Instituto Superior Técnico
Departamento de Matemática

Riemannian Geometry

Exam 2 – July 3, 2024

Show all the computations and relevant justifications.

Duration: 2 hours

(2 val.) 1. Show that the vector field $X = x^2 \frac{\partial}{\partial x}$, where $x \in \mathbb{R}$, is not complete.

2. Let M be a compact oriented manifold with boundary and $\omega \in \Omega^n(M)$ a volume element. The *divergence* of a vector field X is the function $\operatorname{div}(X)$ such that $L_X\omega = (\operatorname{div}(X))\omega$.

(2 val.) (a) Show that

$$\int_M \operatorname{div}(X) = \int_{\partial M} \iota_X \omega.$$

Hint: recall that $L_X\omega = \iota_X d\omega + d(\iota_X\omega)$ *(Cartan formula).*

(2 val.) (b) Using the previous result show that

$$\int_{S^2} x\,dy \wedge dz - y\,dx \wedge dz + z\,dx \wedge dy = 4\pi.$$

Hint: consider $X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}$.

3. Let (M, g) be a Riemannian manifold where M is a compact and simply connected 3-dimensional manifold and g is defined by

$$g = \frac{1}{4}\left[(d\psi - \cos\theta d\varphi) \otimes (d\psi - \cos\theta d\varphi) + d\theta \otimes d\theta + \sin^2\theta\, d\varphi \otimes d\varphi\right],$$

where $(\psi, \theta, \varphi) \in (0, 4\pi) \times (0, \pi) \times (0, 2\pi)$ are local coordinates in M. These coordinates cover an open set whose closure is M.

(2 val.) (a) Show that the connection forms ω_2^1, ω_1^3 and ω_2^3 associated to the orthonormal frame

$$E_1 = 2\frac{\partial}{\partial\psi}, \quad E_2 = 2\frac{\partial}{\partial\theta} \quad \text{and} \quad E_3 = \frac{2}{\sin\theta}\left(\cos\theta\frac{\partial}{\partial\psi} + \frac{\partial}{\partial\varphi}\right),$$

and correspondent orthonormal coframe

$$\omega^1 = \frac{1}{2}(d\psi - \cos\theta\, d\varphi), \quad \omega^2 = \frac{1}{2}d\theta \quad \text{and} \quad \omega^3 = \frac{1}{2}\sin\theta\, d\varphi,$$

are given by

$$\omega_2^1 = \frac{1}{2}\sin\theta\, d\varphi, \quad \omega_1^3 = \frac{1}{2}d\theta \quad \text{and} \quad \omega_2^3 = \frac{1}{2}(d\psi + \cos\theta\, d\varphi).$$

(3 val.) (b) Determine the curvature forms Ω_i^j.

(2 val.) (c) Show that M has constant curvature $K = 1$.

(1,5 val.) (d) Prove that (M, g) is geodesically complete and identify it.

(1,5 val.) (e) Show that the vector field $\frac{\partial}{\partial\psi}$ is *geodesic*, that is, its integral curves are geodesics.

(2 val.) (f) Consider the cylinder $C \subset M$ given in local coordinates by

$$\{(\psi, \theta, \varphi) \mid \psi \in (0, 4\pi);\ \theta \in [a, b] \subset (0, \pi); \varphi = \varphi_0 \in (0, 2\pi)\}$$

equipped with the metric induced by the metric g of M. Show that there is at least one point of C where its curvature must vanish. (Remark: C is a cylinder because the closure of the integral curves of $\frac{\partial}{\partial \psi}$ are circles).

(2 val.) (g) Consider the torus $T^2 \subset M$ given in local coordinates by

$$\{(\psi, \theta, \varphi) \mid \psi \in (0, 4\pi);\ \theta = \frac{\pi}{2}; \varphi \in (0, 2\pi)\}$$

equipped with the metric induced by the metric g of M. Show that the following relation between the curvatures of T^2 and M holds:

$$K^{T^2} - K^M = \langle B(X_p, X_p), B(Y_p, Y_p) \rangle - \|B(X_p, Y_p)\|^2,$$

where $p \in T^2$, $\{X, Y\}$ is a field of orthonormal frames and B is the second fundamental form of T^2.

Cartan structure equations for a field of orthonormal frames:

(i) $d\omega^i = \sum_{j=1}^n \omega^j \wedge \omega_j^i$,

(ii) $\omega_j^i + \omega_i^j = 0$,

(iii) $d\omega_i^j = \Omega_i^j + \sum_{k=1}^n \omega_i^k \wedge \omega_k^j$.