Instituto Superior Técnico
Departamento de Matemática
Secção de Álgebra e Análise

Riemannian Geometry

Homework 8

due on Wednesday, May 10

1. *Divergence and Stokes Theorems in* $\mathbb{R}^3$.
   Let $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be a smooth vector field in $\mathbb{R}^3$ and define

$$\omega_{\mathbf{F}} := F^1 dx + F^2 dy + F^3 dz \quad \text{and} \quad \Omega_{\mathbf{F}} := F^1 dy \wedge dz + F^2 dz \wedge dx + F^3 dx \wedge dy.$$

   (a) Show that if $X, Y \in \mathcal{X}(\mathbb{R}^3)$ then $\omega_{\mathbf{F}}(X) = \langle \mathbf{F}, X \rangle$ and $\Omega_{\mathbf{F}}(X, Y) = \langle \mathbf{F}, X \times Y \rangle$, where $\langle \cdot, \cdot \rangle$ and $\times$ are the usual inner and cross products in $\mathbb{R}^3$.

   (b) Show that $d\omega_{\mathbf{F}} = \Omega_{\mathrm{rot}\,\mathbf{F}}$ and $d\Omega_{\mathbf{F}} = (\mathrm{div}\,\mathbf{F})\, dx \wedge dy \wedge dz$.

   (c) Show that if L is a smooth 1-manifold in $\mathbb{R}^3$ and $\mathbf{g}$ is a positive parameterization of L then

$$\int_L \omega_{\mathbf{F}} = \int_L \langle \mathbf{F}, d\mathbf{g} \rangle$$

   is the line integral of $\mathbf{F}$ along L.

   (d) Show that if S is a smooth 2-manifold in $\mathbb{R}^3$ and $\mathbf{g}$ is a positive parameterization of S then

$$\int_S \Omega_{\mathbf{F}} = \int_S \langle \mathbf{F}, \mathbf{n} \rangle$$

   is the flow of $\mathbf{F}$ through S, where $\mathbf{n}$ denotes a normal vector to S.

   (e) From the general Stokes Theorem deduce the Divergence Theorem and Stokes Theorems in $\mathbb{R}^3$ learned in Calculus courses.

2. Let G be a Lie group with Lie algebra $\mathfrak{g}$.

   (a) Let $\beta : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ be a bilinear map. Prove there exists a unique connection $\nabla = \nabla^\beta$ on G which satisfies the following condition: if $v, w \in \mathfrak{g}$ and X^v, X^w denote the corresponding left-invariant vector fields then

$$\nabla_{X^v} X^w = X^{\beta(v,w)}.$$

   Hint: Consider a basis $\{X^{v_1}, \ldots, X^{v_n}\}$ *of* $\mathcal{X}(G)$ *and compute* $\nabla_X Y$ *in this basis.*

   (b) Prove that this connection is left-invariant in the sense that

$$(L_g)_* \nabla_X Y = \nabla_{(L_g)_* X} (L_g)_* Y, \quad \forall X, Y \in \mathcal{X}(G), g \in G.$$

   Deduce that the parallel transport determined by this connection is left invariant in the sense that if Y is parallel along a curve c then $(L_g)_* Y$ is parallel along $L_g \circ c$.

   (c) Prove that any connection ∇ on G determines a bilinear map β on $\mathfrak{g}$ via

$$\beta(v, w) = (\nabla_{X^v} X^w)_e.$$

   Hence, there is a bijective correspondence between bilinear maps from $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ and left invariant connections on G.