Instituto Superior Técnico
Departamento de Matemática

Riemannian Geometry

Exam 1 – June 19, 2024

Show all the computations and relevant justifications.

Duration: 2 hours

1. Consider the set
$$H = \{(x, y) \in \mathbb{R}^2 : y > 0\}$$
and identify each point in H with the invertible map $h : \mathbb{R} \to \mathbb{R}$ defined by $h(t) = yt+x$. The set containing these maps is a group, with the operation of composition of maps, so the previous indentification induces a group structure on H.

(2 val.) (a) Show that the group operation induced on H is given by
$$(x, y) \cdot (z, w) = (yz + x, yw),$$
and that H equipped with this operation is a Lie group.

(2 val.) (b) Show that the derivative of the left translation $L_{(x,y)} : H \to H$ at a point $(z, w) \in H$ is represented in the coordinates above by the matrix
$$(dL_{(x,y)})_{(z,w)} = \begin{pmatrix} y & 0 \\ 0 & y \end{pmatrix}.$$
Conclude that the left-invariant vector field determined by the vector
$$V = \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y} \in \mathfrak{h} \equiv T_{(0,1)}H,$$
where $\xi, \eta \in \mathbb{R}$, is the vector field
$$X^V = \xi y \frac{\partial}{\partial x} + \eta y \frac{\partial}{\partial y} \in \chi(H).$$

(2 val.) (c) Given $V, W \in \mathfrak{h}$, compute $[V, W]$.

(2 val.) (d) Compute the flow of X^V and write an expression for the exponential map $\exp : \mathfrak{h} \to H$.

2. Let (M, g) be a Riemannian manifold defined by $M = \mathbb{R}^2$ and
$$g = \frac{1}{\sigma^2(x, y)}(dx \otimes dx + dy \otimes dy),$$
where $\sigma : \mathbb{R}^2 \to \mathbb{R}$ is a positive function.

(2,5 val.) (a) Show that the connection form ω_1^2 associated to the orthonormal frame $E_1 = \sigma \frac{\partial}{\partial x}$, $E_2 = \sigma \frac{\partial}{\partial y}$ is given by
$$\omega_1^2 = \frac{\sigma_y}{\sigma} dx - \frac{\sigma_x}{\sigma} dy,$$
where $\sigma_x = \frac{\partial \sigma}{\partial x}$ and $\sigma_y = \frac{\partial \sigma}{\partial y}$.

(1,5 val.) (b) Show that the Gauss curvature of (M, g) is given by

$$K = \sigma(\sigma_{xx} + \sigma_{yy}) - (\sigma_x^2 + \sigma_y^2).$$

(c) Consider $\sigma(x, y) = \cosh(x)$.

(1 val.) (i) Justify that in this case (M, g) is not geodesically complete.

(2 val.) (ii) Determine the vector field $V(t)$, parallel along the curve $c(t) = (x_0, t)$ (for some $x_0 > 0$), satisfying $V(0) = \frac{\partial}{\partial x}$.

(1 val.) (iii) Show that the distance between the points $(1, 0)$ and $(1, 1)$ is less or equal than $\frac{2}{e}$.

3. Consider a 2-dimensional Riemanniann manifold (M, g) with constant curvature such that M is diffeomorphic to the connected sum of n tori, $T^2\#\ldots\#T^2$, where $n \geq 2$.

(3 val.) (a) Show that a simple closed geodesic on M cannot be homotopic to a point.
Hint: recall that $\chi(M\#N) = \chi(M) + \chi(N) - 2$.

(1 val.) (b) Is there an isometric embedding of (M, g) into $\mathbb{R}^3$ with the Euclidean metric?

Cartan structure equations for a field of orthonormal frames:

(i) $d\omega^i = \sum_{j=1}^n \omega^j \wedge \omega_j^i$,

(ii) $\omega_j^i + \omega_i^j = 0$,

(iii) $d\omega_i^j = \Omega_i^j + \sum_{k=1}^n \omega_i^k \wedge \omega_k^j$.