Instituto Superior Técnico
Departamento de Matemática
Secção de Álgebra e Análise

Differential Geometry

Homework 7

due on Wednesday, November 9

A **symplectic manifold** is a pair (M, ω), where M is a differentiable manifold and $\omega \in \Omega^2(M)$ is a **closed** 2-form and **non–degenerate** (the map $T_pM \to T_p^*M$ given by $v \mapsto i_v\omega$ is an isomorphism of vector spaces for all $p \in M$). Show that:

1. The dimension d of M is even $d = 2n$;
2. $\mu = \omega^n$ is a volume form in M (hence any symplectic manifold is orientable);
3. A surface admits a symplectic structure if and only if it is orientable;
4. Every function $f \in C^\infty(M)$ is constant on the flux of its **symplectic gradient** X_f, which is defined by $i_{X_f}\omega = df$;
5. The flux of the symplectic gradient X_f preserves the symplectic form, i. e. $\mathcal{L}_{X_f}\omega = 0$;
6. The **Poisson parenthesis** $\{\cdot,\cdot\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$, defined by the formula $\{f, g\} = \omega(X_f, X_g)$, determines a Lie algebra structure in $C^\infty(M)$;
7. The map $C^\infty(M) \to \mathfrak{X}(M)$, given by $f \mapsto X_f$, is an anti-homomophism of Lie algebras, i. e. $X_{\{f,g\}} = -[X_f, X_g]$.