$$

Now,

$$|s_1\nabla s_0 - s_0\nabla s_1| \leq \frac{\eta^2\sqrt{k}}{2\sqrt{2}} \implies |s_0| \geq \frac{\eta^2}{2\sqrt{2}C}$$

(where C comes from $|\partial s_1| \leq C\sqrt{k}$). This is because,

$$|s_0| \leq \frac{\eta^2}{2\sqrt{2}C} \implies |s_0| \leq \eta$$

which in turn implies that $|\nabla s_0| \geq \eta\sqrt{k}$. On the other hand, because $\eta^2/2\sqrt{2}C$ is $O(\eta/\sqrt{2})$, this also implies that $|s_1| \geq \eta/\sqrt{2}$, so that, when $|s_0| \leq \eta^2/(2\sqrt{2}C)$,

$$|s_1\nabla s_0 - s_0\nabla s_1| \geq \frac{\sqrt{k}\eta^2}{\sqrt{2}} - \frac{\eta^2}{2\sqrt{2}C}C\sqrt{k} = \frac{\sqrt{k}\eta^2}{2\sqrt{2}}.$$

Assume that $|s_1\nabla s_0 - s_0\nabla s_1| \leq \eta^8\sqrt{k}$, then (if η is small enough to make $\eta^8 \leq \frac{\eta^2}{2\sqrt{2}}$) we have,

$$\left|\nabla\left(\frac{s_1}{s_0}\right)\right| = \frac{|s_1\nabla s_0 - s_0\nabla s_1|}{|s_0^2|} \leq a\sqrt{k}\eta^6$$

therefore, $|\partial\partial(s_1/s_0)| \geq k\eta$. Then,

$$|\nabla(s_1\nabla s_0 - s_0\nabla s_1)| \geq a\eta^4 k\eta - \frac{\sqrt{k}\eta^8 2C\sqrt{k}}{\eta^2} \geq \eta^8$$

when η is sufficiently small. We have shown that the sequence of sections of $L^k \otimes L^k \otimes T^*X$, $s_1 \nabla s_0 - s_0 \nabla s_1$ is $\epsilon = \eta^8$ transverse to zero. The zeroes of this sequence of sections form the set of branch points of the Lefschetz pencil. We then conclude:

Proposition 5 *The set of branch points of a Lefschetz pencil coming from Theorem 7 is asymptotically uniformly distributed.*

On S^2, the uniform distribution assumes a particularly simple expression since the fibers of the pencil are sets of k points, which we denote by $\{x_i^k\}$, satisfying

$$\left| \frac{1}{k} \sum_{i=1}^{k} f(x_i^k) - \frac{1}{|S^2|} \int_{S^2} f \right| \leq C \frac{|df|_\infty}{\sqrt{k}}, \tag{5.2}$$

for all $\mathcal{C}^\infty$ functions f. The branch point set is simply a set of $2k-2$ points, $\{p_i^k\}$. It satisfies

$$\left| \frac{1}{2k-2} \sum_{i=1}^{2k-2} f(p_i^k) - \frac{1}{|S^2|} \int_{S^2} f \right| \leq C \frac{|df|_\infty}{\sqrt{k}},$$

for all $\mathcal{C}^\infty$ functions f.

Asymptotically uniformly distributed sets of points on S^2 are important in many areas of mathematics and are the object of a lot of work in potential theory (see [RSZ] and analytic number theory (see [LPS], [BSS]). Therefore, these properties are very helpful in trying to explicitly build the sequence of polynomials (p_k, q_k). On the other hand, they are certainly not enough to characterize these polynomials. In fact, asymptotic uniform distributiveness with estimate (5.2) would allow for a point with multiplicity, whereas Donaldson's construction clearly does not.

A natural question to ask at this point is: How about the fibers above points close to ∞ on S^2? We have not shown that these are asymptotically uniformly distributed since our method failed for them. Next, we show that in fact they are, in the case of S^2, although in a slightly different sense as stated in the introduction. What is more surprising, we show (using the methods developed in the previous section) that the condition of transversality of the pair is enough to ensure the asymptotic uniform distribution of all the fibers and also of the branch points, for pencils in S^2. As before, we identify the sections of $\mathcal{O}(k)$ with homogeneous polynomials and write $||.||$ for their norm.

Theorem 9 *Let (p_k, q_k) be a sequence of pairs of homogeneous polynomials as above. The map p_k/q_k, thought of as a degree k map of $\mathbb{CP}^1$ to itself, has asymptotically uniformly distributed fibers, in the sense that, if x_i^k denote the points in one fiber for $i = 1, \cdots k$, counted with multiplicity, and f is a $\mathcal{C}^2$ function on S^2, then*

$$\left| \frac{1}{k} \sum_{i=1}^{k} f(x_i^k) - \frac{1}{|S^2|} \int_{S^2} f \right| \leq \frac{C||\triangle f||_\infty}{k}, \tag{5.3}$$

and, in particular, tends to zero.

proof: We will start by showing that the zero set of p_k is uniformly distributed in the sense that, if x_i^k denote the points in this set for $i = 1, \cdots k$, counted with multiplicity, and f is a $\mathcal{C}^2$ function on S^2, then

$$\left| \frac{1}{k} \sum_{i=1}^{k} f(x_i^k) - \frac{1}{|S^2|} \int_{S^2} f \right| \leq \frac{C||\triangle f||_\infty}{k}, \tag{5.4}$$

1. The first step is to show that

$$\int_{S^2} \log ||p_k||^2$$

is bounded independently of k. We will do this by contradiction. Let us assume that it is not. For each k, it is easy to find a partition of S^2 by k sets A_i such that $\mathrm{diam}(A_i) = c/\sqrt{k}$ (c is independent of k). By assumption, there is a subsequence n_k, such that $\int_{S^2} \log ||p_{n_k}||^2$ tends to $-\infty$ as k tends to $+\infty$ (note that $\log ||p_{n_k}||^2 \leq 0$) . Now take $i = i(k)$ to be the index of the element in the partition of S^2 were the integral of $\log ||p_{n_k}||^2$ is the smallest. We have

$$n_k \int_{A_i} \log ||p_{n_k}||^2 \leq \int_{S^2} \log ||p_{n_k}||^2,$$

so that the left hand side tends to $-\infty$ as well. But $A_i \subset B(z_k, c/\sqrt{k})$ so

$$n_k \int_{B(z_k, c/\sqrt{n_k})} \log ||p_{n_k}||^2$$

tends to $-\infty$ as well. Suppose we knew that z_k was zero for all k. Then, by using the rescaled coordinate chart around zero which we described in the proof of Proposition 4, we would have for this integral

$$\int_{B(0,c)} \log \left(\frac{|\tilde{p}_{n_k}|^2}{\left(1 + \frac{|z|^2}{n_k}\right)^{n_k}} \right) \frac{dz d\bar{z}}{\left(1 + \frac{|z|^2}{n_k}\right)^2}$$

(were $\tilde{p}_k$ is simply p_k in the rescaled coordinates as in the proof of Proposition 4). We showed before that $\{\tilde{p}_{n_k}\}$ is bounded and has bounded derivatives in $B(0,c)$ and therefore has a convergent subsequence. The limit, p, is a holomorphic function on $B(0,c)$. We will check ahead that it cannot be identically zero and therefore has a finite number of zeroes, with finite multiplicity, so that $\log|p|^2$ is integrable. The integral above converges for that subsequence to the finite number

$$\int_{B(0,c)} \log\left(\frac{|p|^2}{e^{|z|^2}}\right) dz d\bar{z}$$

and we get a contradiction. Let us try to reproduce this argument for a general sequence of points z_k. Given any point p on S^2 there is a map $\phi : S^2 \to S^2$ such that

(a) $\phi^*\omega_{FS} = \omega_{FS}$,

(b) ϕ takes the north pole to p,

(c) the inverse of ϕ, ψ is such that $|\psi^*\mathbf{w}| = |\mathbf{w}| \circ \psi$ were $[\mathbf{z} : \mathbf{w}]$ are homogeneous coordinates on S^2, $\mathbf{w}$ denotes the same section of $\mathcal{O}(1)$ as before and $|.|$ is the usual norm in $\mathcal{O}(1)$.

Such a map can be taken to simply be a rotation. Now consider the composition of this map for $p = z_k$ with the inverse of the usual chart centered at the north pole, l, $\phi \circ l : \mathbb{C} \to S^2$ (we omit the dependence on k). Its inverse defines a chart, centered at the point z_k. Write p_k as a holomorphic function on $\mathbb{C}$ times the section of $\mathcal{O}(k)$, $\psi^* \mathbf{w}^k$. Let f_k denote this holomorphic function composed with $\phi \circ l$. We have,

$$|\psi^* \mathbf{w}^k| \circ \phi \circ l = |\mathbf{w}^k| \circ \psi \circ \phi \circ l = |\mathbf{w}^k| \circ l = \frac{1}{(1+|z|^2)^{\frac{k}{2}}}.$$

As a consequence

$$|f_k|^2 \leq \left(1+|z|^2\right)^k.$$

The same can be done for q_k, so that we obtain two sequences of functions f_k and g_k satisfying

$$\eta \left(1+|z|^2\right)^k \leq |f_k|^2 + |g_k|^2 \leq \left(1+|z|^2\right)^k.$$

If we rescale the z coordinate in $\mathbb{C}$ by $1/\sqrt{k}$ as we did in the proof of Proposition 4, then, we get functions $\tilde{f}_k$ and $\tilde{g}_k$ which are bounded on $B(0,c)$, independently of k, for any fixed c, and whose derivatives are also bounded there. They then have convergent subsequences (in the $\mathcal{C}^\infty$ norm over $B(0,c)$). In fact, we will use that $\tilde{f}_{n_k}$ and $\tilde{g}_{n_k}$ have convergent subsequences. We call p and q their limits. As before,

$$n_k \int_{B(z_k, c/\sqrt{n_k})} \log ||p_{n_k}||^2 = \int_{B(0,c)} \log \left(\frac{|\tilde{f}_{n_k}|^2}{\left(1+\frac{|z|^2}{n_k}\right)^{n_k}} \right) \frac{dz d\bar{z}}{\left(1+\frac{|z|^2}{n_k}\right)^2}.$$

The sequences $\tilde{f}_{n_k}$ and $\tilde{g}_{n_k}$ are subconvergent on compact subsets of $\mathbb{C}$ to p and q. These satisfy

$$\eta e^{|z|^2} \leq |p|^2 + |q|^2 \leq e^{|z|^2}.$$

This inequality implies that p cannot be identically zero. If it were, there would be a holomorphic function q satisfying

$$\eta e^{|z|^2} \leq |q|^2 \leq e^{|z|^2}$$

so that q could not vanish in $\mathbb{C}$. Twice the real part of its log, u, would satisfy

$$\log \eta + |z|^2 \leq u \leq |z|^2.$$

The function $u - |z|^2$ would be bounded on $\mathbb{C}$ and therefore would have a minimum, but its Laplacian is $-2 \leq 0$, so it could not have a minimum. For k sufficiently big, on $B(0, c)$, since (a subsequence of) $\{\tilde{f}_{n_k}\}$ converges uniformly to p

$$\log \frac{|\tilde{f}_{n_k}|^2}{\left(1 + \frac{|z|^2}{n_k}\right)^{n_k}} \geq \frac{1}{2} \log \frac{|p|^2}{e^{|z|^2}},$$

and therefore

$$0 \geq n_k \int_{B(z_k, c/\sqrt{n_k})} log||p_{n_k}||^2 \geq \frac{1}{2} \int_{B(0,c)} \log \frac{|p|^2}{e^{|z|^2}},$$

for that same subsequence. This is a contradiction since we are assuming the left hand side integral converges to $-\infty$.

2. Given the zeroes of p_k as a function on S^2, x_i (we omit the k dependence here), for each of these, consider the function on S^2 given by

$$\log \left(d^2(x, x_i) \frac{e}{4} \right),$$

where d is the chordal distance in S^2, i.e., the usual distance in $\mathbb{R}^3$. The Laplacian of this function is simply $4\pi\delta_{x_i} - 1$ and its integral is zero (hence the $e/4$). Let z_i be the complex coordinates of the points x_i, trough the usual chart centered at the north pole. Now consider the homogeneous polynomial

$$\hat{p_k} = \prod_i \left(\frac{e^{1/2}(z - z_i)}{(1+|z|^2)^{1/2}} \right) \mathbf{w}^k.$$

As a section of $\mathcal{O}(1)$, the norm square of this is

$$\prod_i \left(d^2(x, x_i) \frac{e}{4} \right).$$

This is because, given two points on the sphere, x and y with complex coordinates z and w

$$d^2(x, y) = \frac{4|z - w|^2}{(1+|z|^2)(1+|w|^2)}.$$

Such a $\hat{p_k}$ satisfies $\int \log ||\hat{p_k}||^2 = 0$. The polynomials p_k and $\hat{p_k}$ are related by $\hat{p_k} = c_k p_k$, for some sequence of numbers c_k. The fact that $\int \log ||p_k||^2$ is bounded independently of k, implies that this sequence is bounded.

3. Finally, let f be a $\mathcal{C}^2$ function on S^2 and consider the difference

$$\frac{\sum f(x_i)}{k} - \frac{1}{4\pi} \int f.$$

This can alternatively be written as

$$\frac{1}{4\pi} \int f \frac{\sum (4\pi\delta_{x_i} - 1)}{k},$$

which, in view of a previous remark, is

$$\int f \frac{\sum \triangle \log \left(d^2(x, x_i) \frac{e}{4} \right)}{k} = \frac{1}{k} \int f \triangle \log ||\hat{p_k}||^2.$$

Integrating by parts, we get,

$$\frac{1}{k}\int \triangle f \log ||\hat{p}_k||^2.$$

To prove the proposition, it is then sufficient to show that $\int |\log ||\hat{p}_k||^2|$ is bounded, independently of k. This integral is simply bounded by

$$4\pi \log |c_k|^2 - \int \log ||p_k||^2 = -2\int \log ||p_k||^2,$$

which is bounded, as we have already seen.

In the same way, to prove that any fiber is uniformly distributed in the sense described previously, it is enough to see that the integral

$$\int_{S^2} \left|\log ||p_k + \lambda q_k||^2\right|$$

is bounded independently of k for any λ. Assume that it is not. By the same reasoning as before, we can conclude that, for a sequence of complex numbers z_k,

$$\int_{B(0,c)} \left|\log \frac{|\tilde{f}_{n_k} + \lambda \tilde{g}_{n_k}|^2}{\left(1+\frac{|z|^2}{n_k}\right)^{n_k}}\right| \frac{dzd\overline{z}}{\left(1+\frac{|z|^2}{n_k}\right)^2} \to \infty,$$

where $\tilde{f}_k$ and $\tilde{g}_k$ are defined as before using the z_k. But, again as before, $\{\tilde{f}_{n_k}\}$ and $\{\tilde{g}_{n_k}\}$ subconverge uniformly on compact subsets of $\mathbb{C}$, so that, for each λ, $\{\tilde{f}_{n_k} + \lambda\tilde{g}_{n_k\}}$ subconverges to $p+\lambda q$, uniformly on compact subsets of $\mathbb{C}$. To be able to apply the same reasoning as before, we need only check that $p+\lambda q$ is not identically zero. If it were, there would be a holomorphic function, g satisfying

$$\frac{\eta}{(1+|\lambda|^2)}e^{|z|^2} \leq |g|^2 \leq \frac{1}{(1+|\lambda|^2)}e^{|z|^2},$$

which is impossible. As for the branch points, we need to show that

$$\int_{S^2} \left| \log \frac{||p_k \nabla q_k - q_k \nabla p_k||^2}{k} \right| \tag{5.5}$$

is bounded independently of k. Note that $p_k \nabla q_k - q_k \nabla p_k$ is a section of $T^*S^2 \otimes \mathcal{O}(k) \otimes \mathcal{O}(k)$, which is simply $\mathcal{O}(2k-2)$ and $||.||$ refers to the usual norm in that bundle. Letting f_k and g_k be the representations of p_k and q_k respectively in the trivialization $\psi^* \mathbf{w}^k$ used before, we can write

$$p_k \nabla q_k - q_k \nabla p_k = (f_k g_k' - f_k' g_k) dz \otimes \psi^* \mathbf{w}^k \otimes \psi^* \mathbf{w}^k.$$

The norm square of this section is given by

$$\frac{|f_k g_k' - f_k' g_k|^2}{(1+|z|^2)^{2k}} . \frac{1}{2} \left(1+|z|^2\right)^2 = \frac{|f_k g_k' - f_k' g_k|^2}{2\left(1+|z|^2\right)^{2k-2}}.$$

By rescaling the coordinate z, we get

$$\tilde{f}_k'(z) = \frac{1}{\sqrt{k}} f_k' \left(\frac{z}{\sqrt{z}} \right).$$

Assuming that the sequence of integrals in (5.5) is not bounded, we conclude that there must be complex numbers z_k such that

$$\int_{B(0,c)} \left| \log \frac{|\tilde{f}_{n_k} \tilde{g}_{n_k}' - \tilde{f}_{n_k}' \tilde{g}_{n_k}|^2}{2\left(1+\frac{|z|^2}{n_k}\right)^{2n_k-2}} \right| \frac{dz d\bar{z}}{\left(1+\frac{|z|^2}{n_k}\right)^2}$$

tends to infinity. The sequence $\{\tilde{f}_{n_k} \tilde{g}_{n_k}' - \tilde{f}_{n_k}' \tilde{g}_{n_k}\}$ is uniformly subconvergent on compact subsets of $\mathbb{C}$ to $pq' - p'q$. We will be done if we check that this function cannot be identically zero. If it was, then p/q would be constant, which is impossible, as we saw above (note that q is not identically zero either).

5.2 An estimate for Donalson's map

Consider a sequence of polynomials (p_k, q_k) coming from Theorem 7. Then, their quotient $F_k = p_k/q_k$ defines a sequence of rational maps of S^2 of degree k. In what follows, we drop the k index although there is still k dependence. We will show that this sequence of maps satisfies the following estimate:

$$\mid \frac{F^*\omega}{\omega} \mid \leq Ck,$$

for some constant C, independent of k. Consider the bundle $F^*\mathcal{O}(1)$, with the pullback metric. Let us call the homogeneous coordinates on S^2, $[\mathbf{z} : \mathbf{w}]$. We can then consider the section of $\mathcal{O}(1)$, z and its pullback, F^*z, which is a section of $F^*\mathcal{O}(1)$. Since the curvature of $F^*\mathcal{O}(1)$ is $F^*\omega$ we have:

$$\bar{\partial}\partial \log(|F^*z|^2) = \frac{F^*\omega}{2\pi}.$$

Also,

$$|F^*z|^2 = \frac{|p|^2}{|p|^2 + |q|^2} = \frac{|p|^2}{(|\mathbf{z}|^2 + |\mathbf{w}|^2)^k} \frac{(|\mathbf{z}|^2 + |\mathbf{w}|^2)^k}{|p|^2 + |q|^2} = \frac{||p||^2}{||p||^2 + ||q||^2},$$

and

$$\bar{\partial}\partial \log(|F^*z|^2) = \bar{\partial}\partial \log(||p||^2) - \bar{\partial}\partial \log(||p||^2 + ||q||^2).$$

Since p defines a section of $\mathcal{O}(k)$ whose curvature is $k\omega$,

$$\bar{\partial}\partial \log(||p||^2) = k\omega.$$

Putting these together

$$F^*\omega = k\omega - \triangle \left(\log(||p||^2 + ||q||^2)\right) \omega,$$

and

$$\frac{F^*\omega}{\omega} = k - \triangle \log(||p||^2 + ||q||^2),$$

as we have seen before. Also,

$$\bar{\partial}\partial \log(||p||^2 + ||q||^2) = \frac{\bar{\partial}\partial(||p||^2 + ||q||^2)}{||p||^2 + ||q||^2} - \frac{\bar{\partial}(||p||^2 + ||q||^2) \wedge \partial(||p||^2 + ||q||^2)}{(||p||^2 + ||q||^2)^2}$$

so that the bounds on ∇p and $\nabla\nabla p$ (and the corresponding bounds for q), together with the lower bound on $||p||^2 + ||q||^2$ show that $\bar{\partial}\partial \log(||p||^2 + ||q||^2)$ is bounded by a constant times k, giving the desired statement.

This estimate for $F^*\omega/\omega$ is optimal, but, again it is not enough to characterize the maps coming from Donaldson's construction as we show by building a sequence of rational maps of degree k which verify the same estimate but do not come from the same construction. Let z be the usual complex coordinate, obtained by stereographic projection, on S^2 around a neighborhood of the north pole. Let $\Lambda = \mathbb{Z} \oplus i\mathbb{Z}$. Consider the map

$$\xi_k(z) = \sum_{\lambda\in\Lambda, |\lambda|\leq k^{1/2}} \frac{1}{(z-\lambda)^3},$$

so that

$$\xi_k'(z) = -3 \sum_{\lambda\in\Lambda, |\lambda|\leq k^{1/2}} \frac{1}{(z-\lambda)^4}.$$

We want to show that the quantity

$$\frac{|\xi_k'|}{1+|\xi_k|^2}\left(1 + \frac{|z|^2}{k}\right)$$

is bounded independently of k. Given $z \in \mathbb{C}$, let $\lambda(z)$ be the element of Λ

which is closest to z (there may be several at the same distance, if so choose one). There are 3 cases to consider:

1. Suppose first that z is such that $|z| \leq 2\sqrt{k}$ and that $|z - \lambda(z)| \geq c$ (c is a fixed constant, independent of k and z). The condition $|z| \leq 2\sqrt{k}$ ensures that, the quantity

   $$\left(1 + \frac{|z|^2}{k}\right)$$

   is bounded, independently of k. Consider a $\lambda \in \Lambda$, different from $\lambda(z)$. Notice that $|\lambda(z) - \lambda| \geq 1$ and $|\lambda(z) - z| \leq \sqrt{2}/2$ and therefore

   $$|\lambda(z) - \lambda| \geq \frac{\sqrt{2}}{2}|\lambda(z) - z|.$$

   This implies

   $$|z - \lambda| \geq |\lambda(z) - \lambda| - |\lambda(z) - z| \geq \frac{2 - \sqrt{2}}{2}|\lambda(z) - \lambda|.$$

   Then

   $$|\xi_k'| \leq \frac{3}{|\lambda(z) - z|^4} + 3\left(\frac{2}{2 - \sqrt{2}}\right)^4 \sum_{\lambda \neq \lambda(z), |\lambda| \leq k^{1/2}} \frac{1}{|\lambda(z) - \lambda|^4}$$

   which is bounded above by the constant

   $$\frac{3}{c^4} + 3\left(\frac{2}{2 - \sqrt{2}}\right)^4 \sum_{\mu \neq 0, \mu \in \Lambda} \frac{1}{|\mu|^4}$$

   and we are done in this case.

2. Suppose next that z is such that $|z| \leq 2\sqrt{k}$ but $|z - \lambda(z)| \leq c$. Then, again,

   $$\left(1 + \frac{|z|^2}{k}\right)$$

is bounded independently of k and

$$\frac{|\xi_k'|}{1+|\xi_k|^2} = \frac{\left|3\frac{1}{(z-\lambda(z))^4} + 3\sum_{\lambda\neq\lambda(z),|\lambda|\leq k^{1/2}} \frac{1}{(z-\lambda)^4}\right|}{1+\left|\frac{1}{(z-\lambda(z))^3} + \sum_{\lambda\neq\lambda(z),|\lambda|\leq k^{1/2}} \frac{1}{(z-\lambda)^3}\right|^2}.$$

Multiplying denominator and numerator by $|z-\lambda(z)|^6$

$$\frac{|\xi_k'|}{1+|\xi_k|^2} = \frac{\left|3(z-\lambda(z))^2 + 3(z-\lambda(z))^6\sum_{\lambda\neq\lambda(z)} \frac{1}{(z-\lambda)^4}\right|}{|z-\lambda(z)|^6+\left|1+|z-\lambda(z)|^3\sum_{\lambda\neq\lambda(z)} \frac{3}{(z-\lambda)^3}\right|^2}.$$

The numerator is bounded by

$$3c^2 + 3c^6\sum_{\mu\neq 0}\frac{1}{|\mu|^4}.$$

As for the denominator, it is bounded from below by

$$1 - c^3\sum_{\mu\neq 0}\frac{1}{|\mu|^3},$$

which, if c is small enough, is greater than 1/2.

3. The case where $|z| \geq 2\sqrt{k}$ can be treated by noting that, in the sum defining ξ_k, only λ's with $\lambda \leq |z|/2$ appear so that $|z-\lambda| > |z|/2$ and $|\xi_k'| \leq Ck/|z|^4$. So,

$$|\xi_k'|\left(1+\frac{|z|^2}{k}\right) \leq \frac{C}{|z|^2} \leq \frac{C}{k}$$

and we are done in this case as well.

We then take $F_k(z) = \xi_k(\sqrt{k}z)$ and

$$\frac{F_k^*\omega}{\omega} = k\frac{|\xi_k'|^2(\sqrt{k}z)}{\left(1+|\xi_k|^2(\sqrt{k}z)\right)^2}\left(1+|z|^2\right)^2.$$

The result follows from substituting $\sqrt{k}z$.

Although the magnitude of $F^*\omega/\omega$ is not enough to characterize Donaldson's maps one might think that the maps are likely to make this quantity as small as possible in some sense. Consider the following function on Rat_k (the space of rational maps on S^2 of degree k).

$$L(F) = \int_{S^2} \left(\frac{F^*\omega}{\omega} \right)^2 .$$

We claim that this function has a minimum in Rat_k. More specifically:

Proposition 6 *The function L is proper on Rat_k and therefore has a minimum in that space.*

proof: To prove this, we need only see that, given a sequence of rational maps of degree k on S^2, say F_i, converging to the boundary of Rat_k, $L(F_i)$ converges to infinity. Let us assume that our sequence of maps converges to an element of Rat_{k-1}, the highest dimensional strata in the boundary of Rat_k. The other cases could be treated in a similar way. Without lost of generality, we can suppose that F_i is given by a quotient of two degree k polynomials

$$p = (z - \lambda_i)p_0, \quad q = (z - \mu_i)q_0$$

where $\{\lambda_i\}$ and $\{\mu_i\}$ are two sequences of numbers converging to $\lambda \in \mathbb{C}$. The polynomials p_0 and q_0 can be assumed to be independent of i and non vanishing at λ. Let B_i denote a ball of radius r_i (to be determined later) and center λ_i.

Let us call $|\lambda_i - \mu_i| = d_i$ and assume that $r_i^2 \leq d_i/2$. We then have,

$$p'q - pq' = (z - \lambda_i)(z - \mu_i)(p_0'q_0 - p_0q_0') + (\lambda_i - \mu_i)p_0q_0,$$

so that, on B_i, we have

$$|p'q - pq'| \geq C(d_i - r_i^2) \geq Cd_i,$$

for a different C. On the other hand

$$|p|^2 + |q|^2 = |z - \lambda_i|^2|p_0|^2 + |z - \mu_i|^2|q_0|^2 \leq C(r_i^2 + (r_i + d_i)^2) \leq C(r_i + d_i)^2.$$

This is relevant because it implies that, on B_i,

$$\left|\frac{F^*\omega}{\omega}\right| = \frac{|p'q - pq'|^2(1 + |z|^2)^2}{(|p|^2 + |q|^2)^2} \geq C\frac{d_i^2}{(r_i + d_i)^4},$$

and choosing any $r_i \leq d_i$ and with $r_i/d_i^2 \to \infty$,

$$\int_{B_i} \left(\frac{F^*\omega}{\omega}\right)^2 \geq C\frac{r_i^2}{d_i^4} \to \infty.$$

This proves the desired result.

Now

$$\log(||p||^2 + ||q||^2) = G\left(\frac{F^*\omega}{\omega}\right),$$

and Donaldson's construction gives a sequence of pairs of polynomials that make the above quantity pointwise bounded, independent of k, therefore making its L^2 norm bounded independently of k as well. We then make the following conjecture:

Conjecture 1 *The minimizer of* $\int_{S^2} \left(\frac{F^*\omega}{\omega}\right)^2$ *in* Rat_k*, for large* k*, arises as a quotient of polynomials satisfying the estimated transversality property for the pair.*

Chapter 6

Optimally distributed points on S^2

6.1 Logarithmic equilibrium points on S^2

The problem of finding an optimal way of distributing points on the sphere has been studied in several areas of mathematics. An instance of this problem appears in potential theory. Namely, let $\{x_i\}$ be a set of k points on S^2. One can consider the function $\Pi_{i<j} d(x_i, x_j)$, where $d(x, y)$ denotes the distance in $\mathbb{R}^3$, between any pair of points x and y of the sphere. This function achieves its minimum when two of the points of the set coincide. The question is: for which sets of points does it attain its maximum? Maximizing $\Pi_{i<j} d(x_i, x_j)$ is the same as minimizing the logarithmic potential,

$$\sum_{i<j} \log \frac{1}{d(x_i, x_j)}$$

and the points which achieve the minimum are called logarithmic equilibrium points. It is not known what this (these) configuration(s) is (are), but some things are know (or conjectured) about the minimum value of the logarithmic potential. As a motivation for what follows, let us first see how this problem is related to that of determining the best η for which

$$||p_k|| \leq \eta \implies ||\nabla p_k|| \geq \eta\sqrt{k}.$$

Given a set of points $\{x_i\}$, by considering the complex coordinates z_i of these points trough stereographic projection, we can form a complex polynomial of degree k, whose zeroes are the z_i's. It is determined up to a multiplicative constant. Consider as before

$$p_k(z) = e^{k/2} \prod_i \frac{(z - z_i)}{(1 + |z_i|^2)^{1/2}}.$$

Let x denote the point on S^2 whose complex coordinate is z. As a section of $\mathcal{O}(k)$ (i.e., as $p_k \mathbf{w}^k$), the norm of p_k at x is

$$||p_k||^2 = \prod_i \frac{e}{4} d^2(x, x_i). \tag{6.1}$$

There are two things to note here. First recall that, given two points x and y on S^2, if you denote their complex coordinates by z and w respectively,

$$d(x, y) = \frac{2|z - w|}{(1 + |z|^2)^{1/2}(1 + |w|^2)^{1/2}}.$$

Second, using the metric on S^2 given by $2idzd\bar{z}/(1 + |z|^2)^2$, whose volume is 4π,

$$\frac{1}{4\pi} \int_{S^2} \log d(x, y) dx = \frac{1}{2} \log \frac{4}{e}$$

which corresponds to the fact that the integral of the logarithm of the norm square of p_k is zero.

In order for $\{p_k\}$ to define an η transverse sequence of sections of $\mathcal{O}(k)$, we need $||\nabla p_k(z_i)||$ to be as big as possible (greater than $\eta\sqrt{k}$ at least), where, again, p_k is thought of as section of $\mathcal{O}(k)$. We have

$$||\nabla p_k||(z_i) = \frac{1}{\sqrt{2}} \left(\frac{e}{4}\right)^{k/2} \Pi_{i\neq j} d(x_i, x_j).$$

Here we used the norm of dz which is $(1+|z|^2)/2\sqrt{2}$. Having $\prod_i ||\nabla p_k||(z_i)$ big would be to our advantage. But this product is simply a constant (depending only on k) times $\Pi_{i<j} d(x_i, x_j)$. The problem of maximizing this quantity is therefore related to our own problem. In fact, we will see ahead, that a good approximation of these logarithmic points gives rise to a pairs of polynomials which satisfy Donaldson's constraints, as in Proposition 1, or at least seem to, experimentally. The fact that, in some sense, this distribution is "better" than its approximation leads to the conjecture that these polynomials define sections that do, in fact, satisfy the properties in Theorem 7. In particular, we conjecture that $||p_k||$ is bounded by a constant independent of k. Assuming that $\liminf \max ||p_k||$ is not zero, we can then find an upper bound for η in terms of $\liminf \max ||p_k|| = \alpha$. We will show that, if η is such that

$$||p_k|| \leq \eta \implies ||\nabla p_k|| \geq \eta\sqrt{k},$$

then

$$\eta \leq \frac{\sqrt{\pi}}{\sqrt{e}}(1 - e^{-a})^{b/2},$$

where

$$a = \frac{2\sqrt{2\pi}}{\sqrt{27}}(\sqrt{2\pi} + \sqrt{2\pi + \sqrt{27}}),$$

$$b = \frac{\sqrt{2\pi + \sqrt{27}} - \sqrt{2\pi}}{\sqrt{2\pi + \sqrt{27}} + \sqrt{2\pi}}.$$

So, the real upper bound for Donaldson's η, is this quantity divided by $\underline{inf}\max ||p_k||$. (This is really just the η corresponding to the first condition). Suppose that the sequence of polynomials p_k defines an η transverse sequence. At z_i, we must have $||\nabla p_k||(z_i) \geq \eta\sqrt{k}$. Since p_k vanishes at z_i, we have

$$||\nabla p_k(z_i)|| = \frac{1}{\sqrt{2}} \prod_{i \neq j} \left(\frac{e}{4}\right)^{1/2} d(x_i, x_j),$$

so that

$$\prod_i ||\nabla p_k(z_i)|| = \left(\frac{1}{2}\right)^{k/2} \prod_{i<j} \frac{e}{4} d(x_i, x_j)^2,$$

which is related to the logarithmic potential. There are know estimates for this potential, namely:

Theorem 10 (Rakhmanov, Saff, Zhou, [RSZ]) *Logarithmic equilibria points satisfy:*

$$\prod_{i<j} \frac{e}{4} d(x_i, x_j)^2 \leq \left(\frac{4\pi}{2e}\right)^{k/2} k^{k/2}(1 - e^{-a} + \epsilon_k)^{bk/2},$$

where a and b are as above and $\epsilon_k \to 0$ as $k \to \infty$.

On the other hand, if we are assuming η transversality, we have

$$\prod_i ||\nabla p_k(z_i)|| \geq \eta^k k^{k/2}.$$

Combining these two estimates we obtain the desired result.

6.2 Generalized spiral points

Even though the exact configuration of logarithmic points is not know, in [RSZ], Rakhmanov, Saff and Zhou describe a set of points they call generalized spiral that give extremely good numerical estimates for the maximum of $\prod_{i\leq j} d(x_i, x_j)$. We will use a modification of these points to construct a pair of polynomials, for which we can verify Donaldson's conditions experimentally. We also have some partial results towards proving that these indeed satisfy the conditions in Theorem 7 (and in particular Proposition 1). Consider the points with cylindrical coordinates (h, θ) on S^2 given by:

$$h_i = -1 + \frac{2i-1}{k} \qquad i = 1, \cdots k$$

$$\theta_0 = 0, \;\; \theta_i = \theta_{i-1} + \frac{3.6}{\sqrt{k}\sqrt{1-h_i^2}} \; mod 2\pi \qquad i = 1, \cdots k.$$

Let z_i be the complex coordinates of these points. As before, we consider the polynomials

$$p_k = e^{k/2} \prod_i \frac{(z - z_i)}{(1+|z_i|^2)^{1/2}},$$

and

$$q_k = e^{k/2} \prod_i \frac{(z + z_i)}{(1+|z_i|^2)^{1/2}}.$$

Conjecture 2 *Up to dividing by a constant that will make*

$$\max\left(||p_k||^2 + ||q_k||^2\right)$$

equal to 1*, the sections* $(p_k \mathbf{w}^k, q_k \mathbf{w}^k)$ *satisfy the conditions in Theorem 7.*

We have verified this experimentally by verifying that:

- $\max\left(||p_k||^2 + ||q_k||^2\right)$ is indeed bounded independently of k. The experimental values found for this maximum are

k	50	100	150	170	180	190	200
max	4.7828	4.8272	4.8368	4.8432	4.8373	4.8364	4.8460

- $\min\left(||p_k||^2 + ||q_k||^2\right)$ in bounded from below independently of k. The experimental values found for the minimum are

k	50	100	150	170	180	190	200
min	0.7194	0.7093	0.7085	0.7085	0.7125	0.7089	0.7085

- $\min ||\nabla p_k||(z_i)$ is bounded independently of k and the same for q_k. Below is a table of the values of the minimum of $||\nabla p_k||^2(z)$ (normalized by $2/\sqrt{k}$) taken among the z_i's

k	100	200	500	700	900	1000
min ∇	1.6963	1.6998	1.7020	1.7024	1.7026	1.7027

These values give an experimental η close to 0.15. We can also show that the sequences $||p_k(z/\sqrt{k})||^2$ for $|z| \leq 1$ are bounded (the same holds true for the corresponding q sequence). We start by showing that $||p_k(0)||^2$ is bounded. Now $d(x, x_i)^2 = 2-2\langle x, x_i\rangle$, because x_i and x have norm 1. If x has coordinates $(0, 0, -1)$, then $\langle x, x_i\rangle$ is simply minus the last coordinate of x_i, that is $-h_i$. Equation (6.1) becomes

$$||p_k(0)||^2 = \prod \frac{e}{2}(1 + h_i) = \prod_{i=1}^{k} \frac{e(2i-1)}{2k} = 2\frac{e^k(2k-1)!}{4^k k^k (k-1)!}.$$

We use Stirling's approximation for factorial, $k! \sim \sqrt{2\pi k}k^k e^{-k}$, to see that $||p_k(0)||^2$ actually converges to $\sqrt{2}$. As for $||p_k(z/\sqrt{k})||^2$,

$$||p_k(\frac{z}{\sqrt{k}})||^2 = e^k \prod \frac{|z_i|^2}{1+|z_i|^2} \prod \left| \frac{z}{z_i\sqrt{k}} - 1 \right|.$$

The first product is simply $||p_k(0)||^2$, which we know is bounded. As for the second product, if one considers the inequality $|\log(1-z)| \leq 4|z|$, which holds true for all z with $|z| \leq 1$, then one can see that,

$$\left|\log \left| \prod \left(1 - \frac{z}{z_i\sqrt{k}}\right)\right|\right| \leq 4 \sum \frac{|z|}{\sqrt{k}|z_i|}.$$

In cylindrical coordinates,

$$|z_i| = \sqrt{\frac{1-h_i}{1+h_i}} = \sqrt{\frac{2k-2i-1}{2i-1}}.$$

Now

$$\sum_{i=1}^{k} \frac{1}{\sqrt{k}|z_i|} = \sum \frac{\sqrt{2i-1}}{\sqrt{k}} \frac{1}{\sqrt{2k-2i+1}} \leq \sqrt{2} \sum_{i=1}^{k} \frac{1}{\sqrt{i}},$$

but $\sum 1/\sqrt{i}$ is divergent. If instead of having a single point with given h_i, we had 3 (that differed from each other by multiplication by a cube root of unity in complex coordinates), then our initial product would be

$$\prod \left(1 - z^3/(z_i\sqrt{k})^3\right).$$

The norm of the logarithm of this product is bounded by a constant times $\sum 1/i^{3/2}$. This new set of points would have $3k$ elements instead of k. This would just mean constructing a subsequence of a Donaldson sequence. We have thus showed the following:

Proposition 7 *Consider the set of $3k$ points on S^2, whose h coordinates are given by*

$$h_{3i-2} = h_{3i-1} = h_{3i} = -1 + \frac{2i-1}{k} \qquad i = 1, \cdots k$$

and whose θ coordinates are

$$\theta_{3i-2} = \theta_{3i-5} + \frac{3.6}{\sqrt{k}\sqrt{1-h_i^2}}, \qquad \theta_{3i-1} = \theta_{3i-2} + \frac{\pi}{3}, \qquad \theta_{3i} = \theta_{3i-2} + \frac{2\pi}{3},$$

defined $\mod 2\pi$ *for $i = 1, \cdots k$ (where $\theta_0 = 0$). Then, letting z_i be the complex coordinates of these points trough stereographic projection, and*

$$p_k = e^{3k/2} \prod_i \frac{(z - z_i)}{(1 + |z_i|^2)^{1/2}},$$

$p_k(z/\sqrt{k})$ is bounded, in any fixed neighborhood of 0.

We cannot do this calculation for the other points of S^2, in fact we cannot prove that $||p_k||^2$ is bounded. This is the main ingredient missing. We should remark that, the experimental data shows that the maximum of $||p_k||^2$ occurs close to the north pole, so that this calculations is more relevant to prove what we want, than what it may seem.

Supposing that the sequence of functions $p_k/(1 + |z|^2)^{k/2}$, rescaled, converges, then it is simple to determine what type of limit they should have. To see this, consider one of the spiral points with coordinates (h_i, θ_i). Suppose we take a small neighborhood, of radius of the order $k^{-1/2}$, around that point. Will there be other spiral points in the neighborhood? The next spiral point, close to (h_i, θ_i), will appear after a variation in θ which is approximately 2π, that is, after approximately

$$j = 2\pi\sqrt{k}\frac{\sqrt{1-h_i^2}}{3.6}$$

steps. Then, the h will be approximately

$$h_i + \frac{4\pi\sqrt{1-h_i^2}}{3.6\sqrt{k}}.$$

When one considers what this configuration of points looks like, as k tends to ∞, one sees that the limit of $p_k/(1+|z|^2)^{k/2}$, rescaled by $\sqrt{k}$ around any point distinct from the poles, can only be a function whose zeroes lie in $\{n+\lambda_m+im, m, n \in \mathbb{Z}\}$, where $0 \leq \lambda_n \leq 1$. We may assume that $\lambda_0 = 0$. The first thing to check is that such a function actually exits.

Proposition 8 *Given a sequence $\lambda = \{\lambda_k\}$ of numbers in $[0, 1[$, there is a function P_λ with zeros at $\{n + \lambda_m + im, m, n \in \mathbb{Z}\}$, such that $|P_\lambda|(z) \leq e^{c|z|^2}$.*

proof: In the same way as one defines a theta function, we set

$$P_\lambda(z) = \prod_{m=0}^{\infty}\left(1 - e^{-2\pi m}e^{2\pi i\lambda_m}e^{-2\pi iz}\right)\prod_{m=1}^{\infty}\left(1 - e^{-2\pi m}e^{2\pi i\lambda_{-m}}e^{2\pi iz}\right).$$

Both products are convergent for any given $z \in \mathbb{C}$. In fact, the logarithm of the norm of each of them is bounded by

$$\sum_m \log\left(1 - e^{-2m}e^{2\pi|z|}\right),$$

which is a convergent series, because the series $\sum e^{-2m}$ is itself convergent. P_λ clearly has zeroes at $n + \lambda_m + im$, as wanted. It remains to be seen that P_λ is of order less than 2. First note that $P_\lambda(z+1) = P_\lambda(z)$. It is also true that

$$P_\lambda(z+i) = -e^{-2\pi iz}e^{2\pi}P_{\sigma(\lambda)}(z),$$

where $\sigma(\lambda)$ is simply a shift left in λ. This implies

$$P_\lambda(z+mi) = (-1)^m e^{-2\pi imz}e^{\pi(m^2+m)}P_{\sigma^m(\lambda)}(z)$$

and from this formula, we can conclude that P_λ is of order (less than) 2.

Conjecture 3 *Given z_0 in $\mathbb{C}^*$, $p_k(z_0 + \frac{z}{\sqrt{k}})$ has a subsequence converging to a normalization of P_λ, for some λ.*

There is a corresponding conjecture for q_k and Q_λ where $Q_\lambda(z) = P_\lambda(-z)$. For the rescaling around zero, the sublimit is a function with zeroes in a spiral. To prove Conjecture 2 using Conjecture 3, it is enough to establish that $(P_\lambda(z), P_\lambda(-z))$ "satisfies" Theorem 7. This comes down to checking that P_λ has no double zeroes and that $P_\lambda(z)$ and $P_\lambda(-z)$ have no common zeroes (for the first two conditions).

Chapter 7

An upper bound for η

7.1 The Kälher case

The techniques used in section 4.2 can actually be generalized to determine upper bounds for how much transversality can achieved for a linear system of dimension $n+1$, on an n dimensional Kähler manifold. We prove the following:

Proposition 9 *Let X be a Kähler manifold of dimension n, with symplectic form ω, such that $[\omega/2\pi]$ lies in $H^2(X,\mathbb{Z})$. Let $L \to X$ be a line bundle whose Chern class is $[\omega/2\pi]$. There exits $\eta_0 < 1$, such that, if we have n sequences of holomorphic sections $s_0, \cdots, s_n$ of L^k satisfying*

$$\eta \leq ||s_0||^2 + \cdots + ||s_n||^2 \leq 1,$$

then $\eta < \eta_0$.

proof: We will use the same idea as we used to prove this proposition for S^2. Namely, the sections $s_0, \cdots, s_n$ define a holomorphic map from X to $\mathbb{CP}^n$. The rescaled map subconverges as k tends to infinity, to a map of a disc of fixed radius in $\mathbb{C}^n$ to $\mathbb{CP}^n$. If η were very close to 1, then the pullback metric from $\mathbb{CP}^n$ would be very close to flat, but this can't be true because its curvature is actually the pullback of the Fubini-Study metric.

To make this precise we start by recalling that there exists ,on X, complex coordinates $z = (z_1, \cdots, z_n)$, such that ω differs from the standard symplectic form on $\mathbb{C}^n$ by a small amount. Choosing an appropriate primitive of ω, we showed before that there exists a holomorphic section of L, σ, whose norm is $e^{-|z|^2}\sqrt{u}$ and $|\log u| \leq c|z|^4$ (see the proof of Lemma 2). We will show that (in some sense) the rescaled map to $\mathbb{CP}^n$ subconverges. In a neighborhood of radius $ck^{-1/2}$ of a given point p, in X, σ^k trivializes L^k and we can write

$$s_i = f_i^k \sigma^k \quad i = 1, \cdots, n.$$

Consider also the map $\delta_k : B(c) \to B(p, ck^{-1/2})$ (the c's are actually different and both different from the c appearing in the estimate for $\log u$) which sends z to the point whose coordinate is $z/\sqrt{k}$. Set

$$\tilde{f}_i^k = f_i^k \circ \delta_k.$$

The condition on the s_i becomes,

$$\frac{e^{|z|^2}\eta}{u^{k/2}(\frac{z}{\sqrt{k}})} \leq |\tilde{f}_0^k|^2 + \cdots + \leq |\tilde{f}_n^k|^2 \leq \frac{e^{|z|^2}}{u^{k/2}(\frac{z}{\sqrt{k}})}.$$

Since $-c|z|^4 \leq \log u \leq c|z|^4$, we have

$$u^{k/2}(\frac{z}{\sqrt{k}}) \geq e^{\frac{-c|z|^4}{2k}} > \frac{1}{2},$$

for sufficiently big k, so that each

$$\tilde{f}_i^k \leq 2e^{|z|^2}.$$

As before, this implies that each $\tilde{f}_i^k$ has a convergent subsequence in $B(c)$, for the $\mathcal{C}^r$ norm, for any fixed r we may wish to choose (actually the ball may have to shrink here). Let us call the limits f_i. Taking limits in the inequalities above, we get for the f_i,

$$\eta e^{|z|^2} \leq |f_0|^2 + \cdots + |f_n|^2 \leq e^{|z|^2}.$$

The above implies that at least one of the f_i is non zero at p. Say it is f_0. On a maybe smaller ball, f_0 is nowhere zero and the same holds for the subsequence that converges to it, for k big enough. The sequence of maps defined by the sections s_i, $F_k = [s_0 : \cdots : s_n]$ can also be seen (by composing with δ_k) as a sequence of maps from $B(c)$ to $\mathbb{CP}^n$ whose image is contained in the subset $\{[\mathbf{z}_0 : \cdots : \mathbf{z}_n] : \mathbf{z}_0 \neq 0\}$, and can be written in coordinates as

$$(\frac{\tilde{f}_1^k}{\tilde{f}_0^k}, \cdots, \frac{\tilde{f}_n^k}{\tilde{f}_0^k}).$$

Since each of these quotients subconverges, so does the rescaled F_k.

Next we see how $\tilde{F}_k$, the rescaled F_k, behaves with respect to the metric on $\mathbb{CP}^n$. Away from its branch points, $\tilde{F}_k$ pulls back the Fubini-Study metric on $\mathbb{CP}^n$ to a metric on $B(c)$, whose curvature 2-form is $\tilde{F}_k^* \omega_{FS}$. On the other hand, this metric is the metric associated to the 2-form $\tilde{F}_k^* \omega_{FS}$. On $\{[\mathbf{z}_0 : \cdots :$

$\mathbf{z}_n] : \mathbf{z}_0 \neq 0\}$, there are inhomogeneous coordinates

$$(z_1 = \frac{\mathbf{z}_1}{\mathbf{z}_0}, \cdots, z_n = \frac{\mathbf{z}_n}{\mathbf{z}_0}),$$

and the Fubini-Study metric can be written as

$$\bar{\partial}\partial \log(1 + |z_1|^2 + \cdots + |z_n|^2).$$

Because $\tilde{F}_k$ is holomorphic, we conclude that $F_k^* \omega_{FS}$ is

$$\bar{\partial}\partial \log(1 + |z_1|^2 + \cdots + |z_n|^2) \circ \tilde{F}_k.$$

Now

$$\log(1 + |z_1|^2 + \cdots + |z_n|^2) \circ \tilde{F}_k = \log(|\tilde{f}_0^k|^2 + |\tilde{f}_1^k|^2 + \cdots + |\tilde{f}_n^k|^2) - \log(|\tilde{f}_0^k|^2),$$

so

$$\tilde{F}_k^* \omega_{FS} = \bar{\partial}\partial \log(|\tilde{f}_0^k|^2 + |\tilde{f}_1^k|^2 + \cdots + |\tilde{f}_n^k|^2).$$

Next. we prove the following lemma:

Lemma 9 *Consider the metric on $B(c)$, the ball of radius c around the origin in $\mathbb{C}^n$, associated with the 2-form $\bar{\partial}\partial u$, for some plurisubharmonic function u on $B(c)$. Then, its curvature 2-form is*

$$\bar{\partial}\partial \log \det \mathit{Hess}(u)$$

where $Hess(u)$ denotes the matrix $(u_{z_i \bar{z}_j})$.

proof: On the bundle $TB(c)$ we have a trivialization given by

$$(\partial/\partial z_1, \cdots, \partial/\partial z_n).$$

With respect to this trivialization the matrix representation of the metric is $h = Hess(u)$. There is a unique connection on $TB(c)$ which is compatible with the metric associated with $\bar{\partial}\partial u$ and the complex structure. Its matrix representation, with respect to the mentioned trivialization of the bundle, is $\theta = \partial h h^{-1}$. The curvature matrix representation is $d\theta + \theta \wedge \theta$. The curvature 2-form is the trace of this two by two matrix of 2-forms. Since $\theta \wedge \theta$ is traceless, the curvature is $trd\theta = dtr\theta$. But we know that

$$tr(\partial h h^{-1}) = \partial \log(\det h).$$

The curvature becomes $d\partial \log(\det h) = \bar{\partial}\partial \log(\det h)$, as we wished to prove.

Going back to our problem, we can apply this lemma to $u = \log(|\tilde{f}_0|^2 + \cdots + |\tilde{f}_n|^2)$, to conclude that the curvature of the pullback by F of the Fubini-Study metric is

$$\bar{\partial}\partial \log(\det Hess(\log(|\tilde{f}_0^k|^2 + \cdots + |\tilde{f}_n^k|^2)),$$

away from the branch points of $\tilde{F}_k$ and the zeroes of $\tilde{f}_0^k$. We therefore conclude that

$$\bar{\partial}\partial \log(\det Hess(\log(|\tilde{f}_0^k|^2 + \cdots + |\tilde{f}_n^k|^2)) = \bar{\partial}\partial \log(|\tilde{f}_0^k|^2 + \cdots + |\tilde{f}_n^k|^2),$$

which is now valid on the whole $B(c)$. Taking limits in this equation (after passing to a subsequence) we get

$$\bar{\partial}\partial \log(\det Hess(\log(|f_0|^2 + \cdots + |f_n|^2)) = \bar{\partial}\partial \log(|f_0|^2 + \cdots + |f_n|^2).$$

This implies that

$$\triangle \log(\det Hess(\log(|f_0|^2 + \cdots + |f_n|^2)) = \triangle \log(|f_0|^2 + \cdots + |f_n|^2),$$

Where $\triangle$ denotes the usual flat Laplacian on $\mathbb{C}^n$.

We are now in a position to use equation (7.1) to get a bound on η. Assume that η is very close to 1, then

$$\log(|f_0|^2 + \cdots + |f_n|^2) \simeq |z|^2,$$

so that

$$\triangle \log(|f_0|^2 + \cdots + |f_n|^2) \simeq n,$$

and

$$\log \det Hess \log(|f_0|^2 + \cdots + |f_n|^2) \simeq 0.$$

This gives a contradiction. At first, it seems that the upper bound η_0 obtained for η in this way depends on the particular $f_0, \cdots, f_n$ and therefore on the specific sequences $s_0^k, \cdots, s_n^k$, but this is not so. Using Lemma 8 (and a slight variation of it applicable to $Hess$ instead of $\triangle$), the only thing we need to note here is that there is a bound (independent of the sequences) on

$$d\triangle \log(|f_0|^2 + \cdots + |f_n|^2),$$

$$dHess \log(|f_0|^2 + \cdots + |f_n|^2),$$

and on

$$d\triangle \log \det Hess \log(|f_0|^2 + \cdots + |f_n|^2).$$

This is so because, since $f_0, \cdots, f_n$ are holomorphic, the Cauchy formula and the bounds $|f_i|^2 \leq e^{c^2}$, give bounds on all the derivatives of the f_i depending only on c, and, choosing c appropriately small, the above quantities have bounds depending only on n, just as for $\mathbb{CP}^1$.

Next, we prove a generalization of this proposition to almost holomorphic sections on Kähler manifolds

Proposition 10 *Let X be a Kähler manifold of dimension n, with symplectic form ω, such that $[\omega/2\pi]$ lies in $H^2(X,\mathbb{Z})$. Let $L \to X$ be a line bundle, whose Chern class is $[\omega/2\pi]$. For all C, there exits $\eta_0 < 1$ such that, if we have n sequences of sections $s_0, \cdots, s_n$ of L^k satisfying*

$$\eta \leq ||s_0^k||^2 + \cdots + ||s_n^k||^2 \leq 1,$$

which are C asymptotically holomorphic i.e.

$$\left|\bar{\partial} s\right| \leq C$$

then $\eta < \eta_0$.

proof: The main idea here is the following. Define n sequences of sections, ξ_l^k, by

$$\xi_l^k = -\bar{\partial}^* G \bar{\partial} s_l,$$

where G is the Green's operator, the inverse of Laplacian on $\Lambda^{(0,1)}(L^k)$ and $\bar{\partial}^*$ is the formal adjoint of $\bar{\partial}$. These satisfy two important properties:

- The first is that $\bar{\partial}\xi_l^k = -\bar{\partial}s_l^k$.

- The second is that ξ_l^k are L^2 small in X. This is because

$$\langle -\bar{\partial}^* G\bar{\partial}s_l^k, -\bar{\partial}^* G\bar{\partial}s_l^k\rangle = \langle \bar{\partial}\bar{\partial}^* G\bar{\partial}s_l^k, G\bar{\partial}s_l^k\rangle = \langle \bar{\partial}s_l^k, G\bar{\partial}s_l^k\rangle.$$

  But G has norm $O(k^{-1})$ so that

$$||\xi_l^k||^2_{L^2(S^2)} \leq Ck^{-1}.$$

Write $s_l + \xi_l = t_l$. Then, this new sequence of sections is holomorphic and bounded in L^2. For each point p, in X, there are centered complex coordinates $\chi_p : B(k^{-1/2}) \to B(p, ck^{-1/2})$, and a holomorphic section σ_p of L satisfying

$$|\sigma_p|(z) = e^{-|z|^2}u(z),$$

where $|u|(z) \leq C|z|^4$. There is a partition of X by sets of diameter $ck^{-1/2}$, each of which has volume $ck^{-n/2}$ and contains a ball of radius $k^{-1/2}$. For each k, sufficiently big, there is an element of such a partition A_i for which

$$\int_{A_i} ||\xi_l^k||^2 \leq Ck^{-n/2}k^{-1}$$

so that, for each k, there is p_k for which

$$\int_{B(p_k, k^{-1/2})} ||\xi_l^k||^2 \leq Ck^{-n/2}k^{-1}.$$

Let

$$\chi_k = \chi_{p_k} \circ \delta_k : B(1) \to B(p_k, k^{-1/2})$$

and

$$\sigma^k = \sigma_{p_k}^{\otimes k}.$$

We have

$$||\sigma^k|| \circ \chi_k = e^{\frac{-|z|^2}{2}} u^{k/2}(\frac{z}{\sqrt{k}}).$$

Write $t_l^k = p_l^k \sigma^k$, $s_l^k = f_l^k \sigma_k$ and $\xi_l^k = g_l^k \sigma_k$. As before, we consider the sequences $\{\tilde{p}_l^k\}$, $\{\tilde{f}_l^k\}$ and $\{\tilde{g}_l^k\}$ in the rescaled coordinates. The functions $\tilde{p}_l^k$ are holomorphic in $B(1)$ and their L^2 norms there is bounded independently of k, because $\{\tilde{f}_l^k\}$ has a uniform bound independent of k, and

$$\int_{B(p_k,k^{-1/2})} ||\xi_l^k||^2 = \int_{B(1)} |\tilde{g}_l^k|^2 e^{-|z|^2} |u|^k(\frac{z}{\sqrt{k}}) \frac{dzd\bar{z}}{k^{n/2}} \geq Ck^{-n/2} \int_{B(1)} |\tilde{g}_l^k|^2.$$

Since the left hand side is bounded by $Ck^{-n/2}k^{-1}$, this shows that $\{\tilde{g}_l^k\}$ has L^2 norm tending to zero, and, in particular the L^2 norm of $\{\tilde{p}_l^k\}$ is bounded. This implies that the supremum norm of this sequence is bounded, independently of k, on $B(1/2)$ and so is the supremum norm of the sequence of derivatives of all orders. We conclude that, the sequences $\{\tilde{p_l^k}\}$ are uniformly subconvergent to p_l, for each l in $0, \cdots, n$ and p_l is holomorphic. Now consider the inequality

$$||f_l^k - p_l||_{L^2(B(1/2))} \leq ||g_l^k||_{L^2(B(1/2))} + ||p_l^k - p_l||_{L^2(B(1/2))}$$

and hence tends to zero. By Riesz theorem, we can extract from $\{\tilde{f}_l^k\}$ a subsequence which converges pointwise almost everywhere to p_l, for every l in $\{0, \cdots n\}$. The upshot is that, we can then conclude that the holomorphic functions $p_0, \cdots p_l$ on $B(1/2)$ satisfy

$$\eta e^{|z|^2} \leq |p_0|^2 + \cdots + |p_n|^2 \leq e^{|z|^2}.$$

Note that $u^{k/2}(z/\sqrt{k})$ tends to 1. If η were very close to 1, the function $\log(|p_0|^2 + \cdots + |p_n|^2)$ would be very close to $|z|^2$. Now by considering the holomorphic map from $B(1/2)$ to $\mathbb{CP}^n$ given by $[p_0 : \cdots : p_n]$, and using it to pull back the Fubini-Study metric on $\mathbb{CP}^n$ we conclude

$$\triangle \log(|p_0|^2 + \cdots + |p_n|^2) = \triangle \log \det Hess(|p_0|^2 + \cdots + |p_n|^2).$$

As before, this gives the desired result.

7.2 The symplectic case

We now go on to prove the most general statement stated in the introduction:

Theorem 11 *Let X be a symplectic manifold of dimension $2n$ with symplectic form ω such that $[\omega/2\pi]$ lies in $H^2(X, \mathbb{Z})$, and a compatible almost complex structure. Let $L \to X$ be a Hermitian line bundle whose Chern class is $[\omega/2\pi]$. There exits $\eta_0 < 1$ such that, if we have n asymptotically holomorphic sequences of sections $s_0, \cdots, s_n$ of L^k satisfying $\eta \leq ||s_0||^2 + \cdots + ||s_n||^2 \leq 1$, then $\eta < \eta_0$.*

proof: We will use the same idea as we used to prove this proposition for S^2. Namely, the sections $s_0, \cdots, s_n$ define an asymptotically holomorphic map from X to $\mathbb{CP}^n$. The rescaled map subconverges, (in some sense to be specified ahead) as k tends to infinity, to a map of a disc of fixed radius in $\mathbb{C}^n$ to $\mathbb{CP}^n$. If η were very close to 1, the pullback metric from $\mathbb{CP}^n$ would be very close to flat, but this can't be true because its curvature 2-form is the pullback of the Fubini-Study metric.

To make the argument precise, choose a holomorphic structure J on X, possibly non integrable, compatible with the symplectic form. In this setting, we know that, for each point p in X, there are centered symplectic coordinates, for which J is standard at p and an asymptotically holomorphic sequence of sections σ^k of L^k such that

$$||\sigma^k|| = e^{-k|z|^2},$$

on a ball of radius $k^{-1/2}$. Although this is explained in [Do1], we will briefly recall it here. Let $\{z_i\}$ be symplectic coordinates near p. Choose them so that $J(0) = J_0$ (J_0 denotes the standard complex structure on $\mathbb{C}^n$). Then, in coordinates, $\omega = -i \sum dz_j d\bar{z}_j$ and one can choose for the connection 1-form on L

$$A = \sum z_j d\bar{z}_j - \bar{z}_j dz_j.$$

The complex structure J allows us to decompose $TB(1)$ into $T^{(0,1)}$ and $T^{(1,0)}$, the eigenspaces for J, and to write $d = \partial + \bar{\partial}$. We also have the usual decomposition for d, using J_0, which we write $d = \partial_0 + \bar{\partial}_0$. Let $\mathbf{1}$ denote the unitary section of L, satisfying $\nabla \mathbf{1} = A\mathbf{1}$. Then set

$$\sigma^k = e^{-k|z|^2}\mathbf{1}^k.$$

We have

$$\nabla\sigma^k = -2k \sum \bar{z}_j dz_j e^{-k|z|^2}\mathbf{1}^k,$$

so that $\bar{\partial}_0 \sigma^k = 0$. But

$$|J - J_0| \leq C|z|$$

and

$$\bar{\partial}\sigma^k - \bar{\partial}_0\sigma^k = i\nabla\sigma^k \circ (J - J_0).$$

By taking into consideration $|z_j| \leq Ck^{-1/2}$ we get

$$|\bar{\partial}\sigma^k| \leq C.$$

Write $s_l^k = f_l^k \sigma^k$ and $\tilde{f}_l^k$ for the functions f_l^k on rescaled coordinates, defined over $B(1)$, i.e., $\tilde{f}_l^k = f_l^k \circ \delta_k$, where δ_k is given in coordinates by

$$\delta_k(z) = \frac{z}{\sqrt{k}}.$$

These functions satisfy

$$\eta e^{|z|^2} \leq |\tilde{f}_0^k|^2 + \cdots + |\tilde{f}_n^k|^2 \leq e^{|z|^2}.$$

Also, carrying J over to $B(1)$ by using the rescaled coordinates, we get a new complex structure on $B(1)$, which we again call J. Let J_0 be the standard complex structure on $B(1)$. Because J is J_0 at 0, J and J_0 are close. More precisely, they satisfy

$$|J - J_0| \leq C\frac{|z|}{\sqrt{k}}, \quad z \in B(1).$$

Again the complex structure J allows us to decompose $TB(1)$ into $T^{(0,1)}$ and $T^{(1,0)}$, the eigenspaces for J, and to write $d = \partial + \bar{\partial}$. We also have the usual decomposition for d using J_0 which we write $d = \partial_0 + \bar{\partial}_0$. We have

$$\left|\bar{\partial}\tilde{f}_l^k\right| \leq \frac{C}{\sqrt{k}}.$$

Next, we show that $\{d\tilde{f}_l^k\}$ is L^2 bounded on $B(1/2)$, independently of k. Take β to be a bump function equal to 1 on $B(1/2)$ and 0 outside $B(3/4)$. Set $g_l^k = \beta\tilde{f}_l^k$. It is enough to see that

$$\int_{B(1)} |dg_l^k|^2$$

is bounded. Now,

$$\bar{\partial}g_l^k = \bar{\partial}\beta\tilde{f}_l^k + \beta\bar{\partial}\tilde{f}_l^k$$

and therefore this is uniformly bounded and, as a consequence, L^2 bounded. As for

$$\int_{B(1)} |\partial g_l^k|^2,$$

we want to compare it to

$$\int_{B(1)} |\partial g_l^k|_J^2,$$

where $|.|_J$ denotes the norm obtained from J and ω_0 (the standard symplectic form on $\mathbb{C}^n$). Given a vector field v on $B(1)$,

$$|v|^2 = \omega_0(v, J_0 v), \qquad |v|_J^2 = \omega_0(v, Jv),$$

so

$$\left||v|^2 - |v|_J^2\right| \leq |\omega_0||J - J_0||v|^2 \leq C|v|^2$$

and this implies that the J norm of any tensor which has a bound on its J_0 norm, independent of k, is bounded independently of k as well. For example, ω_0 and $J_0 - J$ have bounded J norms. Conversely, we get

$$\left||v|^2 - |v|_J^2\right| \leq |\omega_0|_J|J - J_0|_J|v|_J^2 \leq C|v|_J^2,$$

therefore, if a tensor has a bound on its J norm, independent of k, it also has a bound (independent of k) on its J_0 norm. Hence it is enough to prove that $\int_{B(1)} |\partial g_l^k|_J^2$ is bounded. Since g_l^k is zero near the boundary of $B(1)$, we can write

$$\int_{B(1)} |\partial g_l^k|_J^2 = \int_{B(1)} \partial^* \partial g_l^k \bar{g}_l^k,$$

where ∂^* means the formal adjoint, with respect to the J metric. The Hodge identities state that $\partial^* = i[\Lambda, \bar{\partial}]$ and $\bar{\partial}^* = -i[\Lambda, \partial]$ (Λ denotes the dual with respect to the J metric of wedging with ω_0) and

$$\partial^* \partial g_l^k = i\Lambda\bar{\partial}\partial g_l^k, \qquad \bar{\partial}^*\bar{\partial} g_l^k = -i\Lambda\partial\bar{\partial} g_l^k.$$

The operators $\bar{\partial}\partial$ and $\partial\bar{\partial}$ are related by

$$\bar{\partial}\partial + \partial\bar{\partial} = N\bar{N} + \bar{N}N,$$

where N is the Nijenhuis tensor for J. The upshot is that we can write

$$\int_{B(1)} |\partial g_l^k|_J^2 = \int_{B(1)} i\Lambda(N\bar{N} + \bar{N}N)|g_l^k|_J^2 + \int_{B(1)} \bar{\partial}^*\bar{\partial} g_l^k \bar{g}_l^k,$$

and the second term in this sum is simply $\int_{B(1)} |\bar{\partial} g_l^k|_J^2$, which we know is bounded. Now N is bounded independently of k and so is Λ (because ω_0 is) and we are done.

This implies that $\{g_l^k\}$ is bounded in $W^1(\mathbb{C}^n)$ and so, by Rellich's lemma, it has a subsequence, converging to f_l in $W^0 = L^2$. Now by Riesz theorem, we can extract from $\{g_l^k\}$ a subsequence converging pointwise a.e. in $B(1)$ and $\{\tilde{f}_l^k\}$ has a pointwise convergent subsequence a.e. in $B(1/2)$ to f_l. These functions satisfy

$$\eta e^{|z|^2} \leq |f_0|^2 + \cdots + |f_n|^2 \leq e^{|z|^2}, \ a.e..$$

The next step is to show that the functions f_l are holomorphic. To do this, note first that $\{\tilde{f}_l^k\}$ has a weakly convergent subsequence in W^1, simply because it is a bounded sequence in this space. Given any L^2 1-form a in $B(1)$,

$$\int_{B(1/2)} \langle d\tilde{f}_l^k - df_l, a \rangle \to 0.$$

In fact, if $a \in L^2(T_0^{(0,1)} B(1))$

$$\int_{B(1/2)} \langle \bar{\partial}_0 \tilde{f}_l^k, a \rangle \to \int_{B(1/2)} \langle \bar{\partial}_0 f_l, a \rangle$$

because $T_0^{(0,1)}$ and $T_0^{(1,0)}$ (the eigenspaces of J_0) are pointwise orthogonal. But

$$\bar{\partial}_0 \tilde{f}_l^k = (\bar{\partial}_0 - \bar{\partial}) \tilde{f}_l^k + \bar{\partial} \tilde{f}_l^k,$$

so, over $B(1/2)$,

$$||\bar{\partial}_0 \tilde{f}_l^k||_{L^2} \leq ||d\tilde{f}_l^k \circ (J - J_0)||_{L^2} + ||\bar{\partial} \tilde{f}_l^k||_{L^2}.$$

The first term is bounded by

$$\frac{C}{\sqrt{k}} \left(\int_{B(1/2)} |d\tilde{f}_l^k|^2 \right)^2 \leq \frac{C}{\sqrt{k}}.$$

We know that the second term is also bounded by $C/\sqrt{k}$. We can conclude that

$$||\bar{\partial}_0 \tilde{f}_l^k||_{L^2} \to 0,$$

so

$$(\bar{\partial}_0 \tilde{f}_l^k, a)_{L^2} \to 0.$$

This in turn proves that

$$(\bar{\partial}_0 f_l, a)_{L^2} = 0, \quad \forall a \in L^2(T_0^{(0,1)} B(1))$$

and, in fact, for all $a \in L^2(T^* B(1))$, so f_l is holomorphic. In particular, it is continuous and the inequality

$$\eta e^{|z|^2} \leq |f_0|^2 + \cdots + |f_n|^2 \leq e^{|z|^2},$$

holds at every point. We conclude that the existence of the n sections $s_0^k, \cdots, s_n^k$ implies the existence of n holomorphic functions $f_0, \cdots, f_n : B(1/2) \to \mathbb{C}$ satisfying

$$\eta e^{|z|^2} \leq |f_0|^2 + \cdots + |f_n|^2 \leq e^{|z|^2}.$$

Next, we show that the existence of such functions implies the equality

$$\triangle \log(|f_0|^2 + \cdots + |f_n|^2) = \triangle \log \det Hess(|f_0|^2 + \cdots + |f_n|^2), \qquad (7.1)$$

at those points where the map $F = [f_0 : \cdots : f_n]$ has injective differential, so that the pullback of the Fubini-Study metric is a metric. Away from its branch points, F pulls back the Fubini-Study metric on $\mathbb{CP}^n$ to a metric on

$B(c)$, whose curvature 2-form is $F^*\omega_{FS}$. On the other hand, this metric is the metric associated to the 2-form $F^*\omega_{FS}$. On $\{[\mathbf{z}_0 : \cdots : \mathbf{z}_n] : \mathbf{z}_0 \neq 0\} \subset \mathbb{CP}^n$ there are inhomogeneous coordinates

$$(z_1 = \frac{\mathbf{z}_1}{\mathbf{z}_0}, \cdots, z_n = \frac{\mathbf{z}_n}{\mathbf{z}_0}),$$

and the Fubini-Study metric can be written as

$$\bar{\partial}\partial \log(1 + |z_1|^2 + \cdots + |z_n|^2).$$

The map F being holomorphic, we conclude that $F^*\omega_{FS}$ is

$$\bar{\partial}\partial \log(1 + |z_1|^2 + \cdots + |z_n|^2) \circ F.$$

Now

$$\log(1 + |z_1|^2 + \cdots + |z_n|^2) \circ F = \log(|f_0|^2 + |f_1|^2 + \cdots + |f_n|^2) - \log(|f_0|^2),$$

so

$$F^*\omega_{FS} = \bar{\partial}\partial \log(|f_0|^2 + |f_1|^2 + \cdots + |f_n|^2).$$

We can apply Lemma 9 to $u = \log(|f_0|^2 + \cdots + |f_n|^2)$, to conclude that the curvature of the pullback by F of the Fubini-Study metric is

$$\bar{\partial}\partial \log(\det Hess(\log(|f_0|^2 + \cdots + |f_n|^2)),$$

away from the branch points of F and the zeroes of f_0. We therefore conclude that

$$\bar{\partial}\partial \log(\det Hess(\log(|f_0|^2 + \cdots + |f_n|^2)) = \bar{\partial}\partial \log(|f_0|^2 + \cdots + |f_n|^2),$$

where F is an embedding. If this set of points is dense in $B(1)$, we are done. In fact, then, equality (7.1) holds true at all points. Assume that η is very close to 1, then

$$\log(|f_0|^2 + \cdots + |f_n|^2) \simeq |z|^2,$$

so that

$$\triangle \log(|f_0|^2 + \cdots + |f_n|^2) \simeq n,$$

and

$$\log \det Hess \log(|f_0|^2 + \cdots + |f_n|^2) \simeq 0.$$

This gives a contradiction. At first, it seems that the upper bound η_0 obtained for η in this way depends on the particular $f_0, \cdots, f_n$ and therefore on the specific sequences $s_0^k, \cdots, s_n^k$, but this is not so. Using Lemma 8 (and a slight variation of it applicable to $Hess$ instead of $\triangle$), the only thing we need to note here is that there is a bound (independent of the sequences) on

$$d\triangle \log(|f_0|^2 + \cdots + |f_n|^2),$$

$$dHess \log(|f_0|^2 + \cdots + |f_n|^2),$$

and on

$$d\triangle \log \det Hess \log(|f_0|^2 + \cdots + |f_n|^2).$$

This is so because, since $f_0, \cdots, f_n$ are holomorphic, the Cauchy formula and the bounds $|f_i|^2 \leq e^{c^2}$, give bounds on all the derivatives of the f_i depending only on c, and, choosing c appropriately small, the above quantities have bounds depending only on n, just as for $\mathbb{CP}^1$.

On a maybe smaller ball, the differential of F is injective exactly where the determinant of the derivative of $(f_1/f_0, \cdots, f_n/f_0)$ is non zero. But this determinant is a holomorphic function, so, if it is not identically zero, its zero set will have zero measure. In the case where the determinant is identically zero, we make use of the following: given any ϵ, there are holomorphic functions $\alpha_1, \cdots, \alpha_n$, each with norm smaller than ϵ, such that

$$\det d\left((\frac{f_1}{f_0}, \cdots, \frac{f_n}{f_0}) + (\alpha_1, \cdots, \alpha_n) \right)$$

is not identically zero. Now set $g_0 = f_0$ and $g_i = f_i + f_0\alpha_i$, for i in $\{1, \cdots n\}$. Then, by choosing ϵ small, we can ensure that

$$(2\eta - 1)e^{|z|^2} \leq |g_0|^2 + \cdots + |g_n|^2 \leq (2 - \eta)e^{|z|^2}.$$

Since the g_i are holomorphic, by the same reasoning, we get an upper bound on η independent of the f_i's.

Chapter 8

Further remarks

Let (p_k, q_k) be polynomials of degree smaller than k in one complex variable z. Set

$$\rho_k(z) = \frac{|p_k|^2(z) + |q_k|^2(z)}{(1+|z|^2)^{2k}}.$$

That is, ρ_k is the sum of the squares of the norms of p_k and q_k, seen as sections of $\mathcal{O}(k)$. If these polynomials satisfy the conditions in Proposition 1 (using inhomogeneous coordinates in $\mathbb{CP}^1$), the function ρ_k is bounded above by 1 and below by η. We have seen that the function $v_k = -\log(\rho_k)$ satisfies

$$\triangle v_k = k - K_k e^{2v_k},$$

where K_k is a positive function vanishing at the branch points of p_k/q_k and completely determined by those branch points. The PDE,

$$\triangle v = c - K e^{2v},$$

has been extensively studied when $c = 1$ (see [KW] and [CY]). It is the equation for prescribing curvature on S^2. There are variational methods adapted to this equation when c is a small constant but these do not apply to our case. Proposition 1 implies:

Proposition 11 *There is a constant C, such that for every k big enough, there is a set of $2k-2$ points such that the equation*

$$\triangle v_k = k - K_k e^{2v_k}$$

has a solution v_k with $||v_k||_\infty \leq C$.

It would be interesting to see if one could recover this result from a PDE theory point of view. This would give an alternative proof of Donaldson's main theorem in [Do3] for S^2 and maybe a way to calculate η, by calculating C.

Consider the following functionals on the space of $\mathcal{C}^\infty$ maps from S^2 to S^2:

$$E_1(F) = \int \left| \frac{F^*\omega}{\omega} \right|^2,$$

and

$$E_2(F) = \int \left| \frac{F^*\omega}{\omega} \right|^4.$$

It is not hard to see that, the restriction of these two functionals to the space of rational maps of degree k from S^2 to itself, is proper and therefore has a minimum. We conjecture that the minimizers of such functionals arise as quotients of polynomials satisfying the property in Proposition 1. Similarly, if one considers the much studied functional,

$$\int |dF|^2 + \epsilon \int |dF|^4,$$

restricted to the space of rational maps of degree k, it is natural to conjecture that its minimum also arises in this way.

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