Differential Geometry

Extra Homework - Connections in Principal Bundles

not mandatory

Let $\xi = (\pi, E, M)$ be a vector bundle of rank r. Given a local frame $\{s_1, \dots, s_r\}$ in a trivializing open set $U \subset M$, we can parameterize all the bases $\{v_1, \dots, v_r\}$ of the fibers E_p over points $p \in U$ by

$$\phi(v_1, \dots, v_r) = (p, B) \in U \times GL(r),$$

where $B \in GL(r)$ is the matrix of change of basis from $s(p)$ to $\{v_1, \dots, v_r\}$. Hence the set F of all bases of E is a differentiable manifold with a natural projection $\pi : F \to M$ and trivializing charts $\phi : \pi^{-1}(U) \to U \times GL(r)$. The triple $\Xi = (\pi, F, M)$ is called the **frame bundle** of ξ; note that the local frame $\{s_1, \dots, s_r\}$ can be seen as a local section $s : U \to F$. Since the fibers F_p are isomorphic to the Lie group $GL(r)$, Ξ is a **principal bundle**.

1. Show that the matrix $B' \in GL(r)$ which represents a basis of E_p in another local frame s' is related with the matrix $B \in GL(r)$ which represents the same basis in s by $B' = S^{-1}B$, where $S : U \to GL(r)$ is the matrix of change of basis from s to s'. Show that then ξ and Ξ are determined by the same cocycle.

2. In $\pi^{-1}(U)$ define the matrix of 1-forms

$$\tilde{\omega} = B^{-1} (\pi^* \omega) B + B^{-1} dB,$$

   where ω is the matrix of the forms of the connection associated to the local frame s. Show that $\tilde{\omega}$ is independent of the choice of frame s, and therefore is globally defined in F. Show also that $\omega = s^* \tilde{\omega}$.

3. In $\pi^{-1}(U)$ define the matrix of 2-forms

$$\tilde{\Omega} = B^{-1} (\pi^* \Omega) B,$$

   where Ω is the matrix of the forms of curvature associated to the local frame s. Show that $\tilde{\Omega}$ is independent of the choice of frame s, and therefore is globally defined in F. Show also that $\Omega = s^* \tilde{\Omega}$.

4. Show that

$$\tilde{\Omega} = d\tilde{\omega} + \tilde{\omega} \wedge \tilde{\omega} = d\tilde{\omega} + \frac{1}{2}[\tilde{\omega}, \tilde{\omega}]$$

   and

$$d\tilde{\Omega} + \tilde{\omega} \wedge \tilde{\Omega} - \tilde{\Omega} \wedge \tilde{\omega} = 0 \Leftrightarrow d\tilde{\Omega} + [\tilde{\omega}, \tilde{\Omega}] = 0.$$

5. Show that the kernel of the 1-forms of the matrix $\tilde{\omega}$ define a distribution H of dimension d in F, and that the curves in F tangent to H correspond to frames that are obtained by parallel transport along the projection of the curve in M. Show also that H is integrable *iff* the connection ∇ is flat.

6. Show that $GL(r)$ acts on the right on F, that the action is free and that $F/G = M$. Show also that if $\Psi(S) : F \to F$ is the action of $S \in GL(r)$ then $\Psi(S)^*\tilde{\omega} = S^{-1}\tilde{\omega}S = \mathrm{Ad}(S^{-1})\tilde{\omega}$, and if $\psi(A) \in \mathfrak{X}(F)$ is the infinitesimal action of $X \in \mathfrak{gl}(r)$ then $\tilde{\omega}(\psi(A)) = A$.

7. Any vector $X \in T_uF$ can be decomposed in a unique way as the sum $X = X^h + X^v$, where $X^h \in H_u$ and $X^v \in \ker d_u\pi$. If $\theta \in \Omega^k(F, \mathfrak{gl}(r))$, define the **covariant exterior derivative** $D\theta \in \Omega^{k+1}(F, \mathfrak{gl}(r))$ by

$$D\theta(X_1, \ldots, X_{k+1}) = d\theta(X_1^h, \ldots, X_{k+1}^h).$$

Show that

$$D\tilde{\omega} = \tilde{\Omega}$$

and

$$D\tilde{\Omega} = 0.$$