Instituto Superior Técnico
Departamento de Matemática
Secção de Álgebra e Análise

Differential Geometry

Homework 6

due on Tuesday, October 31

1. Let $G_k(\mathbb{R}^d)$ denote the set of all linear subspaces of $\mathbb{R}^d$ of dimension k.

   (a) Show tat there is a bijection between $G_k(\mathbb{R}^d)$ and the quotient $O(d)/O(k) \times O(d-k)$ (hence $G_k(\mathbb{R}^d)$has a smooth structure for which this bijection becomes a diffeomorphism. This gives $G_k(\mathbb{R}^d)$ the structure of a manifold called the **Grassmanian manifold** of k-planes in $\mathbb{R}^d$; note that $G_1(\mathbb{R}^{d+1}) = \mathbb{RP}^d$.

   (b) Show that $\dim G_k(\mathbb{R}^d) = k(d-k)$.

   (c) Show that $G_k(\mathbb{R}^d)$ is diffeomorphic to $G_{d-k}(\mathbb{R}^d)$.

2. Let G be a connected Lie group and $\pi : \tilde{G} \to G$ its universal covering. Show that:

   (a) $\ker \pi$ is a normal, closed, discrete subgroup of the center of $\tilde{G}$ (hence it is abelian);

   (b) G is isomorphic to $\tilde{G}/\ker \pi$ (hence $\ker \pi$ is the fundamental group of G).