Instituto Superior Técnico
Departamento de Matemática

Riemannian Geometry

Exam 1 – June 19, 2024

Abbreviated Solution

1. Consider the set
$$H = \{(x, y) \in \mathbb{R}^2 : y > 0\}$$
and identify each point in H with the invertible map $h : \mathbb{R} \to \mathbb{R}$ defined by $h(t) = yt + x$. The set containing these maps is a group, with the operation of composition of maps, so the previous indentification induces a group structure on H.

   (a) Show that the group operation induced on H is given by
   $$(x, y) \cdot (z, w) = (yz + x, yw),$$
   and that H equipped with this operation is a Lie group.

   (b) Show that the derivative of the left translation $L_{(x,y)} : H \to H$ at a point $(z, w) \in H$ is represented in the coordinates above by the matrix
   $$(dL_{(x,y)})_{(z,w)} = \begin{pmatrix} y & 0 \\ 0 & y \end{pmatrix}.$$
   Conclude that the left-invariant vector field determined by the vector
   $$V = \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y} \in \mathfrak{h} \equiv T_{(0,1)}H,$$
   where $\xi, \eta \in \mathbb{R}$, is the vector field
   $$X^V = \xi y \frac{\partial}{\partial x} + \eta y \frac{\partial}{\partial y} \in \chi(H).$$

   (c) Given $V, W \in \mathfrak{h}$, compute $[V, W]$.

   (d) Compute the flow of X^V and write an expression for the exponential map $\exp : \mathfrak{h} \to H$.

Solution:

(a) If (x, y) and (z, w) are identified with $h(t)$ and $g(t)$, respectively, then their composition is given by $(h \circ g)(t) = ywt + zy + x$, so we conclude that
$$(x, y) \cdot (z, w) = (zy + x, yw).$$
H is clearly a smooth manifold, therefore to prove that it is a Lie group we need to show that the operations of product and inverse are smooth. It is clear that the product is smooth, because it is given by polynomial functions. The identity element in H is $(0, 1)$ since $(x, y)\cdot(0, 1) = (0, 1)\cdot(x, y) = (x, y)$, and the inverse map in H is given by $(x, y) \mapsto (-\frac{x}{y}, \frac{1}{y})$, which is clearly smooth, since the denominator in these rational functions is positive. We conclude that H is a Lie group.

(b) Because $L_{(x,y)}(z,w) = (yz + x, yw)$ the matrix representation of $(dL_{(x,y)})_{(z,w)}$ is

$$(dL_{(x,y)})(z,w) = \begin{pmatrix} y & 0 \\ 0 & y \end{pmatrix}.$$

Therefore $X^V_{(x,y)}$ has components

$$\begin{pmatrix} y & 0 \\ 0 & y \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} y\xi \\ y\eta \end{pmatrix},$$

that is

$$X^V_{(x,y)} = (dL_{(x,y)})_{(0,1)} V = \xi y \frac{\partial}{\partial x} + \eta y \frac{\partial}{\partial y} \in \chi(H).$$

(c) If $V = \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}$ and $W = \xi' \frac{\partial}{\partial x} + \eta' \frac{\partial}{\partial y}$ then

$$\begin{aligned}[X^V, X^W] &= \left[\xi y \frac{\partial}{\partial x} + \eta y \frac{\partial}{\partial y}, \xi' y \frac{\partial}{\partial x} + \eta' y \frac{\partial}{\partial y}\right] \\ &= (\xi' \eta - \eta' \xi) y \frac{\partial}{\partial x}.\end{aligned}$$

Therefore

$$[V, W] = [X^V, X^W]_{(0,1)} = (\xi' \eta - \eta' \xi) \frac{\partial}{\partial x}.$$

(d) The flow ψ_t of X^V is given by the solution of the system of ODE's

$$\begin{cases} \dot{x} = \xi y \\ \dot{y} = \eta y \end{cases}$$

with initial condition $\psi_0(x,y) = (x,y)$. The solution is given by

$$\psi_t(x,y) = \left(\frac{\xi}{\eta} y (e^{\eta t} - 1) + x, y e^{\eta t}\right) \quad \text{if} \quad \eta \neq 0$$

and

$$\psi_t(x,y) = (\xi y t + x, y) \quad \text{if} \quad \eta = 0.$$

Since the exponential map is given by $\exp(V) = \psi_1(0,1)$ it follows that

$$\exp(V) = \left(\frac{\xi}{\eta}(e^{\eta} - 1), e^{\eta}\right) \quad \text{if} \quad \eta \neq 0$$

and

$$\exp(V) = (\xi, 1) \quad \text{if} \quad \eta = 0.$$

2. Let (M, g) be a Riemannian manifold defined by $M = \mathbb{R}^2$ and

$$g = \frac{1}{\sigma^2(x,y)}(dx \otimes dx + dy \otimes dy),$$

where $\sigma : \mathbb{R}^2 \to \mathbb{R}$ is a positive function.

(a) Show that the connection form ω_1^2 associated to the orthonormal frame $E_1 = \sigma\frac{\partial}{\partial x}$, $E_2 = \sigma\frac{\partial}{\partial y}$ is given by

$$\omega_1^2 = \frac{\sigma_y}{\sigma}dx - \frac{\sigma_x}{\sigma}dy,$$

where $\sigma_x = \frac{\partial \sigma}{\partial x}$ and $\sigma_y = \frac{\partial \sigma}{\partial y}$.

(b) Show that the Gauss curvature of (M, g) is given by

$$K = \sigma(\sigma_{xx} + \sigma_{yy}) - (\sigma_x^2 + \sigma_y^2).$$

(c) Consider $\sigma(x, y) = \cosh(x)$.

(i) Justify that in this case (M, g) is not geodesically complete.

(ii) Determine the vector field $V(t)$, parallel along the curve $c(t) = (x_0, t)$ (for some $x_0 > 0$), satisfying $V(0) = \frac{\partial}{\partial x}$.

(iii) Show that the distance between the points $(1, 0)$ and $(1, 1)$ is less or equal than $\frac{2}{e}$.

Solution:

(a) The coframe $\{\omega^1, \omega^2\}$ associated to this frame is given by $\omega^1 = \frac{1}{\sigma}\,dx$ and $\omega^2 = \frac{1}{\sigma}\,dy$. Moreover,

$$d\omega^1 = \frac{\sigma_y}{\sigma^2}\,dx \wedge dy = \sigma_y\,\omega^1 \wedge \omega^2 \quad \text{and} \quad d\omega^2 = -\frac{\sigma_x}{\sigma^2}\,dx \wedge dy = -\sigma_x\,\omega^1 \wedge \omega^2.$$

The first Cartan Structure equations,

$$d\omega^1 = \omega^2 \wedge \omega_2^1 \quad \text{and} \quad d\omega^2 = \omega^1 \wedge \omega_1^2,$$

imply that

$$d\omega^1(E_1, E_2) = -\omega_2^1(E_1) = \omega_1^2(E_1)$$

and

$$d\omega^2(E_1, E_2) = \omega_1^2(E_2).$$

Therefore the connection form ω_1^2 is given by

$$\omega_1^2 = d\omega^1(E_1, E_2)\omega^1 + d\omega^2(E_1, E_2)\omega^2 = \frac{\sigma_y}{\sigma}\,dx - \frac{\sigma_x}{\sigma}dy.$$

(b) From the previous question we have

$$d\omega_1^2 = \frac{\sigma_x^2 + \sigma_y^2 - \sigma(\sigma_{xx} + \sigma_{yy})}{\sigma^2}\,dx \wedge dy = (\sigma_x^2 + \sigma_y^2 - \sigma(\sigma_{xx} + \sigma_{yy}))\,\omega^1 \wedge \omega^2.$$

Since $d\omega_1^2 = \Omega_1^2 = -K\,\omega^1 \wedge \omega^2$ where K is the Gauss curvature, we conclude the desired result.

(c) (i) In this case we obtain $K = 1$. If (M, g) was geodesically complete, since M is simply connected then, by the Killing–Hopf Theorem, it would be isometric to the 2-sphere S^2, which is compact. Since M is not compact this is a contradiction.

(ii) Consider the vector field

$$V(t) = V^1(t)\frac{\partial}{\partial x} + V^2(t)\frac{\partial}{\partial y} = \frac{V^1(t)}{\sigma}E_1 + \frac{V^2(t)}{\sigma}E_2.$$

We have

$$\begin{aligned}
\nabla_{\dot c} V &= \nabla_{\dot c}\left(\frac{V^1}{\sigma}E_1 + \frac{V^2}{\sigma}E_2\right) \\
&= \dot c\left(\frac{V^1}{\sigma}\right)E_1 + \frac{V^1}{\sigma}\nabla_{\dot c}E_1 + \dot c\left(\frac{V^2}{\sigma}\right)E_2 + \frac{V^2}{\sigma}\nabla_{\dot c}E_2 \\
&= \frac{\dot V^1}{\sigma}E_1 + \frac{V^1}{\sigma}\omega_1^2(\dot c)E_2 + \frac{\dot V^2}{\sigma}E_2 + \frac{V^2}{\sigma}\omega_2^1(\dot c)E_1 \\
&= \left(\frac{\dot V^1}{\sigma} + \frac{\sigma_x V^2}{\sigma^2}\right)E_1 + \left(\frac{\dot V^2}{\sigma} - \frac{\sigma_x V^1}{\sigma^2}\right)E_2.
\end{aligned}$$

Hence $V(t)$ is parallel along the curve $c(t)$ if

$$\begin{cases} \dfrac{\dot V^1}{\sigma} + \dfrac{\sigma_x V^2}{\sigma^2} = 0 \\[2ex] \dfrac{\dot V^2}{\sigma} - \dfrac{\sigma_x V^1}{\sigma^2} = 0 \end{cases} \Leftrightarrow \begin{cases} \dot V^1 + \tanh(x_0)V^2 = 0 \\ \dot V^2 - \tanh(x_0)V^1 = 0 \end{cases}$$

Ths solution of this system of ODE's with initial condition $V(0) = \frac{\partial}{\partial x}$ is given by

$$V(t) = \cos((\tanh x_0)t)\frac{\partial}{\partial x} + \sin((\tanh x_0)t)\frac{\partial}{\partial y}.$$

(iii) Consider the curve $\gamma(t) = (1, t)$ with $0 \le t \le 1$ which connects the two points. Then the length of γ is given by

$$\ell(\gamma) = \int_0^1 \langle \dot\gamma(t), \dot\gamma(t)\rangle^{\frac{1}{2}}\,dt = \int_0^1 \langle (0,1),(0,1)\rangle^{\frac{1}{2}}\,dt = \int_0^1 \frac{1}{\cosh 1}\,dt = \frac{2}{e + e^{-1}}.$$

Therefore the distance between the points $(1, 0)$ and $(1, 1)$ is less or equal than $\ell(\gamma)$, which is clearly less than $\dfrac{2}{e}$.

3. Consider a 2-dimensional Riemanniann manifold (M, g) with constant curvature such that M is diffeomorphic to the connected sum of n tori, $T^2\#\ldots\#T^2$, where $n \ge 2$.

   (a) Show that a simple closed geodesic on M cannot be homotopic to a point.
   Hint: recall that $\chi(M\#N) = \chi(M) + \chi(N) - 2$.

   (b) Is there an isometric embedding of (M, g) into $\mathbb{R}^3$ with the Euclidean metric?

Solution:

(a) Using the hint and the fact that $\chi(T^2) = 0$, it follows that if M is diffeomorphic to $T^2\#\ldots\#T^2$ then $\chi(M) = \chi(T^2\#\ldots\#T^2) = 2(1-n)$. Then applying the Gauss-Bonnet Theorem we obtain

$$\int_M K = 2\pi\chi(M) = 4\pi(1-n).$$

We conclude that the Gauss curvature K of (M, g) is a negative constant if $n \geq 2$. On the other hand, if γ is a simple closed geodesic which is homotopic to a point then it bounds a region D which is homeomorphic to a disc. Applying the Gauss–Bonnet Theorem for a manifold with boundary to D we obtain

$$\int_D K + \int_{\partial D} k_g = 2\pi\chi(D),$$

where k_g is the geodesic curvature. Since $\chi(D) = 1$ and ∂D is a geodesic it follows that $k_g = 0$ and

$$\int_D K = 2\pi,$$

which is a contradiction because $K < 0$. Therefore a simple closed geodesic on M cannot be homotopic to a point.

(b) Suppose there is an isometric embedding and consider planes $z = C$, with $C \in \mathbb{R}$. Because M is compact there is a maximum value of C, for which the plane is tangent to M at a point p and the surface is below the plane. Since all points of M are on the same side of the tangent plane at p, it follows that both principal curvatures have the same sign (or one or both are equal to 0) at this point p, and consequently the Gauss curvature is great or equal to zero at the point p. This is a contradiction because the Gauss curvature is negative at every point of M. We conclude that there is no isometric embedding of (M, g) into $\mathbb{R}^3$ with the Euclidean metric.