Symplectic Geometry

2^{nd} Semester 2012/2013

Homework 10

(**due on May 10th**)

1. Consider the standard $(S^1)^3$-action on $\mathbb{P}^3$:

$$(e^{i\theta_1}, e^{i\theta_2}, e^{i\theta_3}) \cdot [z_0 : z_1 : z_2 : z_3] = [z_0 : e^{i\theta_1} z_1 : e^{i\theta_2} z_2 : e^{i\theta_3} z_3] \ .$$

(a) What is the moment polytope for this action ?

(b) Exhibit explicitly the subsets of $\mathbb{P}^3$ for which the stabilizer under this action is $\{1\}$, S^1, $(S^1)^2$ and $(S^1)^3$. Show that the images of these subsets under the moment map are the interior, the facets, the edges and the vertices, respectively.

2. What is the moment polytope for the $\mathbb{T}^2$–action on $\mathbb{CP}^1 \times \mathbb{CP}^1$ as

$$(e^{i\theta}, e^{i\eta}).([z_0 : z_1], [w_0 : w_1]) = ([z_0 : e^{i\theta} z_1], [w_0 : e^{i\eta} w_1]) \ ?$$

3. Show that an effective hamiltonian action of a torus $\mathbb{T}^n$ on a $2n$–dimensional symplectic manifold gives rise to an integrable system.

Hint. Show that the coordinates of the moment map give n linearly independent commuting integrals of motion.

4.a) Classify all 2-dimensional Delzant polytopes with 3 vertices, i.e., triangles, up to translation, change of scale and the action of $\mathrm{SL}(2;\mathbb{Z})$.

Hint. By a linear transformation in $\mathrm{SL}(2;\mathbb{Z})$, we can make one of the angles in the polytope into a square angle. How are the lengths of the two edges forming that angle related?

4.b) Classify all 2-dimensional Delzant polytopes with 4 vertices, up to translation and the action of $\mathrm{SL}(2;\mathbb{Z})$.

Hint. By a linear transformation in $\mathrm{SL}(2;\mathbb{Z})$, we can make one of the angles in the polytope into a square angle. Check that automatically another angle also becomes 90^o.

4.c) What are all the 4-dimensional symplectic toric manifolds that have four fixed points?