Homological mirror symmetry

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Outline

Gromov-Witten invariants

Mirror symmetry 1.0 – closed string

Mirror symmetry 2.0 – open string, or ‘Homological’

Calabi-Yau hypersurfaces in projective space

Holomorphic curves

- Let (M, ω) be a **Kähler manifold**: a complex manifold with a compatible symplectic form ω.
- Given a Riemann surface Σ, we consider the moduli space of **holomorphic curves**:

$$\{u : \Sigma \to M \text{ holomorphic}\} \, / \sim,$$

  where $u_1 \sim u_2$ if they are related by a reparametrization.
- Gromov realized (1985) that holomorphic curves come in compact, finite-dimensional families.

Counting curves

The **Gromov-Witten invariants** are counts of the zero-dimensional part of such a moduli space. E.g.:

- Number of degree-1 curves (lines) $u : \mathbb{CP}^1 \to \mathbb{CP}^n$, passing through two generic points: 1.
- Number of degree-2 curves (conics) $u : \mathbb{CP}^1 \to \mathbb{CP}^2$, passing through five generic points: 1.
- Number of lines on a cubic surface: 27.

Curve-counting on the quintic three-fold

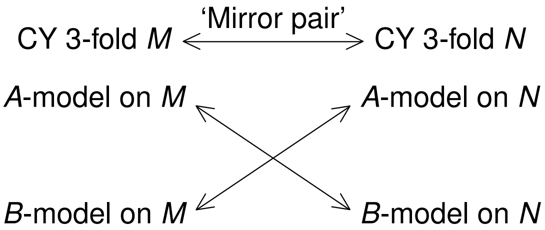
- Number of lines on a quintic three-fold: 2875.
- Number of conics: 609250.
- Number of cubics: 317206375.
- In 1991, the number of degree-d rational curves on the quintic three-fold was unknown, for $d \geq 4$.

A and *B* models

- Physics: study string theory on a **Calabi-Yau** Kähler manifold (M, ω, Ω).
- Calabi-Yau means there is a holomorphic volume form $\Omega \in \Omega^{n,0}(M)$.
- There are two models for closed-string theory on (M, ω, Ω):
  - The '*A*-model' = Gromov-Witten invariants (depend on symplectic structure (M, ω));
  - The '*B*-model' = periods of Ω (depend on complex structure (M, Ω)).

Mirror symmetry 1.0

Physicists noticed (mid-80s) that there are many pairs of Calabi-Yau three-folds on which *A*- and *B*-models are exchanged:



Application to the quintic three-fold

In 1991, string theorists Candelas, de la Ossa, Green and Parkes used mirror symmetry to predict curve counts on the quintic three-fold M:

- They constructed a mirror N to M;
- The A-model (Gromov-Witten invariants) on M should correspond to the B-model on N;
- They explicitly computed the B-model on N (periods of the holomorphic volume form).

The results

- This gave a prediction for the number of degree-d curves on the quintic three-fold for **any** d.
- Their predictions agreed with the known results for $d = 1, 2, 3$.
- In 1996, Givental and Lian-Liu-Yau proved this version of mirror symmetry for all Calabi-Yau (and Fano) complete intersections in toric varieties, using equivariant localization.

Cohomological Field Theory (Kontsevich-Manin)

- The A-model on M should be a **Cohomological Field Theory**: an R-module $\mathcal{C} = H^*(M)$ together with maps

$$I_{g,n} : \mathcal{C}^{\otimes n} \otimes H^*(\overline{\mathcal{M}}_{g,n}) \to R,$$

  where $\overline{\mathcal{M}}_{g,n}$ denotes the Deligne-Mumford compactification of the moduli space of genus-g, n-pointed closed Riemann surfaces.
- These must be compatible with the inclusions of boundary strata

$$\mathcal{M}_{g_1,n_1+1} \times \mathcal{M}_{g_2,n_2+1} \to \mathcal{M}_{g_1+g_2,n_1+n_2}$$

  (+ other axioms).

CohFT from Gromov-Witten invariants

- To define Gromov-Witten invariants of M, consider moduli spaces of stable holomorphic maps

$$\overline{\mathcal{M}}_{g,n,\beta}(M) := \{u : \Sigma_g \to X \text{ holomorphic}, u_*[\Sigma_g] = \beta\}/\sim .$$

- There are maps

$$\overline{\mathcal{M}}_{g,n} \leftarrow \overline{\mathcal{M}}_{g,n,\beta}(M) \to M^n; \text{ hence}$$

$$I_{g,n,\beta} : H^*(M)^{\otimes n} \otimes H^*(\overline{\mathcal{M}}_{g,n}) \to \mathbb{C};$$

then the maps

$$I_{g,n} := \sum_{\beta \in H_2(M)} r^{\omega(\beta)} I_{g,n,\beta}$$

form a $\mathbb{C}[\![r]\!]$-linear Cohomological Field Theory.

Open-closed TCFT (cf. Costello)

- For an R-linear open-closed TCFT, one needs:
  - An R-module $\mathcal{C}$ ('closed-string states');
  - A set of objects L;
  - For each pair of objects L_0, L_1, an R-module $\mathcal{O}(L_0, L_1)$ ('open string states');
- One defines algebraic operations like for a CohFT, where now your Riemann surfaces have boundary, and both internal and boundary marked points:
  - Internal marked points are labelled by $\mathcal{C}$;
  - Boundary components are labelled by objects L_i;
  - Boundary marked points are labelled by $\mathcal{O}(L_{left}, L_{right})$.

A_∞ categories

If we only look at the part of the open-closed TCFT corresponding to disks with no internal marked points, we get an A_∞ **category**:

- There are maps

$$\mu^s : \mathcal{O}(L_0, L_1) \otimes \ldots \otimes \mathcal{O}(L_{s-1}, L_s) \to \mathcal{O}(L_0, L_s);$$

- These maps μ^s satisfy the A_∞ **relations**:

$$\sum_{i,j} \mu^{s+1-j}(p_1, \ldots, \mu^j(p_i, \ldots, p_{i+j}), \ldots, p_s) = 0$$

for all $s \geq 1$.

What the A_∞ relations mean

- When $s = 1$, this means

$$\mu^1 : \mathcal{O}(L_0, L_1) \to \mathcal{O}(L_0, L_1)$$

  is a differential.
- When $s = 2$, this means

$$\mu^2 : \mathcal{O}(L_0, L_1) \otimes \mathcal{O}(L_1, L_2) \to \mathcal{O}(L_0, L_2)$$

  satisfies the Leibniz rule (hence descends to a product on the cohomology of μ^1).
- When $s = 3$, this means the product μ^2 is associative.
- This (+ identity morphisms) means we can define an honest category, with morphism spaces

$$\mathrm{Hom}(L_0, L_1) := H^*(\mathcal{O}(L_0, L_1), \mu^1)$$

The Fukaya category $\mathcal{F}(M)$

- A submanifold $L \subset M$ is called **Lagrangian** if $\omega|_L = 0$, and $\dim(L) = \dim(M)/2$.
- Objects of $\mathcal{F}(M)$ are Lagrangian submanifolds of M.
- Morphism spaces are generated by intersection points:

  $$\mathcal{O}(L_0, L_1) := R\langle L_0 \cap L_1 \rangle$$

  (where R is the algebraic closure of $\mathbb{C}[\![r, r^{-1}]\!]$).
- The A_∞ structure maps μ^s are defined by counting holomorphic disks

  $$u : \mathbb{D} \to M,$$

  with boundary conditions on the Lagrangians $L_0, \ldots, L_s$, weighted by $r^{\omega(u)} \in R$.

From open strings to closed strings

- The Fukaya category 'should' fit into an open-closed TCFT with the Gromov-Witten invariants.
- Kontsevich conjectured that (in good cases) the whole TCFT structure of the Gromov-Witten invariants can be reconstructed from the Fukaya category by taking **Hochschild cohomology**:
  - There ought to be a natural TCFT structure on the Hochschild cohomology of an A_∞ category (Deligne conjecture);
  - There should be an isomorphism

$$H^*(M) \cong HH^*(\mathcal{F}(M))$$

    as R-linear TCFT's.

Homological Mirror Symmetry

- In 1994, Kontsevich introduced a ‘categorified’ version of the mirror symmetry conjecture.
- The A-model should be the **Fukaya category** $\mathcal{F}(M)$, a symplectic invariant.
- The B-model should be (a DG enhancement of) the **category of coherent sheaves** $Coh(M)$, an algebraic invariant.
- Note: the Hochschild-Kostant-Rosenberg isomorphism relates $HH^*(Coh(M))$ to the closed-string B-model.

What HMS means

So, Calabi-Yau Kähler manifolds M and N should be mirror if there are quasi-equivalences of (derived) A_∞ categories:

$$\begin{array}{ccc} D^b\mathcal{F}(M) & & D^b\mathcal{F}(N) \\ & \nwarrow \nearrow & \\ & \swarrow \searrow & \\ D^b Coh(M) & & D^b Coh(N) \end{array}$$

Mirror Symmetry 2.0 should imply Mirror Symmetry 1.0 by taking Hochschild cohomology, but be much stronger!

The A-model

- Let $M^n \subset \mathbb{CP}^{n-1}$ be a smooth hypersurface of degree n. We will think of

$$M^n = \left\{ \sum_{j=1}^{n} z_j^n = 0 \right\} \subset \mathbb{CP}^{n-1}.$$

- M^3 is an elliptic curve, M^4 is the quartic $K3$ surface, and M^5 is the quintic threefold.
- The A-model is the Fukaya category, $\mathcal{F}(M^n)$, which is an R-linear A_∞ category.

The *B*-model

- Define
$$\widetilde{N}^n := \left\{ u_1 \dots u_n + r \sum_j u_j^n = 0 \right\} \subset \mathbb{P}_R^{n-1}.$$
- $G_n \cong (\mathbb{Z}_n)^{n-2}$ acts on $\widetilde{N}^n$ (multiplying coordinates by nth roots of unity), and we define $N^n := \widetilde{N}^n / G_n$.
- Consider the category of coherent sheaves on N^n:
$$Coh(N^n) \cong Coh^{G_n}\left(\widetilde{N}^n\right).$$

Main result

Theorem (S.)

There is a quasi-equivalence of R-linear triangulated A_∞ categories

$$D^\pi \mathcal{F}(M^n) \cong \Psi \cdot D^b Coh(N^n),$$

where Ψ is an automorphism (the 'mirror map')

$$\begin{aligned} \Psi : R &\to R, \text{ sending} \\ r &\mapsto \psi(r) r, \end{aligned}$$

where $\psi(r) \in \mathbb{C}[\![r]\!]$ satisfies $\psi(0) = 1$. We are not yet able to determine the higher-order terms in $\psi(r)$.

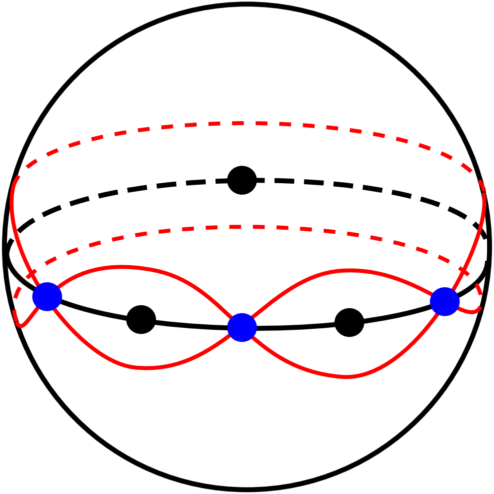
The Lagrangians

We consider the branched cover

$$\begin{array}{rcl} M^n \cong \left\{ \sum_j z_j^n = 0 \right\} & \to & \left\{ \sum_j z_j = 0 \right\} \cong \mathbb{CP}^{n-2} \\ [z_1 : \ldots : z_n] & \mapsto & [z_1^n : \ldots : z_n^n], \end{array}$$

branched along the divisors $D_j = \{z_j = 0\}$. We construct a single Lagrangian $L \subset \mathbb{CP}^{n-2} \setminus \cup D_j$ (the 'pair-of-pants'), and look at all of its lifts to M^n.

The one-dimensional case



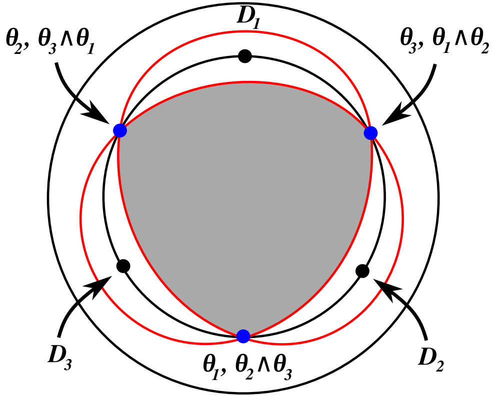
Computing $\mathcal{O}(L, L)$

- $\mathcal{O}(L, L) \cong \Lambda^* R^n$ as an R-vector space.
- $\mu^1 = 0$, μ^2 = wedge product.
- It has higher (A_∞) corrections, which correspond to terms

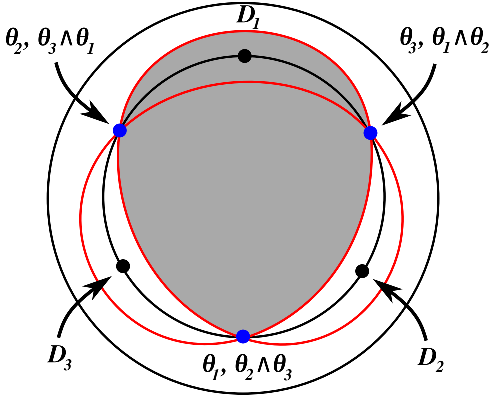
$$\begin{aligned} u_1 \ldots u_n + r \sum_j u_j^n &\in R[\![u_1, \ldots, u_n]\!] \otimes \Lambda^* R^n \\ &\cong HH^*(\Lambda^* R^n) \text{ (HKR isomorphism).} \end{aligned}$$

- They correspond to the defining equation of the mirror N^n.

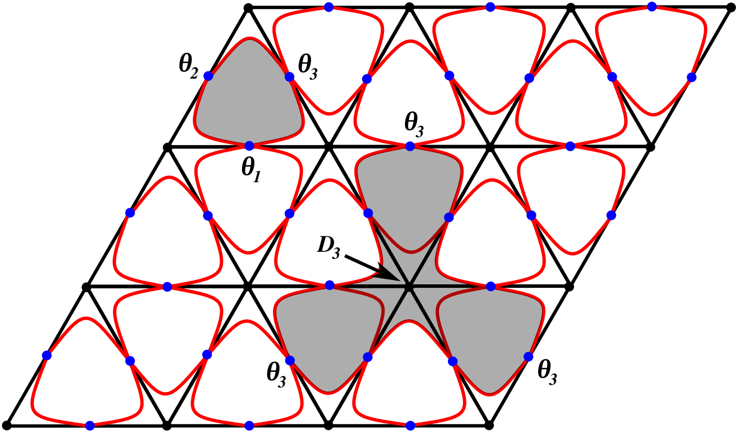
Holomorphic disks giving the exterior algebra



Holomorphic disks giving the higher-order terms



Lifts to N^3 = elliptic curve



Split-generation

- As part of the open-closed TCFT structure, we get an algebra homomorphism

$$\Phi : H^*(M) \to HH^*(\mathcal{F}(M)).$$

- By work of Abouzaid-Fukaya-Oh-Ohta-Ono, if the restriction of this map to the Hochschild cohomology of some subcategory does not kill the top-degree class, then the subcategory split-generates $\mathcal{F}(M)$.
- We apply this to show that the lifts of our Lagrangian split-generate the Fukaya category; so we have 'computed' the Fukaya category, in a sense.

The coherent sheaves

- We consider the restrictions of the Beilinson exceptional collection $\Omega^j(j)$ $(j = 0, \ldots, n-1)$ to $\widetilde{N}^n$.
- There are $|G_n^*| = n^{n-2}$ ways of making each one into a G_n-equivariant coherent sheaf.
- These G_n-equivariant coherent sheaves on $\widetilde{N}^n$ are mirror to the lifts of the Lagrangian L to M^n.
- We can show that their morphisms and compositions agree, and they generate their respective categories.