Instituto Superior Técnico
Departamento de Matemática
Secção de Álgebra e Análise

Riemannian Geometry

Homework 10

due on Wednesday, May 24

1. Recall from Homework 7 that the helicoid is the surface H parameterized by the map $\phi : \mathbb{R}^2 \to \mathbb{R}^3$ given by
$$\phi(u, v) = (u \cos v, u \operatorname{sen} v, v).$$
Consider the metric h induced on H by the Euclidean metric of $\mathbb{R}^3$.
   (a) Show that H is geodesically complete.
   Hint: Consider the diffeomorphism $F : \mathbb{R}^3 \to \mathbb{R}^3$ defined by
$$F(x, y, z) = (x \cos z - y \sin z, x \sin z + y \cos z, z)$$
   and note that $\phi(u, v) = F(u, 0, v)$.
   (b) Determine, without calculating the geodesic equations in local coordinates, two images of distinct geodesics of H passing through $(0, 0, 0)$.

2. Let M be a differentiable manifold of dimension n, ∇ the Levi-Civita connection associated to the metric $\langle \cdot, \cdot \rangle$ and $\widetilde{\nabla}$ the Levi-Civita connection associated to the metric $\langle\langle \cdot, \cdot \rangle\rangle = e^{2\rho} \langle \cdot, \cdot \rangle$, where ρ is a smooth function. (The metrics $\langle \cdot, \cdot \rangle$ and $\langle\langle \cdot, \cdot \rangle\rangle$ are said to be **conformal metrics**).
   (a) Show that
$$\widetilde{\nabla}_X Y = \nabla_X Y + d\rho(X)Y + d\rho(Y)X - \langle X, Y \rangle \operatorname{grad} \rho.$$
   Recall that $\operatorname{grad} \rho$ is the vector field satisfying $\langle \operatorname{grad} \rho, X \rangle = d\rho(X)$, for all $X \in \chi(M)$. *Hint: Use the Koszul formula.*
   (b) The **Laplacian** of $f \in C^\infty(M)$ is defined locally by
$$\Delta f = \sum_{i=1}^{n} dx^i \left(\nabla_{\frac{\partial}{\partial x^i}} \operatorname{grad} f \right).$$
   Show that if $n = 2$ then $\widetilde{\Delta} f = e^{-2\rho} \Delta f$.