Instituto Superior Técnico
Departamento de Matemática
Secção de Álgebra e Análise

Riemannian Geometry

Homework 3

due on Wednesday, March 15

1. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a differentiable function which is homogeneous of degree $k \neq 0$ (that is, $f(tx) = t^k f(x)$ for all $t > 0, x \in \mathbb{R}^n$).
   (a) Show that if $a \neq 0$ and $f^{-1}(a) \neq \emptyset$ then $f^{-1}(a)$ is a submanifold of dimension $n-1$.
   (b) Show that if $ab > 0$ then $f^{-1}(a)$ is diffeomorphic to $f^{-1}(b)$.

2. Consider the following vector fields in $\mathbb{R}^3$

$$X = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}, \qquad Y = z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}, \qquad Z = \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}.$$

   (a) Compute $[X, Y]$ e $[X, Z]$.
   (b) Compute the flow of the vector field $X + Y$.
   (c) Do the flows of Y and Z commute? Justify your answer.
   (d) Are there coordinates (x^1, x^2, x^3) in $\mathbb{R}^3$ such that $Y = \frac{\partial}{\partial x^2}$ and $Z = \frac{\partial}{\partial x^3}$? Justify your answer.