Instituto Superior Técnico
Departamento de Matemática
Secção de Álgebra e Análise

Riemannian Geometry

Homework 7

due on Wednesday, May 3

1. Consider the 2-torus T^2 in the Euclidean space $\mathbb{R}^3$ given by the equation
$$\left(\sqrt{x^2+y^2}-R\right)^2+z^2=r^2,$$
and the following parameterization of a neighborhood of T^2 in $\mathbb{R}^3$ given by
$$p(\rho,\theta,\varphi)=((R+\rho\cos\varphi)\cos\theta,(R+\rho\cos\varphi)\sin\theta,\rho\sin\varphi),$$
where $r-\epsilon<\rho<r+\epsilon$ for some $0<\epsilon<r$, $0<\theta<2\pi$ and $0<\varphi<2\pi$.
   (a) Compute $\omega:=p^*(dx\wedge dy\wedge dz)$, where x,y,z are the Cartesian coordinates in $\mathbb{R}^3$.
   (b) Compute $\eta:=i_X\omega$, where $X=\frac{\partial}{\partial\rho}$ (that is, the interior product of ω by $\frac{\partial}{\partial\rho}$).
   (c) Compute $\int_{p^{-1}(T^2)}\eta$.

2. Let M be a compact oriented manifold with volume form $\omega\in\Omega^n(M)$. Prove that if $f>0$ then $\int_M f\omega>0$. (*Remark: in particular, the volume of a compact manifold is always positive*).

3. Let M be a compact, connected and oriented 2-manifold with boundary, with ∂M diffemorphic to S^1. Let $\omega\in\Omega^1(M)$ be a 1-form whose restriction to ∂M is the 1-form $\sigma=\cos\theta d\theta$, where θ is the usual angular coordinate in S^1. Show that $d\omega$ has at least one zero in $M\setminus\partial M$.

4. Let (M,g) be a Riemannian manifold and $f\in C^\infty(M)$. Show that if $a\in\mathbb{R}$ is a regular value of f then $\operatorname{grad}(f)$ is orthogonal to the submanifold $f^{-1}(a)$.

5. The **catenoid** is the surface of revolution given by
$$C=\{(x,y,z)\in\mathbb{R}^3:x^2+y^2=\cosh^2(z)\}.$$
The **helicoid** is the surface H parameterized by the map $\phi:\mathbb{R}^2\to\mathbb{R}^3$ given by
$$\phi(u,v)=(u\cos v,u\operatorname{sen}v,v).$$
Consider the metrics g and h induced in C and H by the Euclidean metric on $\mathbb{R}^3$.
   (a) Show that
$$g=\cosh^2(z)(d\theta\otimes d\theta+dz\otimes dz),$$
where (θ,z) are the local coordinates in C obtained from the usual cylindrical coordinates $(r,\theta,z)\in\mathbb{R}^3$.
   (b) Show that
$$h=du\otimes du+(1+u^2)\,dv\otimes dv.$$
   (c) Show that C and H are **locally isometric**, that is, show that there exists a local diffeomorphism $\psi:H\to C$ such that $h=\psi^*g$. (*A video of the isometric deformation from catenoid to helicoid is available here: Helicoid-Catenoid - Virtual Math Museum*).