Instituto Superior Técnico
Departamento de Matemática
Secção de Álgebra e Análise

Riemannian Geometry

Homework 2

due on Wednesday, March 8

1. Consider the 2-dimensional differentiable manifold $\mathbb{RP}^2$, the set of lines through the origin in $\mathbb{R}^3$, parameterized by

$$\phi_1(y,z) = [1,y,z], \quad \phi_2(x,z) = [x,1,z], \quad \phi_3(x,y) = [x,y,1],$$

and the map $f : \mathbb{RP}^2 \to \mathbb{RP}^2$, defined by

$$f([x,y,z]) = [x,z,y].$$

Show that f is differentiable.

2. Let (x,y,z) be the Cartesian coordinates in $\mathbb{R}^3$ and consider the usual spherical coordinates (r,θ,φ) defined by

$$\begin{cases} x = r\sin\theta\cos\varphi \\ y = r\sin\theta\,\mathrm{sen}\,\varphi \\ z = r\cos\theta \end{cases}$$

Compute:

(a) The components of the tangent vectors to $\mathbb{R}^3$ $\frac{\partial}{\partial r}, \frac{\partial}{\partial \theta}, \frac{\partial}{\partial \varphi}$ in Cartesian coordinates.

(b) The components of the tangent vectors to $\mathbb{R}^3$ $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$ in spherical coordinates.

3. Let $\Phi : \mathbb{RP}^2 \to \mathbb{R}^3$ be the map defined by

$$\Phi([x,y,z]) = \frac{1}{x^2+y^2+z^2}(yz, xz, xy).$$

Show that Φ is smooth and show that it only fails to be an immersion at 6 points.