

Differential Geometry

Homework 5

due on Tuesday, October 23

1. Let G and H be Lie groups and $\Phi : G \rightarrow H$ a Lie group homomorphism. Show that:

- (a) if $(d\Phi)_e$ is an isomorphism then Φ is a local diffeomorphism;
- (b) if Φ is a surjective local diffeomorphism then Φ is a covering map.

Hint: show that if U is a neighbourhood of $e \in G$ such that the restriction $\Phi|_U$ is a diffeomorphism then

$$\Phi^{-1}(\Phi(U)) = \bigcup_{g \in \ker \Phi} L_g(U)$$

where the open sets $L_g(U)$ are disjoint; and $\Phi : L_g(U) \rightarrow L_{\Phi(g)}(\Phi(U))$ is a diffeomorphism.

2. (a) Show that $\mathbb{R} \cdot SU(2)$ is a 4-dimensional real linear subspace of $\mathcal{M}_{2 \times 2}(\mathbb{C})$, closed under matrix multiplication, with basis

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad i = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad j = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad k = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},$$

satisfying $i^2 = j^2 = k^2 = ijk = -1$. Therefore this space can be identified with the **quaternions**. Show that $SU(2)$ can be identified with the quaternions of Euclidean norm equal to 1, and is therefore diffeomorphic to S^3 .

(b) Show that the Lie algebra of $SU(2)$ consists of the skew-hermitian matrices of trace zero

$$\mathfrak{su}(2) = \left\{ \begin{pmatrix} i\alpha & \beta \\ -\beta & -i\alpha \end{pmatrix} : \alpha \in \mathbb{R}, \beta \in \mathbb{C} \right\}$$

and therefore can be identified with \mathbb{R}^3 .

(c) Show that if $n \in \mathbb{R}^3$ is a unit vector, which we identify with a quaternion with zero real part, then

$$\exp\left(\frac{n\theta}{2}\right) = 1 \cos\left(\frac{\theta}{2}\right) + n \sin\left(\frac{\theta}{2}\right)$$

is also a unit quaternion.

(d) Again, identifying \mathbb{R}^3 with quaternions with zero real part, show that the map $\text{Ad}_{n,\theta} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ given by

$$v \mapsto \exp\left(\frac{n\theta}{2}\right) \cdot v \cdot \exp\left(-\frac{n\theta}{2}\right)$$

is a rotation by an angle θ about the axis defined by n .

(e) Show that there exists a surjective homomorphism $\Phi : SU(2) \rightarrow SO(3)$, and use it to conclude that $SU(2)$ is the universal covering space of $SO(3)$.