Instituto Superior Técnico
Departamento de Matemática
Secção de Álgebra e Análise

Differential Geometry

Homework 9

due on Tuesday, November 21

1. Let M be a connected manifold of dimension d, which is not orientable. Show that $H_c^d(M) = 0$ as follows. Let $\widetilde{M}$ denote the set of orientations for all the tangent spaces T_pM:
$$\widetilde{M} = \{(p, [\mu_p]) \,:\, [\mu_p] \text{ is an orientation for } T_pM\}.$$
Given a chart $\phi : U \subset M \to \mathbb{R}^d$, consider the map $\widetilde{\phi}^{-1} : \phi(U) \to \widetilde{M}$ defined by
$$\widetilde{\phi}^{-1}(x^1, \ldots, x^d) = (\phi^{-1}(x^1, \ldots, x^d), [dx^1 \wedge \ldots \wedge dx^d]).$$
One calls $\widetilde{M}$ the **orientation cover of** M. Show that:
   (a) $\widetilde{M}$ is an orientable manifold of dimension d;
   (b) The map $\pi : \widetilde{M} \to M$, $(p, [\mu_p]) \mapsto p$, is a double covering of M, *i.e.*, show that for each $p \in M$ there exists an open set $U \ni p$ such that $\pi^{-1}(U) = V_1 \cup V_2$, where V_1, V_2 are disjoint open sets and $\pi_{|V_i}$ is a diffeomorphism $(i = 1, 2)$;
   (c) The map $\Phi : \widetilde{M} \to \widetilde{M}$, $(p, [\mu_p]) \mapsto (p, -[\mu_p])$, is a diffeomorphism that changes orientation and satisfies $\pi = \pi \circ \Phi$, $\Phi \circ \Phi = \mathrm{id}$;
   (d) $\widetilde{\omega} \in \Omega^k(\widetilde{M})$ is of the form $\widetilde{\omega} = \pi^*\omega$, for some $\omega \in \Omega^k(M)$, if and only if $\Phi^*\widetilde{\omega} = \widetilde{\omega}$;
   (e) Conclude that we must have $H_c^d(M) = 0$.

2. Compute $H^\bullet(M)$, when:
   (a) $M = \mathbb{P}^d$;
   (b) $M = \mathbb{T}^2$.