Instituto Superior Técnico
Departamento de Matemática
Secção de Álgebra e Análise

Riemannian Geometry

Homework 5

due on Friday, March 31

1. Let G_1 and G_2 be Lie groups and let $f : G_1 \to G_2$ be a homomorphism of Lie groups. Show that given a left-invariant vector field X on G_1 there exists a unique left-invariant vector field Y on G_2 such that $Y = f_*X$.

2. Let M be a smooth n-manifold and let $\omega \in \Omega^n(M)$ be a differential form satisfying $\omega_p \neq 0 \ \forall p \in M$ (ω is called a **volume form**). Given a vector field $X \in \mathcal{X}(M)$ the **interior product** of ω by X is defined by

$$(i_X\omega)_p(v_2, \ldots, v_n) = \omega_p(X_p, v_2, \ldots, v_n)$$

   where $v_i \in T_pM$ com $i = 2, \ldots, n$.

   Show that the map

$$\begin{aligned} \varphi : \mathcal{X}(M) &\longrightarrow \quad \Omega^{n-1}(M) \\ X &\longmapsto i_X\omega \end{aligned}$$

   is an isomorphism of vector spaces, that is, φ is a bijective linear map.

3. Let M be a smooth manifold. Consider $\omega \in \Omega^2(M)$ and $\eta \in \Omega^1(M)$. Show that

$$(\omega \wedge \eta)(X, Y, Z) = \omega(X, Y)\eta(Z) + \omega(Y, Z)\eta(X) + \omega(Z, X)\eta(Y)$$

   where X, Y, Z are vector fields on M.