Instituto Superior Técnico
Departamento de Matemática
Secção de Álgebra e Análise

Riemannian Geometry

Homework 12

due on Wednesday, June 7

1. *(Torus of zero curvature)* Consider the map $f : (0, 2\pi) \times (0, 2\pi) \to \mathbb{R}^4$ given by
$$f(\theta, \varphi) = (\cos\theta, \sin\theta, \cos\varphi, \sin\varphi)$$
which parameterizes the 2-torus
$$T = \{(x, y, z, w) \in \mathbb{R}^4 \, : \, x^2 + y^2 = 1; \, z^2 + w^2 = 1\}.$$
   (a) Show that, in the coordinates (θ, φ), the metric h on T induced by the Euclidean metric on $\mathbb{R}^4$ is given by
   $$h = d\theta \otimes d\theta + d\varphi \otimes d\varphi.$$
   (b) Compute the connection forms associated to the orthonormal frame
   $$E_1 = \frac{\partial}{\partial\theta}, \quad E_2 = \frac{\partial}{\partial\varphi}.$$
   (c) Show that the Gauss curvature of the metric h is zero.

2. Consider the 2-torus T^2 in the Euclidean space $\mathbb{R}^3$ given by
$$T^2 = \left\{(x, y, z) \in \mathbb{R}^3 : z^2 + (\sqrt{x^2 + y^2} - 2)^2 = 1\right\}.$$
and the parameterization of T^2 given by the map $\phi : (0, 2\pi) \times (0, 2\pi) \to \mathbb{R}^3$ defined by
$$\phi(\theta, \varphi) = ((2 + \operatorname{sen}\theta)\cos\varphi, (2 + \operatorname{sen}\theta)\operatorname{sen}\varphi, \cos\theta).$$
   (a) Show that in the coordinates (θ, φ) the metric on T^2 induced by the Euclidean metric on $\mathbb{R}^3$ is given by
   $$g = d\theta \otimes d\theta + (2 + \operatorname{sen}\theta)^2 d\varphi \otimes d\varphi.$$
   (b) Compute the connection forms associated to the orthonormal frame
   $$E_1 = \frac{\partial}{\partial\theta}, \quad E_2 = \frac{1}{2 + \operatorname{sen}\theta}\frac{\partial}{\partial\varphi}.$$
   (c) Show that the Gauss curvature of T^2 is given by
   $$K = \frac{\operatorname{sen}\theta}{2 + \operatorname{sen}\theta}.$$
   (d) Without doing calculations indicate the value of the integral
   $$\int_T \frac{\operatorname{sen}\theta}{2 + \operatorname{sen}\theta}.$$
   Justify your answer.