Instituto Superior Técnico
Departamento de Matemática
Secção de Álgebra e Análise

Riemannian Geometry

Homework 6

due on Wednesday, April 12

1. Let M be a smooth manifold. Show that if $\omega \in \Omega^1(M)$ and $X, Y \in \mathcal{X}(M)$ then
$$d\omega(X,Y) = X \cdot (\omega(Y)) - Y \cdot (\omega(X)) - \omega([X,Y]).$$

2. Consider the following vector field defined on $\mathbb{R}^2 \backslash \{(0,0)\}$,
$$X = -\frac{y}{x^2+y^2}\frac{\partial}{\partial x} + \frac{x}{x^2+y^2}\frac{\partial}{\partial y}$$
and $\omega \in \Omega(\mathbb{R}^2)$ given by $\omega = dx \wedge dy$.
   (a) Compute the Lie derivative of ω along X, $L_X\omega$, using
   $$L_X(\omega_1 \wedge \omega_2) = (L_X\omega_1) \wedge \omega_2 + \omega_1 \wedge (L_X\omega_2)$$
   and
   $$d(L_X\omega) = L_X(d\omega).$$
   (b) Compute $L_X\omega$ using Cartan's formula (*see exercise 3.8(7)*).
   (c) Calculate X and ω in polar coordinates.
   (d) Compute $L_X\omega$ using polar coordinates and $L_X\omega = \frac{d}{dt}(\psi_t^*\omega)_{|t=0}$, where ψ_t is the flow of X.

3. A k-form ω is called **closed** if $d\omega = 0$. If it exists a $(k-1)$-form β such that $\omega = d\beta$ then ω is called **exact**. Note that every exact form is closed. Let Z^k be the set of all closed k-forms on M and define a relation between forms on Z^k as follows: $\alpha \sim \beta$ if and only if they differ by an exact form, that is, if $\beta - \alpha = d\theta$ for some $(k-1)$-form θ.
   (a) Show that this relation is an equivalence relation.
   (b) Let $H^k(M)$ be the corresponding set of equivalence classes (called the k-dimensional **de Rham cohomology space** of M). Show that addition and scalar multiplication of forms define indeed a vector space structure on $H^k(M)$.

4. Consider the 2-form
$$\omega = \frac{x\, dy \wedge dz - y\, dx \wedge dz + z\, dx \wedge dy}{(x^2+y^2+z^2)^{3/2}}$$
defined on $\mathbb{R}^3 \backslash \{(0,0,0)\}$.
   (a) Show that ω is closed.
   (b) Calculate $\int_{S^2} \omega$. Is ω exact?
   (c) Using the previous questions what can you say about $H^2(\mathbb{R}^3 \backslash \{(0,0,0)\})$?