Instituto Superior Técnico
Departamento de Matemática
Secção de Álgebra e Análise

Riemannian Geometry

Homework 9

due on Wednesday, May 17

1. Recall from Homework 7 that the catenoid is the surface of revolution given by

$$C = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = \cosh^2(z)\},$$

and that the expression of the Riemannian metric induced on C by the Euclidean metric of $\mathbb{R}^3$ is

$$g = \cosh^2(z)(d\theta \otimes d\theta + dz \otimes dz),$$

where (θ, z) are the local coordinates in C obtained from the usual cylindrical coordinates $(r, \theta, z) \in \mathbb{R}^3$.

(a) Compute the Christoffel symbols for the Levi-Civita connection in these coordinates.

(b) Show that the geodesic equations in these coordinates are given by

$$\ddot{\theta} + 2\tanh(z)\,\dot{\theta}\,\dot{z} = 0$$
$$\ddot{z} + \tanh(z)(\dot{z}^2 - \dot{\theta}^2) = 0.$$

(c) Show that the circle defined by the equations $x^2 + y^2 = 1$ and $z = 0$ is the image of a geodesic.

(d) Let $c : \mathbb{R} \to C$ be the curve given by $c(t) = (\cosh z_0 \cos t, \cosh z_0 \sin t, z_0)$ where $z_0 \in \mathbb{R}$.

(i) Show that if $z_0 \neq 0$ then c is not a geodesic.

(ii) Let V be a vector field parallel along c such that $V(0) = \frac{\partial}{\partial\theta}$. Compute the angle by which V is rotated when it returns to the initial point.