Instituto Superior Técnico
Departamento de Matemática
Secção de Álgebra e Análise

Riemannian Geometry

Homework 4

due on Wednesday, March 22

1. Consider the vector fields defined on $\mathbb{R}^2$ by

$$X(x,y) = x\frac{\partial}{\partial y}, \quad Y(x,y) = y\frac{\partial}{\partial x}.$$

   (a) Compute the flow of X, $\phi_t(x_0, y_0)$, and the flow of Y, $\psi_t(x_0, y_0)$.

   (b) Compute $(d\phi_{-t})_{\phi_t(x_0,y_0)} Y_{\phi_t(x_0,y_0)}$.

   (c) Compute the Lie derivative $L_X Y$ using (b), and confirm your result computing $[X, Y]$.

2. Consider the set of *symplectic matrices*

$$\mathrm{Sp}(2n) = \left\{ A \in \mathcal{M}_{2n\times 2n}(\mathbb{R}) \mid A^t J A = J \right\},$$

where

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

and I is the $n \times n$ identity matrix.

   (a) Show that $\mathrm{Sp}(2n)$ is a Lie group of dimension $n(2n+1)$.

   (b) Determine its Lie algebra, $\mathfrak{sp}(2n)$.

   (c) Compute the tangent space to $\mathrm{Sp}(2n)$ at J, $T_J \mathrm{Sp}(2n)$.

   (d) Determine a matrix $B \in \mathfrak{sp}(2n)$ such that $e^B = J$. (Note that $J^2 = -I$)