Instituto Superior Técnico
Departamento de Matemática
Secção de Álgebra e Análise

Riemannian Geometry

Homework 1

due on Wednesday, March 1

1. It can be shown that any compact connected 2-dimensional topological manifold which is not homeomorphic to S^2 is either homeomorphic to a connected sum of copies of T^2 or to a connected sum of copies of $\mathbb{RP}^2$ (see [Blo96, Mun00]). Determine which manifold the connected sum of a torus T^2 and a Klein bottle K^2, $T^2 \# K^2$, is homeomorphic to. *Hint: see Exercise 1.8 (4) in [GN14].*

2. Consider the torus T^2 defined as the quotient of the unit square $Q = [0,1]^2 \subset \mathbb{R}^2$ by the equivalence relation

$$(x, y) \sim (x + 1, y) \sim (x, y + 1),$$

equipped with the quotient topology. Give a parameterization of a neighbourhood of $(0, 0)$.

3. Consider the paraboloid

$$P = \{(x, y, z) \in \mathbb{R}^3 : z = x^2 + y^2\}.$$

Show that the parameterizations defined by $\phi : \mathbb{R}^2 \to P$ and $\psi : (0, \infty) \times (0, 2\pi) \to P$ defined by

$$\phi(x, y) = (x, y, x^2 + y^2), \quad \psi(r, \theta) = (r \cos \theta, r \sin \theta, r^2),$$

are compatible.

References

[Blo96] E. Bloch, *A First Course in Geometric Topology and Differential Geometry*, (1996) Birkhäuser, Boston.

[GN14] L. Godinho and J. Natário. *An Introduction to Riemannian Geometry*, (2014) Springer.

[Mun00] J. Munkres. *Topology*, (2000) Prentice-Hall, Upper Saddle River.